Differential geometry
Spring 2012
Exercise 7.

In these exercises there will be 9 **tasks, The first five will test your skills in classical vector analysis which in theory you should already master fluently by now. The classical vector analysis is one area which has laid the foundations for modern differential geometry.**

1. Suppose that $u : \mathbb{R}^3 \mapsto \mathbb{R}$ and $v : \mathbb{R}^3 \mapsto \mathbb{R}$ are smooth vector fields such that $\nabla u \times \nabla v = 0$, and let us look at their restriction to levelset/surface $S = \{(x, y, z) \mid u(x, y, z) = c = \mathit{constant}\}$. Prove that on surface either $v(x, y, z) = \mathit{constant}$ or there exists a smooth function $F : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $F(u, v) = 0$.

2. Suppose that $u, v, w : \mathbb{R}^3 \mapsto \mathbb{R}$ are smooth functions

$$\begin{aligned} u &:= u(x, y, z) \\ v &:= v(x, y, z) \\ w &:= w(x, yz) \end{aligned}$$

   and additionally assume $\nabla u \cdot \nabla v \times \nabla v = 0$. Moreover suppose then that there is a smooth function s.t $f(u, v, w) = 0$ so we can look at the composite function $\tilde{f} = f(u(x, y, z), v(x, y, z), w(x, y, z)) = 0$. Additionally suppose that $(f_x, f_y, f_z) \neq (0, 0, 0)$. Show that then

$$\begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} = 0. \tag{1}$$

   Additionally show if the that the converse is also true if the eq.(1) is valid and if there are smooth functions $a, b : \mathbb{R} \mapsto \mathbb{R}$ such that if $b \neq 0$ and f_v and v_w satisfy the partial differential equation/compatability condition (2)

$$f_v - (a/b) f_w = 0. \tag{2}$$

3. Solve the partial differential equation $\Delta V = \nabla^2 V = 0$ in spherical coordinates assuming that V depends only on the distance from origin $V := V(r)$.

4. Suppose that $A : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a smooth vector field defined by $A := \nabla \times (\psi \mathbf{r})$ and assume that $\Delta\psi = \nabla^2\psi = 0$. Compute $A \cdot \nabla \times A$ in sperical coordinates and show that if ψ is of the product form $\psi = R(r)\Theta(\theta)\Phi(\psi)$ then $A \cdot \nabla \times A = 0$.

   Hint: Remember that in spehrical coordinates the position vector $\mathbf{r}$ is

$$\mathbf{r} = re_r = r(\sin(\theta)\cos(\varphi)e_1 + \sin(\theta)\sin(\varphi)e_2 + \cos(\theta)e_3).$$

   where r represents the length of the vector. Then use the divergence and cross products in spherical coordinates.

5. Let $f := f^1e_1 + f^2e_2 + f^3e_3$ be a smooth vector field and let $A \subset \mathbb{SO}(3)$ be a rotation matrix. Let us then rotate the coordinate axis by A. Show that the divergence has the exactly similar expression in new basis´/coordinates. In other words show that the divergence of a vector field is invariant under rotations.

6. Let us look at the torus $f : [0, 2\pi] \times [0, 2\pi] \mapsto T \subset \mathbb{R}^3$ parametrized as $f := (x(\theta, \psi), y(\theta), \psi), z(\theta))$

$$\begin{aligned} x &= (a + b\cos(\theta)\cos(\psi)) \\ y &= (a + b\cos(\theta))\sin(\psi) \\ z &= b\sin(\theta), \quad a > b,\ \psi \in [0, 2\pi],\ \theta \in [0, 2\pi]. \end{aligned}$$

   Choose for example $a = 3$ and $b = 1$ and formulate the geodesic equations for torus. See how the equations changes when $\theta = constant$ and $\psi = constant$. Can you solve this explicitly. Compute the curvature on the inner "equator" of the torus $\theta = \pi$ and on the outer equator of the torus $\theta = 0$. In inner equator the curvature should be negative and in the outer equator it should be positive.

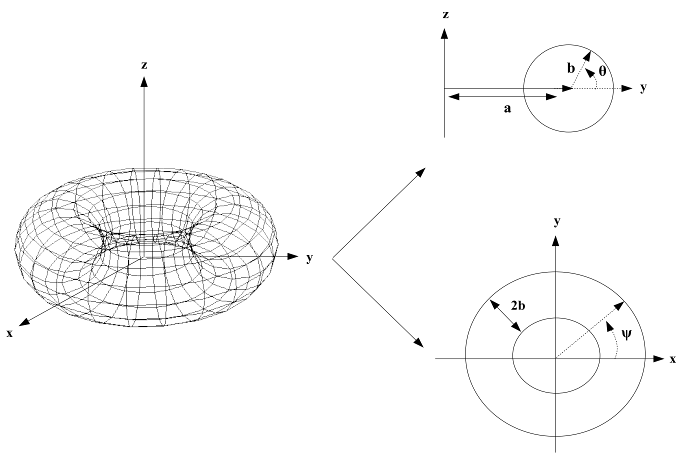

Figure 1: Toruksen hajoituskuva.

7. In theory of classical mechanics in the absence of external force field the particle is either in rest or moves at constant speed in geodesics(straight lines) of $\mathbb{R}^3$. However the law also states that if there are now external force fields present the particle still moves at constant velocity $\|v\|^2 = \|(c'(t))\|^2 = C^2$. Differentiating this with respect to time parameter we get

$$\frac{d}{dt}\|c'(t)\|^2 = \frac{d}{dt}\langle c', c''\rangle + \langle c'', c'\rangle = 2\langle c', c''\rangle = 0,$$

which implies $c'' \perp c'$ on the surface. Now remember the Newton's second law $F(c(t)) = ma(t) = mc''(t)$. We defined the work done by the particle

$$W = \int_{t_0}^{t_1} \langle W(c(t)), c'(t)\rangle = \int_{t_0}^{t_1} \underbrace{\langle c''(t), c'(t)'\rangle}_{=0} = 0.$$

the velocity $c'(t) = v$ is then always perpendicular to the tangent plane T_pM of the surface so we can conclude that the particle moves along geodesics of the surface. There will be no external forces ***but*** there will be a constraint force $F = ma = mc''(t)$ keeping the particle in the surface. By Newton's 3hd law the net force will be zero.

   (a) What you would call the force which is opposite to the direction of the constrained force ?

   (b) Compute the forces explicitly if the particle is restricted to move on the surface of the torus in previous exercise.(You can assume that the mass m of the particle is $m = 1$.)

8. In theory of classical mechanics there is also an equivalent formulation for the equations motion for objects. The equations can be obtained by computing the *Gâteaux derivative* of the *Lagrange function* L which in absence of external forces reduces just to $L = T$ where T is the kinetic energy of let's say one particle. The Gâteaux derivative can be computed perhaps most conviniently by solving the Euler-Lagrange equations for particular system. The kinetic energy and the Lagrange function for one particle of mass $m = 1$ is now just

$$\|c'(t)\|^2 = \|v\|^2 = x'^2 + y'^2 + z'^2 = T,$$

and the Euler-Lagranges equations of motion are

$$\frac{d}{dt}\frac{\partial T}{\partial \theta} - \frac{\partial T}{\partial \theta'} = 0$$
$$\frac{d}{dt}\frac{\partial T}{\partial \psi} - \frac{\partial T}{\partial \psi'} = 0.$$

Suppose then that the particle is restricted to move on the surface of the torus on exercise 6 and verify that that you get the same equations for the motion of the particle as you get for the geodesics of the torus.

9. Report all the possible mistakes in lecture notes on demonstrations.