Differential geometry
Spring 2012
Exercise 2.

1. For example in mechanics the work done between time interval $\Delta t = t_2 - t_1$ by the particle moving along the image C of the smooth regular curve $c : [t_1, t_2] \mapsto \mathbb{R}^3$ on an influence of a smooth force field $F := (f_1, f_2, f_3) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is

$$W = \int_C F \cdot dc$$

Proof that if there exists a smooth function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla f = F$ then

$$W = \int_C F \cdot dc = f(c(t_2)) - f(c(t_1)) \tag{1}$$

Moreover the curve is called *closed* if $c(a) = c(b)$. Show that in this case

$$W = \int_C F \cdot dc = 0$$

2. Consider again a smooth force/vector field $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $F = (f_1, f_2, f_3) : \mathbb{R}^3 \mapsto \mathbb{R}^3$. Show that if the force/vector field F perpendicular to the tangent vector c' of of $c : [t_1, t_2] \mapsto \mathbb{R}^3$ on its image C

$$\int_C F \cdot dc = 0$$

3. Parametrize the following curves by arc length.
a)
A straight line $c_1 : [0, 1] \mapsto \mathbb{R}^2$ between $q, p \in \mathbb{R}^2$,

$$c_1(t) = tq + (1 - t)p$$

b) A circle of radius a and center at $p = (p_1, p_2)$, $c_2 : [0, 2\pi] \mapsto \mathbb{R}^2$

$$c_2(t) = (p_1 + a\cos(t), p_2 + a\sin(t)).$$

4. Compute the curvature and lengths of the curves c_1 and c_2 from last exercise.

5. Let $F = (f_1, f_2) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a vector field and $c : [a, b] \mapsto \mathbb{R}^2$ a smooth and regular curve. If the vector field has a constant length $R > 0$ on the image C of c.

$$\|F\| = R \quad \forall \ (x, y) = c(t) \ t \in [a, b],$$

prove that then $\langle F'(t), F(t) \rangle = 0$.

6. Let C_a be a set of implicitly defined images of curves $f(x, y)_a = 0$

$$\begin{aligned} &f(x, y)_a = y^2 - x^3 - ax = 0 \\ &C_a = \{(x, y) \in \mathbb{R}^2 \mid f(x, y)_a = 0\}. \end{aligned}$$

We say in this case that $p \in C_a$ is a singular point of the curve if $\nabla f = 0$. Find the possible parameters a and points p where the curves are singular.

7. Report again all the mistakes in the lecture notes at demonstrations. I will put them in pdf format to moodle on Monday.