**Differential geometry**
**Spring 2012**
**Exercises** 14.

1. Let us look at the Poincare half plane $M = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 > 0\}$ with metric
$$G = x_2^{-2} I.$$
compute the components of the metric in polar coordinates.

2. Similarly compute the components of the metric for poincare half plane in parabolic coordinates from Ex. 12 and 13.

3. Form the equations for the geodesics for Poincare half-plane in coordinates from tasks 1 and 2.

4. Let us look at the sphere $S^2$ parametrized in polar coordinates. Form the equations for the geodesics and try to get at least two different solutions for the geodesics.

5. Prove the implication (2) $\Rightarrow$ (1) in lemma 4.13 in lecture notes.

6. Consider the theory of special relativity: Assume that we have the identity coordinate system $(t, x, y, z)$ for $M$. The pseudo Riemannian metric in these coordinates is $G = \mathrm{diag}(1, -c^{-2}, -c^{-2}, -c^{-2})$. *The proper time* $\tau$ is then given by
$$\tau = \int_a^b \sqrt{t'(s)^2 - (1/c^2)(x'(s)^2 + y'(s)^2 + z'(s)^2)}ds.$$
Consider then a situation where $t'(s) = 1$ so that $ds = dt$. Suppose then that observer 1. is remains still at point $A$ and the observer 2. takes a trip from $A$ around a close curve returning back to $A$. Suppose that the speed of observer 2. remains constant $\|c'(t)\|^2 = 9c/10$. Assume that when 2. leaves $t_0 = 0$ and when 2. returns observer 1. notices that 2. has been away for 10 years. How long has the trip taken from observer 2. and how much ($\Delta t$) the observer 2. has to move forward his/hers clock to get back in same time with 1. ?

7. In last exercise let the curve $c$ be arbitrary smooth and closed curve. As in previous exercise derive the the general formula for the time dilatation.

8. Let us investigate a special situation in theory of general relativity. The assumptions for the space time $M$ now are

   (a) $M$ is Rotationally invariant (invariant with respect to $\mathbb{SO}(3)$).

   (b) Time invariance (invariant under change $t \to -t$).

   (c) No external gravitational fields (Energy momentum tensor $T_{kl} = 0$).

   We use the polar coordinates chart for space variables and in these coordinates we can put the pseudo Riemannian metric/metric tensor $G$ in form

$$G = \begin{pmatrix} -x(r)^2 & 0 & 0 & 0 \\ 0 & y(r)^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{pmatrix}$$

   From the assumption follows that the Einstein field equations reduces to

$$R_{kl} = 0, \tag{1}$$

   where $R_{kl}$ is the Ricci curvature tensor defined in lectures. Try to solve the components of the metric tensor $-x(r)^2 = g_{11}$ and $y(r)^2 = g_{22}$ from this equation. Notice that $x$ and $y$ only depend on $r$ so you will have a set of ordinary differential equations. Notice that you have to compute Chritoffel symbols also. You can use maple to derive and solve the equations for $x$ and $y$. Remember also that we are now using coordinates $(t, r, \theta, \psi)$.