Differential geometry
Spring 2012
Exercises 12.

1. Let us look at the map $f : M_1 = \mathbb{R}^2 \mapsto \mathbb{R}^2 = M_2$, $f(x,y) = (y,x)$. Compute the pushforward of f_*, when both cases manifolds $\mathbb{R}^2$ are equipped with identity coordinate system $(\mathbb{R}^2, id)$.

2. Let's look at the map from previous exercise, but now with the difference $M_1 = (R^2, y)$ where y^{-1} is the mapping

$$y^{-1}(\theta, r) = (a\cos(\theta), b\sin(\theta)) \quad a, b > 0,$$

and compute the pushforward f_* again for the same mapping.

3. Let's look a the same mapping from exercise 1. but now consider the manifolds $M_1 = (\mathbb{R}^2_+, y)$, $(\mathbb{R}^2_+, x)$, where the mappings x and y are now $y^{-1}(\theta, r) = (r\cos(\theta), r\sin(\theta))$ and $x^{-1}(\tau, \xi) = (\tau\xi, (1/2), ((1/2)(\tau^2 + \xi^2))$ compute the pushforward f_* again for the same mapping.

4. Consider the mapping x^{-1} from previous exercises

$$x^{-1}(\tau, \xi) = r(\tau, \xi) = \tau\xi e_x + \frac{1}{2}(\tau^2 + \xi^2)e_y = x_1 e_x + x_2 e_y$$
$$e_x = (1,0) \; e_y = (0,1).$$

which parts of $\mathbb{R}^2$ you can present diffeomorphically with this mapping ? Next compute $\nabla x_1 = E_\tau$ and $\nabla x_2 = E_\xi$. Then compute the first fundamental form of $\mathbb{R}^2$ in these coordinates. Notice that the diagonal elements are zero. This means that the coordinate system is *orthogonal*. Prove this by computing $\langle E_\tau, E_\xi \rangle$. Then form an orthonormal basis for $\mathbb{R}^2$ using these vectors.

5. compute the *Laplacian*

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial^2 x_1} + \frac{\partial^2}{\partial^2 x_2}$$

for both coordinates (r, θ) and (τ, ξ) from exercise 4.(Hint: Use the lemma 4.3 from the lecture notes.)

6. Prove lemma 4.9 from lecture notes.

7. Prove lemma 4.10 from lecture notes.