Differential geometry
Spring 2012
Exercises 13.

1. Let us look at the manifold $(\mathbb{R}^2 \setminus \{0\}, y)$, $y^{-1} = (r\cos(\theta), r\sin(\theta))$ from last exercises with standard metric. Compute the components of the metric in these coordinates.

2. Similarly compute the components of the metric given by the parabolic coordinates $x^{-1}(\tau, \xi) = (\tau\xi, (1/2)(\tau^2 - \xi^2))$ in $\mathbb{R}^2 \setminus \{0\}$.

3. Compute the components Γ^k_{ij} of the Levi-Civita connection for coordinates in tasks 1. and 2.

4. Let us then look at the spherical coordinates and $(\mathbb{R}^3, y)$, $y_{-1}(r, \theta, \psi)$

$$\begin{aligned} y_1^{-1} &= r\cos(\psi)\sin(\theta) \\ y_2^{-1} &= r\sin(\psi)\sin(\theta) \\ y_3^{-1} &= r\cos(\theta). \end{aligned}$$

   Compute the components of the (standard) metric in $\mathbb{R}^3$ in these coordinates.

5. Compute the components Γ^k_{ij} of the Levi-Civita connection from previous task 4.

6. Let us define the *gradient* ∇f of $f \in C^\infty(M)$ as a vector field in Riemannian manifold (M, g) which satisfies

$$g(\nabla f, X) = X(f) \quad \forall\, X \in \Gamma(M).$$

   compute the gradient in $(\mathbb{R}^2 \setminus \{0\}, y)$ and $(\mathbb{R}^2 \setminus \{0\}, x)$ in standard metric with coordinates from tasks 1. and 2.

7. Let (M, g) be a Riemannian manifold and $[ij, k]$ the Christoffel symbols of the $1st$ kind. Prove

$$\sum_{m=1}^{n} [ij, m] g^{km} = \Gamma^k_{ij}.$$

   Hint: Use the result from proof of Thm. 4.9.

8. Prove that the Levi-Civita connection is symmetric.