Differential geometry
Spring 2012
Exercise 1.

1. Let us look at the vector space $\mathbb{R}^n$. In the cases $n = 2$ and $n = 3$ prove that if we have n linearly independent vectors in $\mathbb{R}^n$ it is a basis of $\mathbb{R}^n$.

2. Let $S = \{x_1, \dots, x_n\} \subset \mathbb{R}^n$ be a set of vectors $x_i \neq 0$ and assume that they are orthogonal. That is

$$\langle x_i, x_j \rangle = 0, \quad i \neq j.$$

   Show that the set S is linearly independent.

3. Let us define

$$\begin{aligned} S_1 &= \{(r\cos(\theta), r\sin(\theta)) \mid r \in [0,1],\ \theta \in [0,\pi]\} \subset \mathbb{R}^2 \\ S_2 &= \{(r\cos(\theta), r\sin(\theta)) \mid r \in [0,1[= [0,1),\ \theta \in [\pi, 2\pi]\} \subset \mathbb{R}^2 \\ S &= S_1 \cup S_2 \subset \mathbb{R}^2 \end{aligned}$$

   Is the set S open or closed ?

4. Can you find a set $S \subset \mathbb{R}^n$ which is open and closed ?

5. Let us look at the curve which is defined by $c : [a,b] \mapsto \mathbb{R}^2$, $c = (x(t), y(t))$, $c \in C^1((a,b))$ and let us then assume that the image of the curve can be represented implicitly by a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f \in C^1(\mathbb{R}^2)$, $f(x,y) = 0$. Show that in points $t \in (a,b)$, $(x,y) = c(t)$

$$\langle \nabla f(x,y), c'(t) \rangle = 0.$$

6. We say that two curves $c_1 : I_1 \mapsto \mathbb{R}^2$ and $c_2 : I_2 \mapsto \mathbb{R}^2$ *osculate* if for some $t_1 \in I_1$ and $t_2 \in I_2$, $c_1(t_1) = r = c_2(t_2)$ and they have the same tangent lines at this point. Let us then assume that we can represent the image of two curves by equations

$$\begin{aligned} f_1(x,y) &= y - x^2 = 0 \\ f_2(x,y) &= ax - y - 2 = 0. \end{aligned}$$

   Find possible parameters a so that the curves osculate.

7. Let us look at the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ and assume that at least $f \in C^2(\Omega)$. Then function f has a Taylor expansion with remainder at least of order 2

$$U(p+h) = U(p) + \nabla U(p) \cdot h + h^T(d^2U(p)h) + \varepsilon(h)\|h\|^2, \quad \lim_{h \to 0} \varepsilon(h) = 0.$$

for all $p \in \Omega$. Where $d^2U(p) \in \mathbb{R}^{3\times 3}$ is the Hessian matrix of f formed from all partial derivatives of order 2. Show that if all the eigenvalues of $d^2U(p)$ are positive and $\nabla f(p) = 0$ then $f(p)$ is a local minimum of f.

Hint: Remeber that if $f \in C^2(\Omega)$ then the Hessian at p is symmetric and so it has the spectral decomposition

$$d^2U(p) = X\Lambda X^T, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3), \; X^T X = I. \tag{1}$$

8. Try to find all the possible spelling or any other mistakes/errors in the lecture notes in Moodle and report them at demonstrations.