Subfamilies of Möbius transformations

Automorphisms of the unit disk $\mathbb{D}$ are those <u>analytic</u> and univalent (one-to-one) functions f in $\mathbb{D}$ with $f(\mathbb{D}) = \mathbb{D}$.

We denote by $Aut(\mathbb{D})$ the family of automorphisms in the unit disk. The "good news" is that $Aut(\mathbb{D})$ consists of Möbius trasformations, as we shall see.

We also need the Schwarz-lemma.

<u>Lemma</u> (Schwarz).

Let φ be an analytic function in $\mathbb{D}$ with $\varphi(0) = 0$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$.

then $\quad |\varphi(z)| \leq |z| \quad \forall z \in \mathbb{D}$

and $\quad |\varphi'(0)| \leq 1.$

Moreover, if $\exists\ z \neq 0$ such that $|\varphi(z)| = |z|$ or $|\varphi'(0)| = 1$, then

$$\varphi(z) = \lambda z, \quad |\lambda| = 1.$$

that is, $\varphi \in Aut(\mathbb{D})$ (and $\varphi(0) = 0$).

the proof should be known for all of us. See, for instance, [Ahlfors, Complex Analysis, 2nd. Ed. McGraw-Hill, p. 135].

We can now prove the main theorem:

<u>theorem</u>.- Let $\varphi \in Aut(\mathbb{D})$. then

$$\varphi(z) = \lambda \varphi_a(z), \text{ for some } |\lambda| = 1 \text{ and } a \in \mathbb{D}.$$

Here $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$.

In other words, $Aut(\mathbb{D}) = \{\lambda \varphi_a(z), |\lambda|=1, a \in \mathbb{D}\}$.

<u>Pf</u>.- Note that φ_a is analytic in $\mathbb{D}$, it is one-to-one ($\varphi_a \in \mathcal{M}$).

Moreover, if $|z| = 1$, $|\varphi_a(z)| = 1$:

$$\left|\frac{a-z}{1-\bar{a}z}\right|^2 = \frac{|a|^2+|z|^2-2\operatorname{Re}\{\bar{a}z\}}{1+|az|^2-2\operatorname{Re}\{\bar{a}z\}} \underset{|z|=1}{\uparrow}{=} 1.$$

In fact, if $|z| < 1$,

$$|a|^2+|z|^2-2\operatorname{Re}\{\bar{a}z\} < 1+|az|^2-2\operatorname{Re}\{\bar{a}z\}$$

$$\boxed{\equiv \ |z|^2+|a|^2 < 1+|az|^2 \equiv |a|^2(1-|z|^2) < 1-|z|^2 \equiv |a|<1 \ \checkmark}$$

So that $\varphi_a(\mathbb{D}) \subset \mathbb{D}$.

Since $\varphi_a^{-1} = \varphi_a$, we have $\varphi_a \in Aut(\mathbb{D})$.

hence $\{\lambda \varphi_a(z), |\lambda|=1, a\in \mathbb{D}\} \subset \mathrm{Aut}(\mathbb{D})$.

Assume now that $\varphi \in \mathrm{Aut}(\mathbb{D})$, $\varphi(0)=\alpha$.

Consider the automorphism (composition of 2 automorphisms)

$$\psi = \varphi_\alpha \circ \varphi$$

then, $\psi(\mathbb{D}) = \mathbb{D}$ & $\psi(0) = \varphi_\alpha(\varphi(0)) = \varphi_\alpha(\alpha) = 0$.

$$\Longrightarrow \quad |\psi(z)| \leq |z| \quad \forall z \in \mathbb{D}.$$

But $\psi^{-1} \in \mathrm{Aut}(\mathbb{D})$ & $\psi^{-1}(0)=0$.

$$\Longrightarrow \quad |\psi^{-1}(\zeta)| \leq |\zeta| \quad \forall \zeta \in \mathbb{D}$$

$$\underset{\zeta=\psi(z)}{\Longrightarrow} \quad |z| \leq |\psi(z)| \quad \forall z \in \mathbb{D}.$$

that is, $|\psi(z)|=|z|$ $\forall z\in\mathbb{D} \Rightarrow \varphi_\alpha\circ\varphi(z) = \mu z$, $|\mu|=1$.

$$\Longrightarrow \varphi(z) = \varphi_\alpha^{-1}(\mu z) = \varphi_\alpha(\mu z) = \frac{\alpha - \mu z}{1-\bar{\alpha}\mu z}$$

$$= \mu \frac{\bar{\mu}\alpha - z}{1-\overline{(\bar{\mu}\alpha)} z} = \lambda \varphi_a(z), \quad \lambda=\mu, \ a = \bar{\mu}\alpha.$$

$\square$.

[Remark: We said that $\varphi_a \in \mathrm{Aut}(\mathbb{D})$, $\varphi_a^{-1} = \varphi_a$, $\varphi_a(a)=0$. Note also that $\varphi_a(0)=a \equiv \varphi_a$ is an involutive automorphism that interchanges the points 0 and a].

The Schwarz-Pick lemma.

Here we have a generalization of the Schwarz lemma

Lemma [Schwarz-Pick]. Let $\varphi \in \mathcal{H}(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then

(i) $d_{ph}(\varphi(z), \varphi(w)) \leq d_{ph}(z,w) \quad \forall z, w \in \mathbb{D}$,

$\varphi(0)=0 \cdot \; |\varphi_0(\varphi(z))| \leq |\varphi_0(z)|$

$w=0 \qquad |\varphi(z)| \leq |z| \quad \forall z \in \mathbb{D}$

where, for $a, b \in \mathbb{D}$,

$d_{ph}(a,b) = |\varphi_a(b)| \equiv$ pseudo-hyperbolic distance between a and b.

(ii) $$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \quad \forall z \in \mathbb{D}.$$

Moreover, equality holds in either (i) for $a \neq b$ or (ii) if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Pf.- Let a be an arbitrary point in $\mathbb{D}$.

Define $\quad \psi_a(z) = \varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z)$

then, $\psi_a \in \mathcal{H}(\mathbb{D})$, $\psi_a(\mathbb{D}) \subset \mathbb{D}$, and $\psi_a(0) = 0$.

$\Rightarrow \quad |\psi_a(z)| \leq |z| \quad \forall z \in \mathbb{D}$.

$$\equiv \quad |\varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z)| \leq |z| \quad \forall z \in \mathbb{D}$$

$$\equiv (z = \varphi_a(b)) \quad |\varphi_{\varphi(a)}(\varphi(b))| \leq |\varphi_a(b)|.$$

And $=$ holds iff. $|\psi_a(z)| = |z|$ for some $a \neq b$.

$$\equiv \quad \varphi_{\varphi(a)} \circ \varphi \circ \varphi_a(z) = \lambda z \equiv \varphi_{\varphi(a)} \circ \varphi(z) = \lambda \varphi_a(z)$$

$\equiv \varphi(z)$ is the composition of 2 automorphisms

$\Rightarrow \varphi \in Aut(\mathbb{D})$.

Also, $\quad |\psi_a'(0)| \leq 1 \quad$ ($=$ iff $\psi_a \in Aut(\mathbb{D})$ $\equiv \varphi \in Aut(\mathbb{D})$, as before).

$$\Updownarrow$$

$$|\varphi_{\varphi(a)}'(\varphi(a))| \cdot |\varphi'(a)| \cdot |\varphi_a'(0)| \leq 1.$$

Since $\quad \varphi_a'(z) = \dfrac{|a|^2 - 1}{(1-\bar{a}z)^2}, \quad |\varphi_a'(0)| = 1 - |a|^2, |\varphi_a'(a)| = \dfrac{1}{1-|a|^2}$

This proves the lemma. $\square$.

Solutions to some related exercises

(1) Let $T(z) = \dfrac{az+b}{cz+d} \in \mathcal{M}$. Prove that the following conditions are equivalent.

(a) $T \in \mathcal{H}(\mathbb{D})$ & $T(\mathbb{D}) \subset \mathbb{D}$.

(b) $|b\bar{d} - a\bar{c}| + |ad-bc| \leq |d|^2 - |c|^2$.

(c) $|\bar{c}d - \bar{a}b| + |ad-bc| \leq |d|^2 - |b|^2$.

(d) $|d| > |c|$ & $2|a\bar{b} - c\bar{d}| \leq |c|^2 + |d|^2 - |a|^2 - |b|^2$.

• (a) $\Leftrightarrow$ (b).

$\Rightarrow$) $T \in \mathcal{H}(\mathbb{D}) \Rightarrow cz+d \neq 0 \ \forall z \in \mathbb{D}. \Rightarrow |d| > |c|$.
(otherwise, $z = -\frac{d}{c}$, $c \neq 0$, belongs to $\mathbb{D}$!

Now, $T(\mathbb{D}) = D(z_0, r) \subset \mathbb{D}$ for some $z_0 \in \mathbb{D}$ & $r > 0$, since $T(\mathbb{D}) \subset \mathbb{D}$. But hence, $\dfrac{T(z) - z_0}{r}$ maps the disk onto itself & is one-to-one. that is, this transf. equals an automorphism of $\mathbb{D}$: $(|z_0| + r < 1)$

$$\frac{T(z) - z_0}{r} = \mu \varphi_\alpha(z), \quad |\mu| = 1, \ |\alpha| < 1, \ \varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

$$\equiv \quad \frac{az+b}{cz+d} = \mu r \frac{\alpha - z}{1 - \bar{\alpha} z} + z_0, \qquad r + |z_0| \leq 1 \ !!!$$

(since, again, $T(\mathbb{D}) \subset \mathbb{D}$!)

$$\equiv \quad \frac{\frac{a}{d} z + \frac{b}{d}}{\frac{c}{d} z + 1} = \frac{(-\mu r - z_0 \bar{\alpha}) z + z_0 + \mu r \alpha}{-\bar{\alpha} z + 1}$$

$$\equiv \begin{cases} \dfrac{a}{d} = -\mu r - z_0 \bar{\alpha} & \to \mu r = \dfrac{c}{d} z_0 - \dfrac{a}{d} \\ \dfrac{b}{d} = z_0 + \mu r \alpha & \to \dfrac{b}{d} = z_0 + \alpha\left(\dfrac{c}{d} z_0 - \dfrac{a}{d}\right) \\ \dfrac{c}{d} = -\bar{\alpha} & \qquad = z_0 - \left(\dfrac{c}{d} z_0 - \dfrac{a}{d}\right)\cdot \overline{\dfrac{c}{d}} \end{cases}$$

$$\Updownarrow$$

$$\alpha = -\overline{\frac{c}{d}} \qquad\qquad = z_0 - \left|\frac{c}{d}\right|^2 z_0 + \frac{a\bar{c}}{|d|^2}$$

$$\equiv \frac{b\bar{d} - a\bar{c}}{|d|^2} = \frac{(|d|^2 - |c|^2)}{|d|^2} z_0 . \quad (*)$$

$$\equiv r = \left|\frac{c}{d}\cdot z_0 - \frac{a}{d}\right| = \left|\frac{c}{d}\cdot \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2} - \frac{a}{d}\right|$$

$$= \frac{1}{|d|} \frac{|bc\bar{d} - \cancel{a|c|^2} - a|d|^2|}{|d|^2 - |c|^2} = \frac{|bc - ad|}{|d|^2 - |c|^2} \quad (**).$$

and since $r + |z_0| \le 1$, we have, from $(*)$ and $(**)$, that (b) holds.

$\Leftarrow$ If (b) holds, then $|d| > |c|$ (since $ad - bc \neq 0$) and the system above can be solved for $r > 0$, $z_0 \in \mathbb{D}$, $\alpha \in \mathbb{D}$, and $|\mu| = 1$ with $r + |z_0| < 1$ & $T(z) - z_0 = \mu r \varphi_\alpha(z)$. Hence (a) holds.

(b) $\Leftrightarrow$ (c). This equivalence is based on the following result.

Lemma: $\varphi(z) = \dfrac{az+b}{cz+d}$ is an analytic function in $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ if and only if

$$\varphi^*(z) = \frac{1}{\overline{\varphi^{-1}\left(\frac{1}{\bar{z}}\right)}} \overset{\circledast}{=} \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$$

has the same properties.

Pf.- $\varphi(\mathbb{D}) \subset \mathbb{D} \Longrightarrow \varphi^{-1}(s) : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$

[If $\exists\, s \in \mathbb{C} \setminus \overline{\mathbb{D}} : \varphi^{-1}(s) \in \mathbb{D}$, then $\varphi(\varphi^{-1}(s)) = s$ would belong to both $\mathbb{D}$ (since $\varphi(\mathbb{D}) \subset \mathbb{D}$) & $\mathbb{C} \setminus \overline{\mathbb{D}}$ by hypothesis $\rightarrow\leftarrow$].

Now, $z \in \mathbb{D} \to \dfrac{1}{\bar{z}} \in \mathbb{C} \setminus \overline{\mathbb{D}} \Rightarrow \varphi^{-1}\left(\dfrac{1}{\bar{z}}\right) \in \mathbb{C} \setminus \overline{\mathbb{D}}$

& $\dfrac{1}{\overline{\varphi^{-1}\left(\frac{1}{\bar{z}}\right)}} \in \mathbb{D}$. Moreover, $\varphi^{-1}\left(\dfrac{1}{\bar{z}}\right) \neq 0$!

This proves the lemma in addition to the identity $\circledast$ which is easily obtained using that $\varphi^{-1}(z) = \dfrac{dz+b}{a-cz}$. $\square$.

To complete the proof of (b) $\Leftrightarrow$ (c) just apply (a) $\Leftrightarrow$ (b) to φ^*.

Finally, (a) $\Leftrightarrow$ (c).

$\Rightarrow$] As mentioned before, $T \in \mathcal{H}(\mathbb{D}) \Rightarrow |d| > |c|$.

Also, $T(\mathbb{D}) \subset \mathbb{D} \equiv |T(z)| \leq 1 \quad \forall z \in \mathbb{D}$ (hence $\forall\ |z| \leq 1$).

$\Rightarrow |az+b|^2 \leq |cz+d|^2 \quad \forall\ |z| = 1$.

$\equiv \quad 2\operatorname{Re}\{(a\bar{b} - c\bar{d})z\} \leq |c|^2 + |d|^2 - |a|^2 - |b|^2 \quad \forall |z| = 1$.

$$\Rightarrow \begin{cases} \cdot\ a\bar{b} - c\bar{d} = 0 \Rightarrow \text{(d) for sure.} \\ \cdot\ a\bar{b} - c\bar{d} \neq 0 \Rightarrow \text{(d) too } \left(\text{take } z = \dfrac{\overline{a\bar{b} - c\bar{d}}}{|a\bar{b} - c\bar{d}|} \in \partial\mathbb{D}\right). \end{cases}$$

$\Leftarrow$] Just undo the previous argument:

- $a\bar{b} - c\bar{d} = 0 \Rightarrow \forall\ |z| = 1 \quad \left|\dfrac{az+b}{cz+d}\right| \leq 1$

$\Rightarrow$ by the Maximum modulus principle, $\left|\dfrac{az+b}{cz+d}\right| \leq 1$ in $\mathbb{D}$.

the same holds if $a\bar{b} - c\bar{d} \neq 0$.

(2) For a given analytic function φ in $\mathbb{D}$ and a positive integer n, define the n-th iterates of φ by the recursive formulas

$$\varphi_1 = \varphi \quad \text{and} \quad \varphi_n = \varphi \circ \varphi_{n-1}.$$

(1) What are the solutions of $\underset{\varphi\circ\varphi}{\underline{\varphi_2}} = T$, $\to \varphi$? where $T \in \mathcal{M}$? Note that $\varphi \in \mathcal{H}(\mathbb{D})$, so that in order to ensure that φ_2 is well-defined, we need $\underline{\varphi(\mathbb{D}) \subset \mathbb{D}}$ (hence $T(\mathbb{D}) \subset \mathbb{D}$!)

(2) And the solutions of $\underline{\varphi_n} = T$, $n \geq 2$?? $\to \varphi$?

$\varphi_n = \underbrace{\varphi\circ\varphi\circ\varphi\circ\varphi \cdots \circ\varphi}_{n \text{ times}}$.

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(1) • $T = Id$ & $\lambda^2 = 1$, $\underline{\varphi(z) = \varphi_p(\lambda\, \varphi_p(z))}$,

$\underline{\varphi_p(z) = \dfrac{p-z}{1-\bar{p}z}}$, $p \in \mathbb{D}$ satisfies

$\varphi_p \circ m_\lambda \circ \cancel{\varphi_p \circ \varphi_p} \circ m_\lambda \circ \varphi_p$

$\varphi \circ \varphi = Id$ $\qquad$ $\varphi_p \circ m_{\lambda^2} \circ \varphi_p = Id$, $\lambda^2 = 1$

$\underline{\varphi_2(z) = T}$

Note that the same holds if $\lambda^n = 1$ with the same φ to solve $\varphi_n = \underline{Id}$!

(1) & (2): Case when $\exists\, p \in \mathbb{D}$: $\boxed{T(p) = p}$

Lemma.- Let $T \in \mathcal{M}$, $T(\mathbb{D}) \subset \mathbb{D}$, $T(p) = p$, $p \in \mathbb{D}$, $T \neq Id$. Then $\hat{T} = \varphi_p \circ T \circ \varphi_p$ has the form

$\hat{T}(0)=0$ | $$\hat{T}(z) = \frac{Az}{Cz+1}, \quad |A|+|C| \leq 1, \quad A \neq 1.$$ (previous ex)

Pf.- $\hat{T}$ is a Möbius transformation that fixes the origin and is analytic in $\mathbb{D}$

$$\equiv \quad \hat{T}(z) = \frac{az}{cz+d} \underset{d\neq 0.}{=} \frac{Az}{Cz+1}.$$

Since $\hat{T}(\mathbb{D}) \subset \mathbb{D}$, by the previous exercise we have $|A|+|C| \leq 1$. & $A = 1 \Rightarrow C = 0 \rightarrow T = Id$. $\square$

Finding φ: $\varphi_n = T$, with T as in the lemma.

$$\boxed{\varphi_p \circ \varphi \circ \varphi \circ \ldots \circ \varphi \circ \varphi_p = \varphi_p \circ T \circ \varphi_p} \equiv \underbrace{\varphi_p \circ \varphi \circ \varphi_p}_{\hat{\varphi}} \circ \underbrace{\varphi_p \circ \varphi \circ \varphi_p}_{\hat{\varphi}} \ldots \circ \varphi_p \circ \varphi \circ \varphi_p = \hat{T}$$

$$\equiv \quad \hat{\varphi}_n = \hat{T}. \quad \& \quad \hat{\varphi}_n(0) = 0 \Rightarrow \hat{\varphi}(0) = 0.$$

$$\Rightarrow \hat{\varphi}(z) = \frac{az}{cz+1}, \quad \hat{\varphi}_2(z) = \frac{a\left(\frac{az}{cz+1}\right)}{c\left(\frac{az}{cz+1}\right)+1} = \frac{a^2 z}{(c+ac)z+1},$$

$$\ldots \quad \boxed{\hat{\varphi}_n(z) = \frac{a^n z}{(c+ac+\ldots+a^{n-1}c)z+1}}.$$

Note that, as before, if $a=1 \Rightarrow \varphi = Id \Rightarrow T = Id$!

thus, we are to solve
$$\begin{cases} a^n = A \\ c(1+a+\ldots+a^{n-1}) = C. \end{cases}$$

$$\equiv \begin{cases} a^n = A \\ c = \dfrac{G}{1+a+\dots+a^{n-1}} \end{cases} \quad \left(\begin{array}{l} \text{Note:} \\ S = 1+a+\dots+a^{n-1} = 0 \\ \Rightarrow S = \dfrac{1-a^n}{1-a} = \dfrac{1-A}{1-a} = 0 \to \underset{\text{NO!}}{A=1} \end{array} \right).$$

$$\therefore \quad \hat{\varphi}(z) = \frac{az}{\dfrac{G}{\sum_{j=0}^{n-1} a^j} z + 1}$$

$\varphi_\mu, \ |\mu| = 1$

(1) & (2): Case when $\Phi(z) \neq z \ \ \forall \ z \in \mathbb{D}$.

Recall that, in this case, $\exists \ \mu \in \partial\mathbb{D} : T(\mu) = \mu$.

this is similar to the previous item but considering now $\hat{T} = T_\mu \circ T \circ T_\mu^{-1}$, $T_\mu = \dfrac{\mu+z}{\mu-z}$ Cayley maps.

We get $\boxed{\hat{T}(\omega) = A\omega + B}$, $\omega \in \mathbb{H} = \mathrm{Re}\{\omega\} > 0\}$.

You can find more details in:

[Contreras, Díaz-Madrigal, M, Vukotic, Contemp. Math., 2012].

Remark – We have seen before that if $T \in \mathcal{M}$ fixes 3 points in $\hat{\mathbb{C}}$, then $T = Id$.

The next result shows that the same result holds if we change the hypotheses $T \in \mathcal{M}$ fixes 3 points in $\mathbb{C}$ by $\varphi \in \mathcal{H}(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and φ fixes 2 points.

Thm.– Let $\varphi \in \mathcal{H}(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$. Suppose that there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, such that $\varphi(z_1) = z_1$ and $\varphi(z_2) = z_2$. Then $\varphi = Id$.

Pf.– Consider, for $a \in \mathbb{D}$, the automorphism

$\varphi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$, $|z| < 1$, and recall that $\varphi_a^{-1} = \varphi_a$.

Now, define $\psi = \varphi_{z_1} \circ \varphi \circ \varphi_{z_1} \in \mathcal{H}(\mathbb{D})$.

Then, on the one side,

$$\psi(0) = \varphi_{z_1} \circ \varphi(\varphi_{z_1}(0)) = \varphi_{z_1}(\varphi(z_1))$$
$$= \varphi_{z_1}(z_1) = 0. \rightarrow |\psi(\xi)| \leq |\xi| \ \forall \xi \in \mathbb{D}.$$

On the other side, for $s = \varphi_{z_1}(z_2)$ ($\neq 0$, since $z_1 \neq z_2$), we have:

$$\psi(s) = \varphi_{z_1} \circ \varphi \circ \varphi_{z_1}(s) \underset{\uparrow}{=} \varphi_{z_1} \circ \varphi(z_2) = \varphi_{z_1}(z_2)$$

$\varphi_{z_1}(s) = \varphi_{z_1} \circ \varphi_{z_1}(z_2) = z_2$

$$= s.$$

Hence $\psi(z) = \lambda z$, $|\lambda| = 1$. But $\lambda = 1$, since $\psi(s) = s$.

$\square$

On the fixed points of a Möbius transformation $T \in \mathcal{M}$, $T(\mathbb{D}) \subset \mathbb{D}$.

Several situations can come up.

→ $T = Id$ $(\Rightarrow T(z) = z \ \forall z \in \mathbb{D})$.

→ T has 1 (unique) fixed point in $\mathbb{D}$.

→ T has no fixed points in $\mathbb{D}$. Then, we will show that $\exists\, \zeta \in \partial\mathbb{D}$: $T(\zeta) = \zeta$.

We need Rouche's thm.

thm [Rouche] Let $f, g \in H(\overline{\mathbb{D}})$. If $|f+g| < |g|$ on $\partial\mathbb{D}$, then f has as many zeros in $\mathbb{D}$ as g has.

$|T_k| = |r_k T(z)| < 1 = |g|$

Now, assume that T has no fixed points in $\mathbb{D}$. Choose an increasing sequence of positive real numbers $\{r_k\} \subset (0,1)$ with $r_k \to 1$.

Define $T_k = r_k T$ and apply Rouché's thm. to $f(z) = T_k(z) - z = 0$ and $g(z) = z$. $(|r_k T(z)| \leq r_k < 1 = |z|$, $|z| = 1)$.

$T_k(z) = z$

So that each T_k has 1 unique fixed point, z_k, say, in $\mathbb{D} \equiv r_k T(z_k) = z_k$.

Passing to a subsequence, if needed, $z_k \to \zeta \in \overline{\mathbb{D}}$ & $T(\zeta) = \zeta$. And since T has no fixed points in $\mathbb{D}$, we are done.