Entire Functions Fall 2017.

## Chapter 1. The Möbius transformations

* the Alexandroff extension theorem.

let $(X, \tau)$ be a topological space.

$\tau \equiv$ collection of subsets of $X$ :

(i) $\phi, X \in \tau$

(ii) $\{X_i\}_{i \in I} \in \tau \Rightarrow \bigcup_{i \in I} X_i \in \tau$.

(iii) $\{X_i\}_{i=1}^{N} \in \tau \Rightarrow \bigcap_{i=1}^{N} X_i \in \tau$

the elements of $\tau$ are called open sets and $\tau \equiv$ topology on $X$.

Example - Given a metric space, $(X, d)$

$$\tau = \{ D(a,r) : a \in X, r > 0 \},$$

$$D(a,r) = \{ x \in X : d(a,x) < r \}$$

So... let $(X, \tau)$ be a topological space. the Alexandroff extension of $X$ is certain compact space $X^*$ together with an open embedding $\varphi : X \to X^*$ such that the complement of $X$ in $X^*$ consists of a single point, denoted by $\infty$.

the map $\varphi$ is a Hausdorff compactification $\Leftrightarrow$ $X$ is locally compact, non-compact, and Haussdorff.

- Open map: $\forall$ open set $U \subset X$, $\varphi(U)$ is open in $X^*$.
- Embedding: $\varphi : X \to \varphi(X)$ homeomorphism.
(≡ injective & continuous)
- $(X, \tau)$ is Haussdorff: $\forall\ x_1, x_2 \in X$, $x_1 \neq x_2$, $\exists\ X_1, X_2 \in \tau$: $X_1 \cap X_2 = \emptyset$, $x_1 \in X_1$, $x_2 \in X_2$.
- $(X, \tau)$ is compact: Each of its open covers has a finite subcover ≡
If $X = \bigcup_{i \in I} X_i$, then, $\exists\ \{i_1, \dots, i_N\} \subset I$:
$$X = \bigcup_{j=1}^{N} X_{i_j}.$$
- Locally compact if every point has a compact neighborhood.

"Our example"

Consider the topological space $(\mathbb{C}, \tau_{|\cdot|}) = \mathbb{C}$

$|\cdot|$ = Euclidean distance

It is clear that $\mathbb{C}$ is not compact

(Use $X_n = D(0,n)$, $n \in \mathbb{N}$)

Take a point $\infty \notin \mathbb{C}$ and define $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\tau_\infty = \tau_{|\cdot|} \cup \Theta_\infty$, where

$$\Theta_\infty = \{ \hat{\mathbb{C}} \setminus K : K \text{ is compact in } \mathbb{C} \}$$

($\equiv$ family of neighborhoods of $\infty$).

$$Id = \varphi : \mathbb{C} \to \hat{\mathbb{C}}$$

then:

→ $\tau_\infty$ is a topology in $\hat{\mathbb{C}}$

→ $(\mathbb{C}, \tau_{|\cdot|})$ is a dense supspace of $(\hat{\mathbb{C}}, \tau_\infty)$

→ $(\hat{\mathbb{C}}, \tau_\infty)$ is compact:

$\hat{\mathbb{C}} = \bigcup_{i \in I} X_i$. Since $\infty \in \hat{\mathbb{C}}$, $\exists\ i_0 \in I : \infty \in X_{i_0}$.

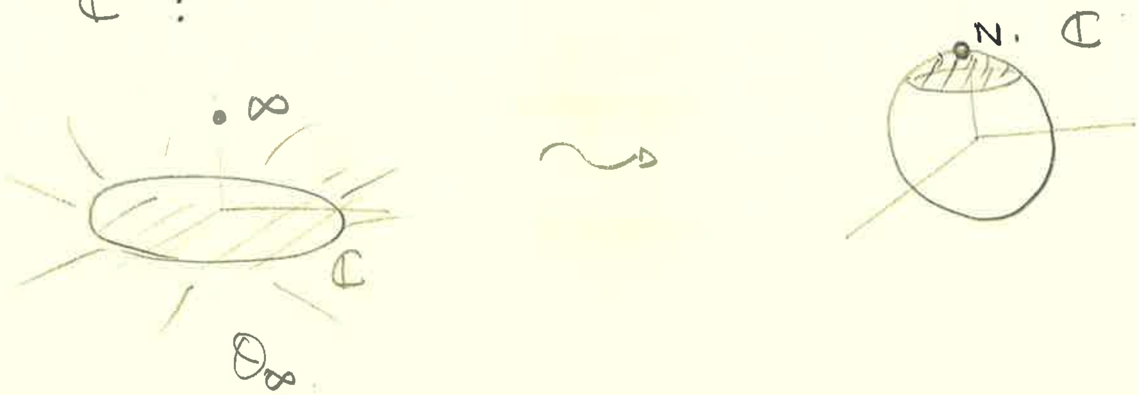
$\Rightarrow \exists\ K_0$, compact in $\mathbb{C}$ : $X_{i_0} = \hat{\mathbb{C}} \setminus K_0$

$$\Rightarrow \hat{\mathbb{C}} = \left( \bigcup_{i \in I \setminus i_0} X_i \right) \cup G_{i_0}.$$

But $K_0 \subset \bigcup_{i \in I \setminus i_0} X_i$ & $K_0$ compact in $\mathbb{C}$ !

$\square$

$\hat{\mathbb{C}}$ :

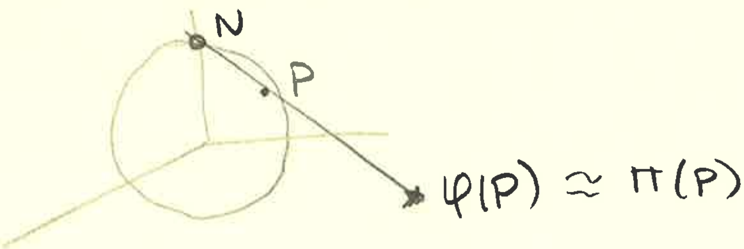



<u>Remark</u>.- There is <u>no</u> metric on $\hat{\mathbb{C}}$ which equals the extension of $d_e$ on $\mathbb{C}$

(Sup) $\rho : \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \mathbb{R}$ is a metric on $\hat{\mathbb{C}}$ with $\rho(z,w) = |z-w| \ \forall\, z,w \in \mathbb{C}$.

then,
$$\underset{\substack{\downarrow n\to\infty \\ \infty}}{n} = \rho(n,0) \le \underset{\substack{\downarrow n\to\infty \\ 0}}{\rho(n,\infty)} + \rho(\infty,0) \qquad \rightarrow\leftarrow$$

the homeomorphism $\varphi$ is given by the stereographic projection: to each point $P$ of the sphere $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$, there corresponds the complex number $x+iy$, where $(x,y,0)$ is the intersection of the line that joints $P$ and $N=(0,0,1)$ with the $(x,y)$-plane



**Theorem** $\pi : S^2 \to \hat{\mathbb{C}}$ is a homeomorphism

given by:

$$\pi(x_1,x_2,x_3) = \begin{cases} \dfrac{x_1+ix_2}{1-x_3}, & x_3 \neq 1 \\ \infty, & x_3 = 1 \end{cases}$$

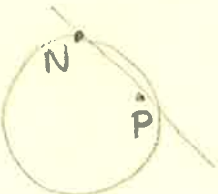
Moreover,

$$\pi^{-1}(z) = \begin{cases} \left( \dfrac{2\operatorname{Re} z}{1+|z|^2}, \dfrac{2\operatorname{Im} z}{1+|z|^2}, \dfrac{|z|^2-1}{1+|z|^2} \right), & z = x+iy \in \mathbb{C} \\ (0,0,1), & z = \infty \end{cases}$$

Pf.- Let $P = (x_1,x_2,x_3) \in S^2$

$x_3 = 0 \rightsquigarrow \pi(P) = (x_1, x_2, 0) \sim x_1 + ix_2$.

$x_3 \neq 0$. the line $PN$ is given by:




$$\begin{cases} c_1 = t\,0 + (1-t)\,x_1 \\ c_2 = t\,0 + (1-t)\,x_2 \\ c_3 = t\,1 + (1-t)\,x_3 \end{cases}$$

$$\equiv \{ (1-t)x_1, (1-t)x_2, t+(1-t)x_3 : t \in \mathbb{R} \}.$$

the intersection with $\{z=0\}$ occurs when

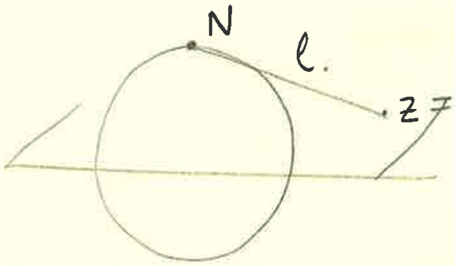
$$t + (1-t)x_3 = 0$$

$$\equiv (1-x_3)t = -x_3 \equiv t = \frac{-x_3}{1-x_3}.$$

So that $1-t = \dfrac{1}{1-x_3}$ and, hence,

$$\pi(P) = \frac{x_1 + i x_2}{1 - x_3}$$

$\pi^{-1}$:




$z = x + iy \in \mathbb{C}$
$\cong$
$(x, y, 0)$

$$\ell = \{((1-t)x\,,\ (1-t)y,\ t) : t \in \mathbb{R}\}.$$

$\ell \cap S^2$: $\quad (1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$

$$\equiv \ t^2 (x^2+y^2+1) - 2t(x^2+y^2) + x^2+y^2-1 = 0.$$

$$\equiv \ (1+|z|^2)\, t^2 - 2t|z|^2 + |z|^2 - 1 = 0$$

$$\equiv \quad t = \frac{2|z|^2 \pm \sqrt{4|z|^4 + 4(1-|z|^4)}}{2(1+|z|^2)}$$

$$= \frac{|z|^2 \pm 1}{1+|z|^2} = \begin{cases} 1 \longrightarrow N!!!! \\ \dfrac{|z|^2 - 1}{1+|z|^2}. \end{cases}$$

Prove the continuity of $\pi$ & $\pi^{-1}$ and check that $\pi(\pi^{-1}(z)) = z$ & $\pi^{-1}(\pi(P)) = P$!

A nice property:

<u>DEF</u>.- A circle on $S^2$ is the intersection of $S^2$ with a plane.

Let $C$ be a circle on $S^2$.

$\equiv \mathcal{C} = \{ (x_1, x_2, x_3) \in S^2 : ax + bx_2 + cx_3 = d \}$.

$\pi(\mathcal{C})$ ?

* $(0,0,1) \notin \mathcal{C}$

$$(x_1, x_2, x_3) \sim \left( \frac{2\operatorname{Re} z}{1+|z|^2}, \frac{2\operatorname{Im} z}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2} \right)$$

$$\rightsquigarrow \quad a\left(\frac{z+\bar z}{1+|z|^2}\right) + b\left(\frac{z-\bar z}{i(1+|z|^2)}\right) + c\left(\frac{|z|^2-1}{1+|z|^2}\right) = d.$$

$$\equiv \quad 2ax + 2by + c(x^2+y^2-1) = d(x^2+y^2+1).$$

$$\equiv \quad (c-d)x^2 + (c-d)y^2 + 2ax + 2by - (c+d) = 0.$$

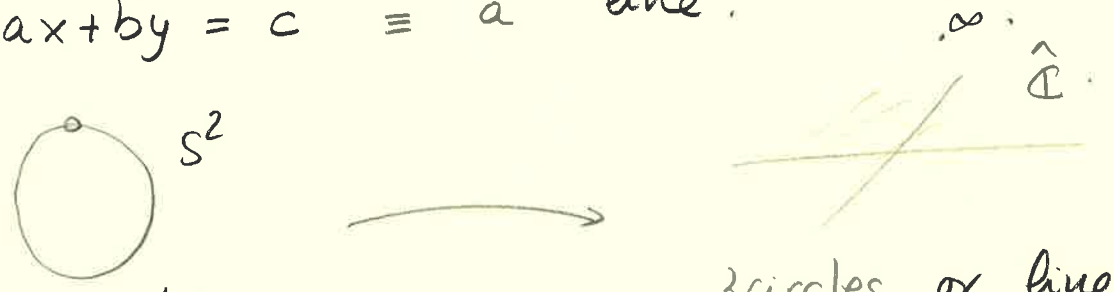
* $c = d \Rightarrow (0,0,1) \in \mathcal{C}$. $\longrightarrow$ later.
* $c \neq d \Rightarrow \left(x + \frac{a}{c-d}\right)^2 + \left(y + \frac{b}{c-d}\right)^2 - \frac{a^2}{(c-d)^2} - \frac{b^2}{(c-d)^2}$

$$- \frac{c+d}{c-d} = 0.$$

$$\equiv \left(x+\frac{a}{c-d}\right)^2+\left(y+\frac{b}{c-d}\right)^2=\frac{a^2+b^2+c^2-d^2}{(c-d)^2}$$

If $(0,0,1)\in \mathcal{C} \Rightarrow$ ($\Leftarrow$ too) $c=d$.

$\Rightarrow \quad ax+by=c \quad \equiv$ a line.

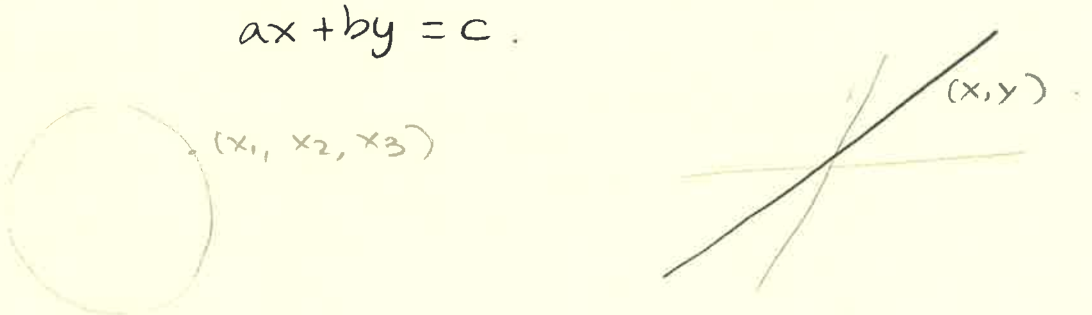



$\{\text{circles}\}$ $\qquad$ $\{\text{circles or lines}\}$.

Reciprocally:

A line in the plane is given by

$$ax+by=c.$$




$$a\left(\frac{x_1}{1-x_3}\right)+b\left(\frac{x_2}{1-x_3}\right)=c$$

$$\equiv \quad ax_1+bx_2+cx_3=c \quad \equiv \text{plane!}$$

Moreover, $(0,0,1)$ satisfies this equation.

Circles ?

$$|z-a| = r \equiv |z-a|^2 = r^2$$

$$(z-a)\,\overline{(z-a)} = r^2 \equiv |z|^2 - a\bar{z} - \bar{a}z + |a|^2 = r^2$$

$$\equiv \quad x^2+y^2 - 2\operatorname{Re}\{a\bar{z}\} = r^2 - |a|^2 .$$

$$\equiv \quad x^2+y^2 - 2(a_1x + a_2y) = r^2 - |a|^2 .$$

$$(x_1, x_2, x_3) \longmapsto (x, y) = x + iy = \frac{x_1}{1-x_3} + i\,\frac{x_2}{1-x_3} .$$

$x_1^2 + x_2^2 + x_3^2 = 1$ !

$$\frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} - 2a_1 \cdot \frac{x_1}{1-x_3} - 2a_2\,\frac{x_2}{1-x_3} = r^2 - |a|^2$$

$$\equiv \quad \frac{1+x_3}{1-x_3} - \frac{2a_1x_1 + 2a_2x_2}{1-x_3} = r^2 - |a|^2$$

$$\equiv \quad Ax_1 + Bx_2 + Cx_3 = D .$$

In other words, $\pi$ (and $\pi^{-1}$) preserve the family of {circles & lines}.

# Möbius transformations.

**DEF.-** A Möbius (or linear fractional) transformation is a function of the form

$$T(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}.$$

Note that $c=d=0$ does not make sense!

Also that

$$\frac{az+b}{cz+d} = \frac{a(cz+d)}{c(cz+d)} - \frac{ad-bc}{c(cz+d)}, \quad c \neq 0.$$

and

$$\frac{az+b}{d} = \frac{b}{d} + \frac{adz}{d^2}.$$

that is, if $ad-bc=0$, $T$ is constant.

Moreover, $T$ is non-constant $\overset{①}{\Longleftrightarrow}$ $ad-bc \neq 0$ $\overset{②}{\Longleftrightarrow}$ $T$ is one-to-one.

Pf.- ① $T'(z) = \dfrac{ad-bc}{(cz+d)^2}$. this proves ① $\Leftarrow$ ].

To show $\Rightarrow$, it suffices to note that the assertion is equivalent to $ad-bc=0 \Rightarrow T$ constant, which was proved above.

② $T(z) = T(w) \Longleftrightarrow \dfrac{az+b}{cz+d} = \dfrac{aw+b}{cw+d}$

$$\iff \cancel{aczw} + adz + bcw + \cancel{bd} = \cancel{aczw} + adw + bcz + \cancel{bd}$$

$$\iff (ad - bc)(z - w) = 0 \implies \text{We have } ② .$$

Some more properties; but first let's agree with the notation

$$\mathcal{M} = \left\{ T(z) = \frac{az+b}{cz+d} : ad - bc \neq 0 \right\}.$$

Now, for $T \in \mathcal{M}$, it is clear that

$$T : \mathbb{C} \to \hat{\mathbb{C}}$$

$$\left( T\left(\frac{-d}{c}\right) = \infty, \ c \neq 0 \right)$$

but we can extend

$$T : \hat{\mathbb{C}} \to \hat{\mathbb{C}} .$$

by defining $T(\infty) = \begin{cases} \infty, & c = 0 \\ \frac{a}{c}, & c \neq 0 \end{cases}$.

thus, every $T \in \mathcal{M}$ defines a function (which is necessarily one-to-one) in $\mathbb{C}$.

But since $T^{-1}(z) = \dfrac{dz+b}{a-cz} \in \mathcal{M}$,

we have that any <u>$T \in \mathcal{M}$ is a</u> <u>bijection on</u> $\hat{\mathbb{C}} \sim S^2$.

It is obvious that if $T, S \in \mathcal{M}$, then $T^{-1}$, $S \circ T \in \mathcal{M}$.

Moreover, $\mathrm{Id}(z) = z \in \mathcal{M}$.

Hence,

<u>Lemma</u> $(\mathcal{M}, \underset{\substack{\uparrow \\ \text{composition}}}{\circ})$ is a group.

though it is not commutative, as the following example shows.

<u>Example</u>

$$T_1(z) = \frac{z}{z+1}, \qquad T_2(z) = \frac{z+1}{z+2}.$$

then, $T_1 \circ T_2(z) = \dfrac{z+1}{2z}$, $\quad T_2 \circ T_1(z) = -2z-1$.

Among all $T \in \mathcal{M}$, let us consider the following groups of (some times called <u>simple</u>) Möbius transformations:

→ Translations: $z \longmapsto z+b$, $b \neq 0$.

→ Rotations: $z \longmapsto az$, $|a|=1$.

→ Homotheties: $z \longmapsto rz$, $r \in \mathbb{R}\setminus\{0\}$.

→ Inversions: $z \longmapsto 1/z$

<u>Theorem</u>.- Every $T \in \mathcal{M}$ is the composition of simple Möbius transformations as above.

<u>Pf</u>.- <u>CASE 1</u>. $c=0$:

$$T(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d} = \underbrace{\left|\frac{a}{d}\right|}_{\text{Homothety}} \underbrace{e^{i \operatorname{Arg} \frac{a}{d}}}_{\text{rotation}} z + \underbrace{\frac{b}{d}}_{\text{translation}}.$$

<u>CASE 2</u>. $c \neq 0$.

Define $f_1(z) = z + \frac{d}{c}$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{bc-ad}{c^2} z$, $f_4(z) = z + \frac{a}{c}$.

then $f_4 \circ f_3 \circ f_2 \circ f_1 (z) = \dfrac{az+b}{cz+d}$ $\square$.

<u>Remark</u> - Theorem 1.2 in [I. Laine, C.A. III] says that any $T \in \mathcal{M}$ preserves the family

$\mathcal{F} = \{$circles or lines in $\mathbb{C}\} = \{$circles in $\hat{\mathbb{C}}\}$.

We will prove this theorem using a different approach related to the cross-ratio.

First, let's prove the following lemma:

**Lemma**.- Given 3 distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, $\exists!\ T \in \mathcal{M}$: $T(z_1) = 1$, $T(z_2) = 0$, $T(z_3) = \infty$.

**Pf.** **EXISTENCE**.

If $z_1, z_2, z_3 \in \mathbb{C}$, $T(z) = \dfrac{z-z_2}{z-z_3} : \dfrac{z_1-z_2}{z_1-z_3}$ satisfies $T(z_1)=1$, $T(z_2)=0$, $T(z_3)=\infty$.

If some $z_i = \infty$, just take limits in the previous expression:

$$T(z) = \frac{z-z_2}{z-z_3}, \ z_1 = \infty, \quad T(z) = \frac{z_1-z_3}{z-z_3}, \ z_2 = \infty$$

& $$T(z) = \frac{z-z_2}{z_1-z_2}, \ z_3 = \infty.$$

**UNIQUENESS**. Assume that $T$ and $S$ satisfy the hypotheses. Define $M = S^{-1} \circ T$.

Then, $M \in \mathcal{M}$ and $M(z_i) = z_i$, $i = 1, 2, 3$.

That is, $M$ has 3 fixed points in $\hat{\mathbb{C}}$.

But how many fixed points a Möbius transformation can have in $\hat{\mathbb{C}}$?

Let's see... we are to solve $M(z) = \frac{az+b}{cz+d} = z$.

CASE 1: $\boxed{c=0}$ In these cases, $\infty$ is a fixed point!

$az+b = z \Leftrightarrow (a-1)z + b = 0$.

this gives different possibilities.

→ $(a,b) = (1,0)$ (i.e., $M = Id$), all points in $\hat{\mathbb{C}}$ are fixed.

→ $a \neq 1 \to z = \frac{-b}{a-1}$. , → $a=1, b\neq 0$ → no fixed points in $\mathbb{C}$

that is, $c = 0 \Rightarrow$
- $M = Id$
- 2 points ($a \neq 1$) are fixed
- 1 point ($a=1, b\neq 0$) is fixed.

$\boxed{c \neq 0}$ $\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0$.

$$z = \frac{a-d \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

$\therefore$ 2 fixed points, at most.

that is, $M \in \mathcal{M}$ fixes 3 points $\Rightarrow M = Id$.

this ends the proof of the lemma. $\square$

# The cross-ratio.

**DEF.-** The cross-ratio of 4 distinct points $z_0, z_1, z_2, z_3$ is the value of $S(z_0)$, where $S$ is the Möbius transformation satisfying $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$. Notation: $(z_0, z_1, z_2, z_3)$

So that (see p. 14), if $z_1, z_2, z_3 \in \mathbb{C}$,

$$(z_0, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} : \frac{z_1 - z_2}{z_1 - z_3}.$$

**Theorem.-** If $T \in \mathcal{M}$,

$$(T(z_0), T(z_1), T(z_2), T(z_3)) = (z_0, z_1, z_2, z_3).$$

**Pf.-** Let $S$ be the Möbius transformation defined by $S(z_1) = 1$, $S(z_2) = 0$, $S(z_3) = \infty$.

And let $M = S \circ T^{-1}$.

Note that
$$M(T(z_1)) = S(z_1) = 1$$
$$M(T(z_2)) = 0$$
$$M(T(z_3)) = \infty.$$

Hence, $(T(z_0), T(z_1), T(z_2), T(z_3)) = M(T(z_0))$

$$= S \circ T^{-1}(T(z_0)) = S(z_0) = (z_0, z_1, z_2, z_3)$$

$\square$.

Another nice result is

**Theorem.** – Let $\mathcal{F} = \{\text{circles in } \hat{\mathbb{C}}\}$ $= \{\text{circles and lines in } \mathbb{C}\}$ and let $T$ be a Möbius transformation. Then $T(\mathcal{F}) \subset \mathcal{F}$.

The proof of this theorem is based on the following lemma.

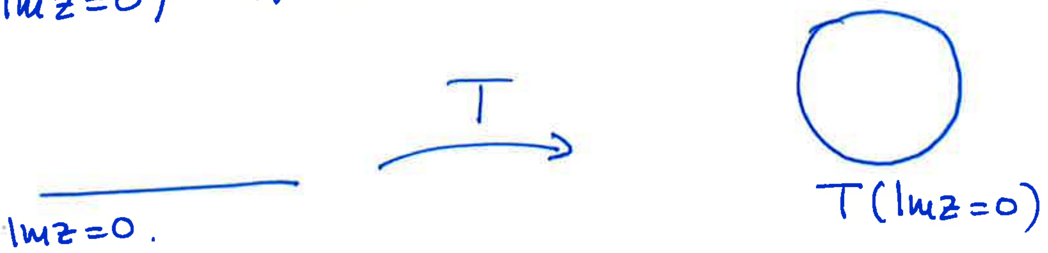
**Lemma.** – The cross ratio $(z_0, z_1, z_2, z_3)$ is real if and only if the four points lie on a circle or on a straight line.

**Pf (Lemma)**

It is clear that if $z_0, z_1, z_2, z_3$ lie on the straight line $\{\operatorname{Im} z = 0\}$, so that the four points are real, then

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} : \frac{z_1 - z_2}{z_1 - z_3} \in \mathbb{R}.$$

We now prove that for any $T \in \mathcal{M}$, $T(\operatorname{Im} z = 0)$ is either a circle or a line.



Since the cross ratio is invariant under Möbius transf. we have

$$(T(z_0), T(z_1), T(z_2), T(z_3)) \in \mathbb{R}.$$

Hence $\overline{(T(z_0), T(z_1), T(z_2), T(z_3))} = (T(z_0), T(z_1), T(z_2), T(z_3))$

$$\equiv \overline{\frac{aw+b}{cw+d}} = \frac{aw+b}{cw+d}$$ (Remember that the cross-ratio is a Möbius map!)

$$\equiv (a\bar{c} - c\bar{a})|w|^2 + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - \bar{b}d = 0. \quad (*)$$

- $a\bar{c} = c\bar{a}$. then, $a\bar{d} - c\bar{b} \neq 0$. (Otherwise, we would have $ad - bc = 0$ $\rightarrow\leftarrow$. The proof would go as follows.

  CASE 1. $c = 0$. $\Rightarrow$ (for sure), $ad \neq 0$. Hence $a\bar{d} \neq 0$.

  CASE 2. $c \neq 0 \Rightarrow a = \frac{c}{\bar{c}}\bar{a}$. So that if

  $a\bar{d} - \bar{b}c = 0$, $\frac{c}{\bar{c}}\bar{a}\bar{d} - \bar{b}c = 0 \Rightarrow \bar{a}\bar{d} - \bar{b}\bar{c} = 0$ $\rightarrow\leftarrow$).

Hence, we have from $(*)$.

$$Aw - \overline{Aw} = B, \quad A = a\bar{d} - c\bar{b} = a_1 + ia_2, \quad B = \bar{b}d - b\bar{d} = b_1 + ib_2 = ib_2!!$$

$$\equiv 2i\,\mathrm{Im}\{Aw\} = B,$$ which is, for $w = x + iy$

$$\equiv 2i \cdot (a_1 x - a_2 y) = ib_2 \equiv a_1 x - a_2 y = \frac{b_2}{2} \rightarrow \text{line!}$$

If $a\bar{c}-c\bar{a} \neq 0$, we divide ⊛ by this number and complete the square to get

$$\left| w + \frac{\bar{a}d-\bar{c}b}{\bar{a}c-\bar{c}a} \right| = \left| \frac{ad-bc}{\bar{a}c-\bar{c}a} \right| \to \text{circle}.$$

this shows that if $(z_0, z_1, z_2, z_3) \in \mathbb{R} \Rightarrow$ $(z_0, z_1, z_2, z_3)$ lie on a straight line or a circle.

To show that the reciprocal also works, notice the following:

Given any line $\mathcal{L} \equiv z = w_0 + re^{it}$, $r \in \mathbb{R}$, the Möbius transformation $T(z) = e^{-it}(z - w_0)$ maps that line onto $\{\operatorname{Im} z\} = 0$.

Given any circle $C: z = w_0 + Re^{it}$, $t \in [0, 2\pi]$,

$T_1(z) = \dfrac{(z-z_0)}{R} : C \to \partial\mathbb{D}$. and if we define $T_2(z) = i\,\dfrac{1+z}{1-z}$, the composition $T_2 \circ T_1$ maps $C$ onto $\{\operatorname{Im} z\} = 0$.

So that given 4 points in either $\mathcal{L}$ or $C$, $z_1, z_2, z_3, z_4$, we have

$$(z_1, z_2, z_3, z_4) = (\underbrace{Tz_1, Tz_2, Tz_3, Tz_4}_{\text{the points are real!}}) \in \mathbb{R}. \quad \square.$$

Now it's trivial ~~to~~ prove the theorem stated on p. 17.

<u>Pf:-</u> Let $\mathcal{C}$ be a circle in $\hat{\mathbb{C}}$ and $z_0, z_1, z_2, z_3 \in \mathcal{C}$.

then $(z_0, z_1, z_2, z_3) \in \mathbb{R}$ and hence so does $(Tz_0, Tz_1, Tz_2, Tz_3)$ $\forall\, T \in \mathcal{M}$. $\Rightarrow$ $Tz_0, Tz_1, Tz_2, Tz_3$ belong to a circle as well. $\square$.

## 3 Remarks before classifying Möbius transformations.

① Recall that any matrix $M \in \mathcal{M}_{2\times 2}(\mathbb{C})$ gives rise to the linear transformation

$$T_M(z_1, z_2) = M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ in } \mathbb{C}^2.$$

And that any linear transf. in $\mathbb{C}^2$ is identified with a matrix $M \in \mathcal{M}_{2\times 2}(\mathbb{C})$.

Moreover, the transformation $T: \mathbb{C}^2 \to \mathbb{C}^2$ is an isomorphism (= linear & bijective) iff the associated matrix has non-zero determinant.

It is usual to denote by

$$GL(2, \mathbb{C}) \equiv \text{linear general group in } \mathbb{C}^2$$
$$= \{\text{automorphisms of } \mathbb{C}^2\} \approx \{M \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \det M \neq 0\}.$$

~~which~~ the multiplication of matrices gives $(GL(2,\mathbb{C}),\cdot)$ the structure of group.

Now, the identification

$$T(z)=\frac{az+b}{cz+d}\in\mathcal{M}\longrightarrow\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{M}_{2\times 2}(\mathbb{C})$$

shows that $(\mathcal{M},\circ)=(GL(2,\mathbb{C}),\cdot)$.

$$\frac{az+b}{cz+d}\circ\frac{Az+B}{Cz+D}=\frac{a\left(\frac{Az+B}{Cz+D}\right)+b}{c\left(\frac{Az+B}{Cz+D}\right)+d}=\frac{(aA+bC)z+aB+bD}{(cA+dC)z+cB+dD}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}aA+bC&aB+bD\\cA+dC&cB+dD\end{pmatrix}.$$

(2) It's usual to use the following argument to find explicitly the Möbius transformation $T$: $T(z_1)=w_1$, $T(z_2)=w_2$, $T(z_3)=w_3$.

Here it goes: solve the equation

$$(T(z),w_1,w_2,w_3)=(z,z_1,z_2,z_3)$$

for $T(z)$.

**Example**. - Find $T$ such that $T(1)=2$, $T(i)=3$, $T(-1)=4$.

We have

$$(T(z),2,3,4) = (z,1,i,-1)$$

$$\equiv \frac{T(z)-3}{T(z)-4} : \frac{2-3}{2-4} = \frac{z-i}{z+1} : \frac{1-i}{1+1}$$

$$\equiv T(z) = \frac{(2-4i)z+(2+4i)}{(1-i)z+(1+i)}.$$

Why does this work? Because of the definition of cross-ratio.

$$(T(z),w_1,w_2,w_3) = (z,z_1,z_2,z_3) \equiv \text{image of } z \text{ under } S: \begin{cases} z_1 \to 1 \\ z_2 \to 0 \\ z_3 \to \infty. \end{cases}$$

$\equiv$ image of $T(z)$ under $M: \begin{cases} w_1 \to 1 \\ w_2 \to 0 \\ w_3 \to \infty. \end{cases}$

then, essentially we do $M \circ T(z) = S(z)$

$$\Rightarrow T(z) = M^{-1} \circ S(z)$$

and, $T(z_1) = M^{-1} \circ S(z_1) = M^{-1}(1) = w_1$

$T(z_2) \qquad = w_2$

$T(z_3) \qquad = w_3.$

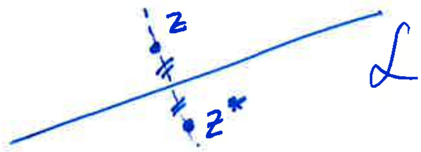
## ③ Symmetric points.

I did not have time to talk about this in our lectures. But I would say it is interesting to say a few words about this topic...

Given a line $\mathcal{L}$ in $\mathbb{C}$ and $z \notin \mathcal{L}$, it is not difficult to find the symmetric point $z^*$ of $z$ with respect to $\mathcal{L}$, so that

$$\text{dist}(z, \mathcal{L}) = \text{dist}(z^*, \mathcal{L}) \text{. \& } z^* \text{ "}\in \mathcal{L}^{\perp}\text{".}$$

In fact, if $z \in \mathcal{L}$, $z^* = z$.




Example.- $z$ & $\bar{z}$ are symmetric with respect to the real axis.

DEF.- If a linear transformation $T$ carries the real axis into a circle $\mathcal{C} \subset \hat{\mathbb{C}}$, we say that $w = Tz$ and $w^* = T\bar{z}$ are symmetric with respect to $\mathcal{C}$. Equivalently,

$z$ and $z^*$ are symmetric with respect to the circle $\mathcal{C}$ through $z_1, z_2, z_3$ if and only if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

Hence, Möbius transformations preserve symmetry!

[See L.V. Ahlfors, Complex Analysis 2nd Ed., McGraw-Hill, pp. 80–82]