Differential geometry
Spring 2012
Exercise 5.

1. In case $n = 2$ show that if the linear mapping $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ is symmetric $\langle Lx, y\rangle = \langle x, Ly\rangle \ \forall \ x, y \in \mathbb{R}^n$ then the matrix L_K of the linear map in basis $K = \{x_1, x_2\}$ is symmetric $L_K = (L_K)^T$.

2. Prove the lemma 3.1 in the lecture notes.

3. Prove the theorem 3.3 in the lecture notes.

4. Let $c : \mathbb{R} \mapsto \mathbb{R}^3$ be a regular smooth curve. *A generalized cone* is a surface M_f defined by
$$f(u_1, u_2) = a + u_2(c(u_1) - a).$$
Draw a picture of the curve. Show that the curvature of this surface is zero by showing that the image of the Gauss map μ is a curve.

5. Let M_f be a smooth surface defined by $f : \Omega \mapsto \mathbb{R}^3$. We say that f is *angle preserving/conformal* if $T(u) = \lambda(u)I$ for some scalar function λ, where I is the identity matrix. Let then
$$f(u) = \frac{1}{\cosh(u_2)}(\cos(u_1), \sin(u_1), \sinh(u_2)).$$
Check that $M_f = f(\Omega)$, $\Omega = [0, 2\pi] \times \mathbb{R}$ is the unit sphere minus north pole $N = (1, 0, 0)$ and south pole $S = (0, 0, -1)$, $M_f = S^2 \setminus \{N, S\}$. Show that f is conformal. The inverse of f is called *Mercator projection*. Compute the inverse.

6. Let $c : I \mapsto \mathbb{R}^2$ be a regular curve. A surface given by
$$f(s, \theta) = (c_1(s)\cos(\theta), c_1(s)\sin(\theta), c_2(s))$$
is called *the surface of revolution*. The plane curve $c(s) = (c_1(s), c_2(s))$ is called *a profile curve* and curves $c_1 : [0, 2\pi] \mapsto M_f$ $c_2 : I \mapsto M_f$ defined by
$$c_1(\theta) = f(s, \theta) \quad (s = \text{const.})$$
$$c_2(s) = f(s, \theta) \quad (\theta = \text{const.})$$
are called *parallels* and *meridians*. Draw a picture which explains the terminology. Assume then that the profile curve is parametrized by arclength and compute the curvature.

7. Report all the possible errors in lecture notes.