Differential geometry
Spring 2012
Exercise 7.

In these exercises there will be 10 tasks, Before moving to general manifolds the first five tasks will test your skills in classical vector analysis which in theory you should already mastered fluently by now. The classical vector analysis is one area which has laid the foundations for modern differential geometry.

Moreover I will try from now on every week arrange additional computer demonstrations. This is only for the students good. It is totally pointless and downright crazy that our University pays huge licence costs for Maple, Mathematica and Matlab and then there are no comprehensive course where we would teach the use of these programs. I would say that instead buying expensive computers and expensive programs that 3 persons know how to use I suggest that we could start to use to use Abacus and Slide rule, logarithm and trigonometric tables a and save the extra money from expensive computers and printers to the salaries of the staff so they could keep their jobs For example for the salaries off secretaries etc...Hopefully you can notice a slight sarcasm in my text, but if you don't see it I could not care less. I also can say from experience that if you want to pursue you currier in private sector, you need programming skills. And my suggestion wold be that after two first years of studies we should aim the researcher to right directions. I mean seriously when you are applying job what can you do if you have done for example Ms.C and Licenciate thesis from example from Blachke products ?

1. Suppose that $u : \mathbb{R}^3 \mapsto \mathbb{R}$ and $v : \mathbb{R}^3 \mapsto \mathbb{R}$ are smooth vector fields such that $\nabla u \times \nabla v = 0$, and let us look at their restriction to levelset/surface $S = \{(x, y, z) \mid u(x, y, z) = c = \mathit{constant}\}$. Prove that on surface either $v(x, y, z) = \mathit{constant}$ or there exists a smooth function $F : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $F(u, v) = 0$.

2. Suppose that $u, v, w : \mathbb{R}^3 \mapsto \mathbb{R}$ are smooth functions

$$\begin{aligned} u &:= u(x, y, z) \\ v &:= v(x, y, z) \\ w &:= w(x, yz) \end{aligned}$$

and additionally assume $\nabla u \cdot \nabla v \times \nabla w = 0$. Moreover suppose then that there is a smooth function s.t $f(u, v, w) = 0$ so we can look at the composite function $\tilde{f} = f(u(x, y, z), v(x, y, z), w(x, y, z)) = 0$. Additionally suppose that $(\tilde{f}_x, \tilde{f}_y, \tilde{f}_z) \neq (0, 0, 0)$. Show that then

$$\begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} = 0. \tag{1}$$

Additionally show that if (1) is valid and there are smooth functions $a, b : \mathbb{R} \mapsto \mathbb{R}$ and that $b \neq 0$ satisying

$$f_v - (a/b) f_w = 0. \quad (2) \tag{2}$$

So that there will be a function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that $\nabla u \cdot \nabla v \times \nabla w$=0

3. Solve the partial differential equation $\Delta V = \nabla^2 V = 0$ in spherical coordinates assuming that V depends only on the distance from origin $V := V(r)$.

4. Suppose that $A : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a smooth vector field defined by $A := \nabla \times (\psi \mathbf{r})$ and assume that $\Delta\psi = \nabla^2\psi = 0$. Compute $A \cdot \nabla \times A$ in sperical coordinates and show that if ψ is of the product form $\psi = R(r)\Theta(\theta)\Phi(\psi)$ then $A \cdot \nabla \times A = 0$.

   Hint: Remember that in spehrical coordinates the position vector $\mathbf{r}$ is

   $$\mathbf{r} = re_r = r(\sin(\theta)\cos(\varphi)e_1 + \sin(\theta)\sin(\varphi)e_2 + \cos(\theta)e_3).$$

   where r represents the length of the vector. Then use the divergence and cross products in spherical coordinates.

5. Let $f := f^1e_1 + f^2e_2 + f^3e_3,\ f_i := f^i(x, y, z)$ be a smooth vector field and let $A \subset \mathbb{SO}(3)$ be a rotation matrix. Let us then rotate the coordinate axis by A. Show that the divergence has the exactly similar expression in new basis/coordinates. In other words show that the divergence of a vector field is invariant under rotations and has the same structure in new coordinates.

6. Let us look at the torus $f : [0, 2\pi] \times [0, 2\pi] \mapsto T \subset \mathbb{R}^3$ parametrized as $f := (x(\theta, \psi), y(\theta, \psi), z(\theta))$

   $$\begin{aligned} x &= (a + b\cos(\theta)\cos(\psi)) \\ y &= (a + b\cos(\theta))\sin(\psi) \\ z &= b\sin(\theta), \quad a > b,\ \psi \in [0, 2\pi],\ \theta \in [0, 2\pi]. \end{aligned}$$

   Choose for example $a = 3$ and $b = 1$ and formulate the geodesic equations for torus. See how the equations changes when $\theta = constant$ and $\psi = constant$. Can you solve this explicitly. Compute the curvature on the inner "equator" of the torus $\theta = \pi$ and on the outer equator of the torus $\theta = 0$. In inner equator the curvature should be negative and in the outer equator it should be positive.

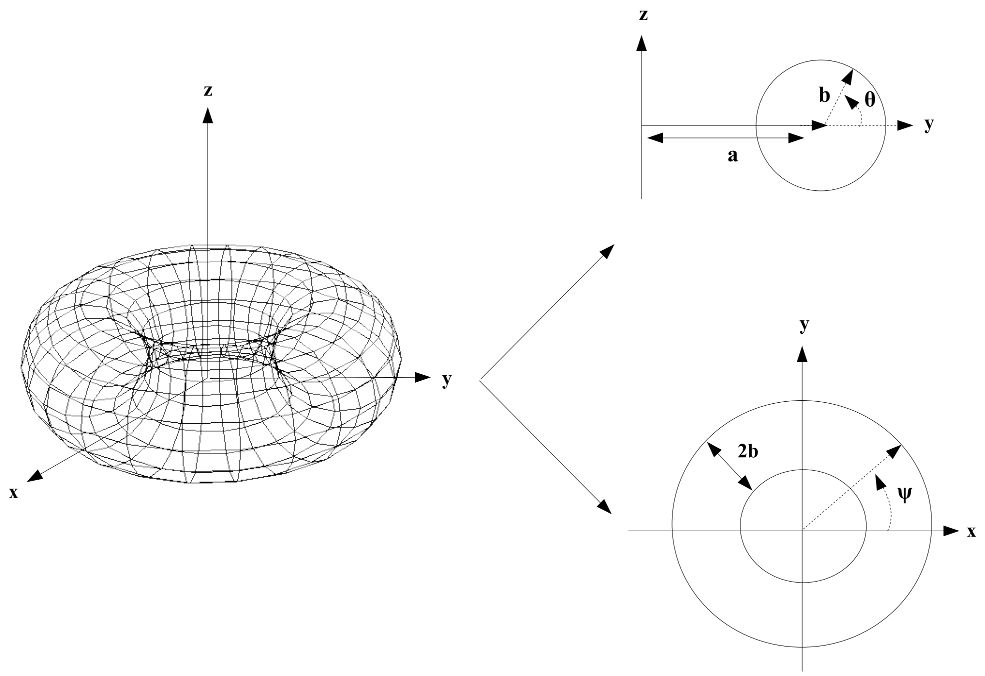

Figure 1: Toruksen hajoituskuva related to Ex.6

7. In theory of classical mechanics in the absence of external force field the particle is either in rest or moves at constant speed in geodesics(straight lines) of $\mathbb{R}^3$. However the law also states that if there are now external force fields present the particle still moves at constant velocity $\|v\|^2 = \|(c'(t))\|^2 = C^2$. Differentiating this with respect to time parameter we get

$$\frac{d}{dt}\|c'(t)\|^2 = \frac{d}{dt}\langle c', c''\rangle + \langle c'', c'\rangle = 2\langle c', c''\rangle = 0,$$

which implies $c'' \perp c'$ on the surface. Now remember the Newton's second law $F(c(t)) = ma(t) = mc''(t)$. We defined the work done by the particle

$$W = \int_{t_0}^{t_1} \langle W(c(t)), c'(t)\rangle dt = \int_{t_0}^{t_1} \underbrace{\langle c''(t), c'(t)\rangle}_{=0} dt = 0.$$

the acceleration $c''(t) = a$ is then always perpendicular to the tangent plane T_pM of the surface M so we can conclude that the particle moves along geodesics of the surface. There will be no external forces ***but*** there will be a constraint force $F = ma = mc''(t)$ keeping the particle in the surface. By Newton's 3hd law the net force will be zero.

(a) What you would call the force which is opposite to the direction of the constrained force ?

(b) Compute the forces explicitly if the particle is restricted to move on the surface of the torus in previous exercise.(You can assume that the mass m of the particle is $m = 1$.)

8. In theory of classical mechanics there is also an other equivalent(in fact many) formulations for the equations of motion for objects. The result can be obtained by

computing the *Gâteaux derivative* and setting it to zero. In special case of classical mechanics solving the Gâteaux derivative will just lead to famous *Euler-Lagranges equations of motion.* The Lagrange functions reduces in the absence of external force fields in absence of external forces reduces just to $L = T$ where T is the kinetic energy of let's say one particle. The Gâteaux derivative can be computed perhaps most conviniently by solving the Euler-Lagrange equations for particular system. The kinetic energy and the Lagrange function for one particle of mass $m = 2$ is now just

$$\|c'(t)\|^2 = \|v\|^2 = x'^2 + y'^2 + z'^2 = T,$$

and the Euler-Lagranges equations of motion are

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \theta} - \frac{\partial T}{\partial \theta'} &= 0 \\ \frac{d}{dt}\frac{\partial T}{\partial \psi} - \frac{\partial T}{\partial \psi'} &= 0. \end{aligned} \tag{3}$$

Suppose then that the particle is restricted to move on the surface of the torus on exercise 6 and verify that that you get the same equations for the motion of the particle as you get for the geodesics of the torus. Can you compute a general solution with special functions ? In this case a good candidate would probably be the Jacobi's Elliptic Functions. Can you reduce the order of the equations by finding some conservation laws/invariants for the equations (3). This will usually greatly simplify the solution process.

9. Transform the equations of motion to four first order equation. Linearize the coefficient matrix at equilibrium points and compute its eigenvalues. How many equilibrium points there will be ? Try to deduce that the inner "equator" of the torus $\theta = \pi$ is unstable equilibrium point and on the outer equator of the torus $\theta = 0$ is stable equilibrium point of the equations of motion of (3).

10. Report all the possible mistakes in lecture notes on demonstrations.