Differential geometry
Spring 2012
Exercise 6.

1. Compute explicitly Christoffel symbols Γ^k_{ij} when surface $M_f \subset \mathbb{R}^3$ is given as a graph

$$f(u_1, u_2) = (u_1, u_2, h(u_1, u_2)).$$

You can do this by hand or use for example Maple or Mathematica.

2. Let T be the first fundamental form whose components E, F, G are functions of $u = (u_1, u_2)$ and let Γ^k_{ij} be the corresponding Christoffel symbols. Show that

$$\frac{\partial}{\partial u_1} \ln \sqrt{\det(T)} = \Gamma^1_{11} + \Gamma^2_{12}$$
$$\frac{\partial}{\partial u_2} \ln \sqrt{\det(T)} = \Gamma^1_{12} + \Gamma^2_{22}.$$

3. Let's look at the surface $M = f(\Omega)$, $\Omega = [0, 2\pi] \times \mathbb{R}$ defined by $f(u_1, u_2) := (\cos(u_1), \sin(u_1), u_2)$. Compute the first and second fundamental forms T, $\tilde{T}$ and curvature κ of the surface M and compute the principal curvatures λ_1 and λ_2 of M.

4. Let us look at the surface M_f given by $f := (u_1, u_2, u_1^2)$. Compute the first and second fundamental forms $T, \tilde{T}$ and curvature κ of M_f.

5. Suppose that $u : \mathbb{R}^3 \mapsto \mathbb{R}$ and $v : \mathbb{R}^3 \mapsto \mathbb{R}$ are smooth functions and assume that $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is smooth function. Show that if $f(u, v) = 0$ then $\nabla u \times \nabla v = 0$.

6. Report all the possible errors in lecture notes.