Classification of Möbius transformations.

Let $T \in \mathcal{M}$, $T(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

then, $$T(z) = \frac{\frac{a}{\sqrt{ad-bc}} z + \frac{b}{\sqrt{ad-bc}}}{\frac{c}{\sqrt{ad-bc}} z + \frac{d}{\sqrt{ad-bc}}} = \frac{Az+B}{Cz+D},$$

$AD - BC = 1$.

From now on, we identify T with its "normalize" transformation so that $ad-bc=1$.

<u>DEF</u>.- Two Möbius transformations T_1 and T_2 are <u>conjugated</u> if $\exists\, S \in \mathcal{M}$:

$$T_1 = S^{-1} \circ T_2 \circ S.$$

Note that conjugation is an equivalence relation:

→ $T \sim T$: $T = Id^{-1} \circ T \circ Id$.

→ $T_1 \sim T_2 \Rightarrow T_2 \sim T_1$:

$$T_1 = S^{-1} \circ T_2 \circ S \Rightarrow T_2 = S \circ T_1 \circ S^{-1}$$
$$= (S^{-1})^{-1} \circ T_1 \circ S^{-1}.$$

→ $T_1 \sim T_2$ & $T_2 \sim T_3 \Rightarrow T_1 \sim T_3$.

$T_1 = S_2^{-1} \circ T_2 \circ S_2$, $T_2 = S_3^{-1} \circ T_3 \circ S_3$

$$\Rightarrow \quad T_1 = S_2^{-1} \circ S_3^{-1} \circ T_3 \circ S_3 \circ S_2$$
$$= (S_3 \circ S_2)^{-1} \circ T_3 \circ (S_3 \circ S_2).$$

Recall also the identification

$$T(z) = \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(And $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$!).

Fixed points.

$$\frac{az+b}{cz+d} = z \Longleftrightarrow cz^2 + (d-a)z \cancel{+} b = 0.$$

$\Rightarrow$ $c \neq 0$, $\quad z = \dfrac{a-d \pm \sqrt{(d-a)^2+4bc}}{2c}$ $\Rightarrow$ 2 fixed points, $z_1, z_2 \in \mathbb{C}$. (maybe $z_1 = z_2$!)

$\Rightarrow$ $c = 0$ $\Rightarrow$ $(d-a)z = +b$.

$\Rightarrow$

① $d \neq a$, $z = \dfrac{b}{d-a}$ & $z = \infty$ fixed:
$T(z) = \dfrac{a}{d} z + \dfrac{b}{d}$.

$d = a \Rightarrow T(z) = z + b$

② 1 fixed point: ∞ if $b \neq 0$.

all fixed points! $b = 0$.

there are 2 cases when a given $T \in M$ can have 1 fixed point in $\mathbb{C}$ (T is <u>parabolic</u>).

$$T(z) = z + b \quad , b \neq 0.$$

$$T(z) = \frac{az+b}{cz+d} \quad , (d-a)^2 + 4bc = 0.$$

Notice that since $ad - bc = 1 \equiv bc = ad - 1$, the second option gives

$$0 = (d-a)^2 + 4ad - 4 \equiv (a+d)^2 = 4.$$

$$\equiv \left[\mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]^2 = 4$$

the first option: $\left(\mathrm{Tr} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right)^2 = 4$ as well.

Hence T is parabolic $\Rightarrow \left[\mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]^2 = 4.$

Does the reciprocal holds?

We will see that the answer is YES. Hence,

$$T \text{ is parabolic} \Longleftrightarrow \left[\mathrm{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]^2 = 4.$$

We just need to see that if $Tr^2 = 4$, then T fixes 1 point. ($T \neq Id$!)

Assume that $T(z_1) = z_1$, $T(z_2) = z_2$ for $z_1, z_2 \in \mathbb{C}$.

let $$S(z) = \frac{z - z_1}{z - z_2} \equiv \frac{\frac{z}{\sqrt{z_1 - z_2}} - \frac{z_1}{\sqrt{z_1 - z_2}}}{\frac{z}{\sqrt{z_1 - z_2}} - \frac{z_2}{\sqrt{z_1 - z_2}}},$$

then $S \circ T \circ S^{-1}$ fixes 0 and ∞.

$$\equiv \quad S \circ T \circ S^{-1} = kz = \frac{\sqrt{k} z + 0}{0 z + \frac{1}{\sqrt{k}}} = R. \quad (k \neq 0!)$$

$$\equiv \quad T \sim R. \qquad (Tr\, R)^2 = \left(Tr \begin{bmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{bmatrix} \right)^2$$

$$= \left(\sqrt{k} + \frac{1}{\sqrt{k}} \right)^2 = 4 \Leftrightarrow k + \frac{1}{k} = 2.$$

$$\Leftrightarrow k^2 - 2k + 1 = 0 \Leftrightarrow k = 1. \Rightarrow R = Id.$$

$\Rightarrow T = Id$.

If $z_2 = \infty$, consider $S(z) = z - z_1$.

Again, $S \circ T \circ S^{-1}$ fixes 0 and ∞ and we can argue as before.

Corollary (of the proof).

If T is not parabolic, $T \sim M_k$, $k \neq 1, 0$.

Moreover, if T is parabolic,

then $T \sim \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $a^2 = 1$.
$2a = \pm 2$.

$$\Rightarrow \quad T(z) \sim \begin{cases} z+1 \\ z-1. \end{cases}$$

And notice that $z+1 \not\sim z-1$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} -a+c & d-b \\ -c & -d \end{pmatrix}$$

$$\Rightarrow \left\{ c = a = 0 \right.$$

NO!

BUT $\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{z+1}^{?} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{z-1}^{?}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

what's going on???

We can also prove the following lemma:

Lemma. – $M_{k_1} \sim M_{k_2}$ $(k_1, k_2 \neq 1, 0) \Leftrightarrow k_1 \in \{k_2, \frac{1}{k_2}\}$.

$$M_{k_1} \sim M_{k_2} \Rightarrow \left(\sqrt{k_1} + \frac{1}{\sqrt{k_1}}\right)^2 = \left(\sqrt{k_2} + \frac{1}{\sqrt{k_2}}\right)^2$$

$$\Rightarrow \quad k_1 + \frac{1}{k_1} = k_2 + \frac{1}{k_2}$$

$$\Leftrightarrow \quad k_1 - k_2 = \frac{1}{k_2} - \frac{1}{k_1} = \frac{k_1 - k_2}{k_1 \cdot k_2} .$$

$$\Rightarrow \begin{cases} * \; k_1 = k_2 \\ * \; k_1 k_2 = 1 \to k_1 = \frac{1}{k_2} \end{cases} .$$

Obviously, if $k_1 \in \{k_2, \frac{1}{k_2}\}$, $m_{k_1} \sim m_{k_2}$.

Since, in the case $k_1 = k_2 \to m_{k_1} = m_{k_2}$
and if $k_1 = \frac{1}{k_2}$ we can write

$$k_1 z = \frac{1}{k_2} z = \sqrt{k_2}\, k_2 \left(\frac{1}{\sqrt{k_2}} z \right) = m_{\sqrt{k_2}}\, m_{k_2}\, m_{\sqrt{k_2}}^{-1} z .$$

Non-parabolic transformations.

We know that $T \sim m_k$, $k \neq 0, 1$.

T is called elliptic if $|k| = 1$. $(\sqrt{k} \neq 1)$.

T is called loxodromic if $|k| \neq 1$.

Moreover, a loxodromic Möbius transformation $T \neq Id$ is hyperbolic if there's a disk D (possibly of radius ∞) such that $T(D) = D$.
Otherwise, T is said to be properly loxodromic

theorem.- Let $T \neq Id$. the T is

(a) Parabolic $\iff \left[Tr \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^2 = 4 . \equiv Tr^2 T = 4$

(b) Elliptic $\iff 0 \leq Tr^2(T) < 4$.

(c) Hyperbolic $\iff Tr^2(T) > 4$

(d) Properly loxodromic $\iff Tr^2(T) \neq |Tr^2(T)|$.
$\quad \& \; |Tr^2(T)| > 4$.

Pf.- (a) already done.

(b) T elliptic $\iff T \sim \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix}$, $|k| = 1, k \neq 1$.

$$\equiv T \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \; \theta \neq 0, \pi \; \left(\begin{array}{l} \sqrt{k} = 1 \Leftrightarrow k = \begin{cases} 1 \\ -1 \end{cases} \\ \Leftrightarrow e^{i\theta} = k, \; k \in \{0, \pi\} \end{array}\right)$$

$$\iff Tr^2(T) = \left(e^{i\theta} + e^{-i\theta}\right)^2 = e^{2i\theta} + e^{-2i\theta} + 2$$

$$= 2 + 2\cos\theta \in [0, 4).$$

(c) & (d) : Loxodromic $\iff |Tr^2(T)| > 4$.

$$T \sim \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix}, \; |k| \neq 1.$$

$$\iff \left|\sqrt{k} + \frac{1}{\sqrt{k}}\right|^2 = |k| + \frac{1}{|k|} + 2, \; (|k| > 1) \quad ,$$

$$> 4.$$

Now, let's see that

$$T \text{ Hyperbolic} \Leftrightarrow Tr^2(T) > 4.$$

(Notice this will end the proof!).

$\Leftarrow \rvert$ $Tr^2(T) = \left(\sqrt{k} + \frac{1}{\sqrt{k}}\right)^2 = k + \frac{1}{k} + 2 > 4$

$\Leftrightarrow$ $k + \frac{1}{k} \in \mathbb{R}_{>1}$! (and $|k|$, say, > 1).

$k = Re^{i\theta}$ then $Re^{i\theta} + \frac{e^{-i\theta}}{R}$

$$= \left(R + \frac{1}{R}\right)\cos\theta + \left(R - \frac{1}{R}\right)\sin\theta\, i \in \mathbb{R}$$

$\Leftrightarrow \theta = \begin{cases} 0 \\ \pi \end{cases}$ $\Rightarrow$ $k \in \mathbb{R}$. and $k > 1$.

that is $T \cancel{=} S^{-1} \circ M_k \circ S$, $k > 1$.

Let $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$, and notice that $m_k(\mathbb{H}) = \mathbb{H}$!.

then, let $D = S^{-1}(\mathbb{H})$ - wich is a circle!)

and note that

$$T(D) = S^{-1}(m_k(\mathbb{H})) = S^{-1}(\mathbb{H}) = D.$$

$\Rightarrow \rvert$ $T \sim M_k$ for some $|k| > 1$.

$$\equiv T = S^{-1} \circ M_k \circ S \equiv M_k = S \circ T \circ S^{-1}.$$

Let D be the disk (or half-plane) with $T(D)=D$. And set $H=S(D)$ - also a disk or a half-plane.

Moreover,

$$M_k(H) = S \circ T(D) = S(D) = H.$$

Hence, in fact, we have $\forall z \in H$, $n \in \mathbb{N}$; $[M_k(z)]^n = k^n z \in H$.

Also, since $M_k(H)=H$, we have

$$H = M_k^{-1}(H) = M_{\frac{1}{k}}(H).$$

And, then, $[M_k(z)]^{-n} = M_{\frac{1}{k}} z = \frac{1}{k^n} z \in H$.

Now, if $z \neq 0, \infty \in H$, we have, since $|k|>1$, $k^n z \to \infty$, $k^{-n} z \to 0$, $n \to \infty$.

$\Rightarrow \infty, 0 \in \overline{H}$ and thus H is a half-plane

But the same holds $\forall z \notin H$, and we conclude that ∂H is a line passing through the origin. Hence k must be positive (>1) and

$$(\mathrm{tr}\, M_k)^2 = (\mathrm{tr}\, T)^2 > 4 \qquad \square.$$