Differential geometry
Spring 2012
Exercises 9 and 10.

1. Proof that the function $f : (-\pi/2, \pi/2) :\mapsto \mathbb{R}$, $f(x) = \tan(x)$ is a diffeomorphism.

2. Let $(a, b) \subset \mathbb{R}$ be an open interval. Proof that it is diffeomorphic to $(-\pi/2, \pi/2)$, that is: Form a diffeomorphism between these two sets. Remember the cases $a = -\infty$, $b = -\infty$ and $a = -\infty$ and $b = -\infty$.

3. A subset S of a normed space M is called convex if all of its points can be connected by a line segment which belongs to S. Let then $U \subset S$ be open and $p \in U$. Show that there is a convex neighborhood of p.

4. Let us look at the functions $f, g : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x, y) = x^2 - y^2$ and $g(x, y) = x^2 - y$. Are the sets $f^{-1}(0)$ and $g^{-1}(0)$ a smooth manifolds ?

5. Form a smooth atlas for circle $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \subset \mathbb{R}^2$, which has three charts.

6. Let us look at smooth mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^k$, $n \geq k$. Show that the directional derivative of f at $x \in \mathbb{R}^n$ to direction $y \in \mathbb{R}^n$ is

$$\lim_{\lambda \to 0} \frac{f(x + \lambda y) - f(x)}{\lambda} = df(x)y,$$

where $df(x)$ is the Jacobian of f at x.

7. Let us look at the set of $\mathbb{R}^{2\times 2} \simeq \mathbb{R}^4$ matrices and its subset

$$\mathrm{SL}(2) = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1\} \subset \mathbb{R}^{2\times 2}.$$

Suppose we have a smooth curve $A : \mathbb{R} \mapsto \mathrm{SL}(2)$, $a_{ij} = a_{ij}(t)$, which satisfies $A(0) = I$. Show that we have a differential expansion of A at $t = 0$

$$A(t) = I + t\mathrm{Tr}(A'(0)) + o(t), \quad o(t) = \varepsilon(t)\|A(t) - A(0)\|, \ \lim_{t \to 0} \varepsilon(t) = 0,$$

where $Tr(X)$ means the trace of the matrix X.

8. Show that $\mathrm{SL}(2) \subset \mathbb{R}^{2\times 2}$ is a smooth submanifold of $\mathbb{R}^{2\times 2}$ and compute its dimension.

9. Suppose that we have a bounded set $S \subset \mathbb{R}^2$ which boundary ∂S is an image of a smooth Jordan curve. Let us then take a linear mapping $A \subset \mathrm{SL}(2)$. Compute the area of $A(S) \subset \mathbb{R}^2$.

10. Let $V = \mathrm{span}\{e_1, \ldots, e_n\}$ be an n-dimensional vector space with coefficient field $\mathbb{R}$. Suppose that $f : V \mapsto \mathbb{R}$ is a linear map and assume $f \neq 0$. Let then $g : V \mapsto \mathbb{R}$ be a linear map and suppose that

$$\ker(f) = \ker(g).$$

Proof that we have to have $f = cg$, where $0 \neq c \in \mathbb{R}$.

11. Prove lemma 4.4 from the lecture notes.