Differential geometry
Spring 2012
Exercise 11.

1. Let us look at the set $S = \{(x, y, z) \mid z = \|x + y\|\}$. Is the set S a smooth manifold ?

2. Let us look at the function $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$,

$$f_1 = x^2 + y^2 + z^2 - 1$$
$$f_2 = (ax)^2 + (ay)^2 - 1$$

   Is the set $f^{-1}\{(0,0)\}$ a smooth manifold ? If yes what is its dimension?

3. Let us look at set of all invertible 2×2 matrices

$$S = \{A \in \mathbb{R}^{2\times 2} \mid A \text{ has an inverse matrix.}\}$$

   If you take a random 2×2 matrix $A \subset \mathbb{R}^{2\times 2}$ what is the probability that $A \in S$.

4. Is the set $M \subset \mathbb{R}^{2\times 2}$ of all singular 2×2 matrices a smooth submanifold of $\mathbb{R}^{2\times 2}$.

5. Suppose that M is a smooth manifold and $\dim(M) = n$. Suppose that (x, U) is a chart of M and $p \in M$. Assume that we have two curves

$$\gamma_1 : (a, b) \mapsto M$$
$$\gamma_2 : (a, b) \mapsto M,$$

   and further $\gamma_1(t_0) = \gamma_2(t_0) = p$. Let us the look at the curves

$$\alpha_1 := x \circ \gamma_1 : (a, b) \mapsto \mathbb{R}^n$$
$$\alpha_2 := x \circ \gamma_2 : (a, b) \mapsto \mathbb{R}^n$$

   We say denote $\gamma_1 \sim \gamma_2$ if

$$\frac{d}{dt}(x \circ \gamma_1)(t_0) = \frac{d}{dt}(x \circ \gamma_2)(t_0).$$

   Prove that $\sim$ is an equivalence relation between curves γ through p.

6. We gave an alternative description for tangent space of M at p as a set of equivalence classes $[\gamma(t_0)]$ of curves defined in previous exercise

$$T_pM = \{[\gamma'(t_0)] \mid \gamma'(t_0) \text{ is a tangent vector}\}.$$

Can you think of an easy way to find a basis for T_pM using this definition.

7. In last lectures I tried to picture S^1 and its tangent bundle. Write down explicitly the tangent bundle of S^1 by the parametrization of its polar coordinates representation and as an implicit representation of a function $f(x, y) = x^2 + y^2 - 1$, $S^1 = f^{-1}(0)$.