Infinite products

In previous courses, we have studied convergence of (numerical) series of real & complex numbers.

DEF.- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Define $s_n = \sum_{j=1}^{n} a_j$, $n \geq 1$.

If $\exists \lim_{n\to\infty} \{s_n\} = s \in \mathbb{C}$, we say that the series $\sum_{n=1}^{\infty} a_n$ converges (to s).

Otherwise, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Criteria of convergence

* $s_n \to s$ if $\forall \varepsilon > 0$, $\exists N : |s_n - s| < \varepsilon$ $\forall n \geq N$.

* $a_n = s_n - s_{n-1} \longrightarrow 0$. (in case of convergence)

  Hence, if $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

* Comparison test. Given two sequences $\{a_n\}$ and $\{b_n\}$.

  If $\forall n$ $0 \leq a_n \leq b_n$ an $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

  And if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

* Ratio test. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ and

  → $r<1$, $\sum_{n=1}^{\infty} a_n$ converges

  → $r>1$, $\sum_{n=1}^{\infty} a_n$ diverges.

* Root test. Let $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

  → $r<1$, $\sum_{n=1}^{\infty} a_n$ converges

  → $r>1$ $\sum_{n=1}^{\infty} a_n$ diverges.

...

DEF.- Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

The infinite product $\prod_{j=1}^{\infty} b_j$ converges if

$$\exists \lim_{n\to\infty} \prod_{j=1}^{n} b_j \neq 0.$$

Otherwise, the product is divergent.

Remark.- Define $P_n = \prod_{j=1}^{n} b_j$. Note that if $\prod_{j=1}^{\infty} b_j$ converges, then $P_n \neq 0$ $\forall n$.

Hence, $b_n = \frac{P_n}{P_{n-1}} \longrightarrow 1$. (in case of convergence)

So that...

Lemma.- Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Suppose that $\lim_{n\to\infty} b_n \neq 1$. Then $\prod_{j=1}^{\infty} b_j$ is divergent.

If view of the previous lemma, it's customary to use the notation

$$b_n = 1 \pm a_n \ , \quad a_n \xrightarrow[n\to\infty]{} 0 .$$

Theorem.- Let $a_j \geq 0 \ \forall j \in \mathbb{N}$. Then

$$\prod_{j=1}^{\infty} (1+a_j) \text{ converges} \iff \sum_{j=1}^{\infty} a_j \text{ converges.}$$

Pf.- Since $a_j \geq 0$, $P_n = \prod_{j=1}^{n} (1+a_j)$ is a non-decreasing sequence.

Therefore, $\exists \lim_{n\to\infty} P_n = \begin{cases} p \in \mathbb{R} \\ \infty \end{cases}$.

Now,

$$a_1 + a_2 + \dots + a_n \leq (1+a_1)(1+a_2)\cdot \ldots \cdot (1+a_n)$$

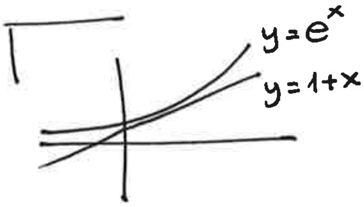
$\Big[$ $n=1$ ✓.

$$a_1 + \dots + a_{n-1} + a_n \leq \overbrace{(1+a_1)\cdots(1+a_{n-1})}^{A \geq 1!} + a_n$$

$$A + a_n \leq A(1+a_n) \leq (1+a_1)\cdots(1+a_{n-1})(1+a_n)$$

$$\iff a_n \leq A a_n$$

On the other hand, since $e^x \geq 1+x \ \forall\, x \geq 0$.



or $\varphi(x) = e^x - 1 - x$.

$\varphi(0) = 0$.

$\varphi'(x) = e^x - 1 = 0 \iff x = 0$ & $>0 \ \forall x>0$

$\to \varphi'(x) > 0 \quad \forall x > 0$.

$\Rightarrow \varphi(x) \geq 0 = \varphi(0)$

$$(1+a_1)\cdots(1+a_n) \leq e^{a_1}\cdots e^{a_n} = e^{\sum_{j=1}^n a_j}.$$

That is, we have

$$s_n = \sum_{j=1}^n a_j \leq \prod_{j=1}^n (1+a_j) \leq e^{\sum_{j=1}^n a_j}$$

So that, if $\{s_n\}$ converges, P_n converges (to a non-zero number!) & the converse also holds. $\square$.

<u>thm</u>.- Let $a_j \geq 0$, $a_j \neq 1 \ \forall j \in \mathbb{N}$.

$$\prod_{j=1}^{\infty} (1-a_j) \text{ converges} \iff \sum_{j=1}^{\infty} a_j \text{ converges}.$$

<u>Pf</u>. <u>$\Leftarrow$)</u> $\exists N : \sum_{j=N}^{\infty} a_j < \frac{1}{2}$. $(\Rightarrow a_j < 1 \ \forall j \geq N)$

Note that

$$(1-a_N)(1-a_{N+1}) = 1 - a_N - a_{N-1} + a_N a_{N+1}$$
$$\geq 1 - \underbrace{(a_N + a_{N-1})}_{<\frac{1}{2}} > \frac{1}{2}.$$

Assume that the following equality holds:

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n) \geq 1-(a_N+a_{N+1}+\dots+a_n).$$

Then

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n)(1-a_{n+1})$$

$$\geq \left(1-a_N-a_{N+1}-\dots-a_n\right)(1-a_{n+1})$$

$$= \left(1-a_N-a_{N+1}-\dots-a_n\right)-a_{n+1}+\underbrace{(a_N+a_{N+1}+\dots+a_n)a_{n+1}}_{\geq 0}.$$

Hence, $\forall n$,

$$\underbrace{(1-a_N)(1-a_{N+1})\cdots(1-a_n)}_{\prod_{j=N}^{n}(1-a_j) =: q_n} \geq 1-\underbrace{(a_N+\dots+a_n)}_{<\frac{1}{2}} > \frac{1}{2}.$$

So that q_n (being decreasing) must converge to a limit $Q \geq \frac{1}{2}$. (and $Q \leq 1$!!: $a_j \xrightarrow[j\to\infty]{} 0$, $a_j \neq 1$! $\Rightarrow 0<(1-a_j)<1$)

Now, for $n \geq N$.

$$P_n = \prod_{j=1}^{n}(1-a_j) = P_{N-1}\prod_{j=N}^{n}(1-a_j)$$

$$\Rightarrow \lim_{n\to\infty} P_n = P_{N-1}\cdot \lim_{n\to\infty}\prod_{j=N}^{n}(1-a_j) = P_{N-1}\cdot Q \neq 0.$$

$$\therefore \prod_{j=1}^{\infty}(1-a_j) \text{ converges.}$$

$\Rightarrow$) In order to get a contradiction, let us assume that $\sum_{j=1}^{\infty} a_j$ diverges. ($\equiv \sum_{j=1}^{\infty} a_j = \infty$!)

Note that if $a_j \not\to 0$, then $1-a_j \not\to 1$ and hence, $\prod_{j=1}^{\infty}(1-a_j)$ diverges $\rightarrow\leftarrow$.

So that let us assume that $a_j \xrightarrow[j\to\infty]{} 0$. Choose N: $0 \leq a_j < 1 \quad \forall j \geq N$ and notice that

$$1-x \leq e^{-x} \quad \forall\ 0 \leq x < 1$$

$\varphi(x) = 1-x-e^{-x}$.
$\varphi(0) = 0$ & $\varphi'(x) = -1+e^{-x}$.
$e^{-x} \leq 1$ if $-x \leq 0 \equiv x \geq 0$

therefore,

$$0 \leq \prod_{j=N}^{n}(1-a_j) \leq \prod_{j=N}^{n} e^{-a_j} = e^{-\sum_{j=N}^{n} a_j}, \quad n > N.$$

$$\downarrow_{n\to\infty}$$

$$0.$$

this means,

$$\lim_{n\to\infty} \prod_{j=N}^{n}(1-a_j) = 0.$$

$$\Rightarrow \quad \lim_{n\to\infty} \prod_{j=1}^{n}(1-a_j) = 0 \quad \rightarrow\leftarrow. \qquad \square.$$

Absolute convergence

DEF.- $\prod_{j=1}^{\infty}(1+a_j)$ is absolutely convergent if $\prod_{j=1}^{\infty}(1+|a_j|)$ is convergent

$\left(\Longleftrightarrow \sum_{j=1}^{\infty}|a_j| \text{ converges } \equiv \sum_{j=1}^{\infty} a_j \text{ is absolutely convergent}\right)$

Theorem. An absolutely convergent product is convergent.

Pf.- Let us use the notation

$$P_n = \prod_{j=1}^{n}(1+a_j) \quad , \quad q_n = \prod_{j=1}^{n}(1+|a_j|)$$

Note that for $m>n$,

$$P_m - P_n = \prod_{j=1}^{m}(1+a_j) - \prod_{j=1}^{n}(1+a_j)$$

$$= \prod_{j=1}^{n}(1+a_j)\left[\prod_{j=n+1}^{m}(1+a_j) - 1\right]$$

$$\Rightarrow |P_m - P_n| \overset{*}{\leq} \prod_{j=1}^{n}(1+|a_j|)\left[\prod_{j=n+1}^{m}(1+|a_j|) - 1\right].$$

$$= q_m - q_n$$

* $m = n+1$.

$$\left|\prod_{j=n+1}^{n+1}(1+a_j) - 1\right| = |a_{n+1}| = \prod_{j=n+1}^{n+1}(1+|a_j|) - 1.$$

$$\left|\prod_{j=n+1}^{n+k}(1+a_j) - 1\right| \leq \prod_{j=n+1}^{n+k}(1+|a_j|) - 1.$$

then, $$\left|\prod_{j=n+1}^{n+k+1}(1+a_j) - 1\right| = \left|(1+a_{n+k+1})\prod_{j=n+1}^{n+k}(1+a_j) - 1\right|$$

$$= \left|\prod_{j=n+1}^{n+k}(1+a_j) - 1 + a_{n+k+1}\cdot\prod_{j=n+1}^{n+k}(1+a_j)\right|$$

$$\leq \prod_{j=n+1}^{n+k}(1+|a_j|) - 1 + |a_{n+k+1}|\prod_{j=n+1}^{n+k}(1+|a_j|)$$

$$= \prod_{j=n+1}^{n+k+1}(1+|a_j|) - 1.$$

And since we are assuming that the sequence of complex numbers $\{q_n\}$ converges, it is a Cauchy sequence.

therefore, by $*$, $\{P_n\}$ is a Cauchy sequence as well $\Rightarrow \exists \lim_{n\to\infty} P_n$.

Now, since $\sum_{j=1}^{\infty} |a_j|$ converges, $\lim_{n\to\infty} 1 + a_n = 1$

$\Rightarrow |1+a_j| \geq \frac{1}{2}$ for j large enough.

therefore, $\sum_{j=N}^{\infty} \left|\frac{a_j}{1+a_j}\right| \leq 2 \sum_{j=N}^{\infty} |a_j|$ gives

that $\sum_{j=1}^{\infty} \left|\frac{a_j}{1+a_j}\right|$ converges $\overset{**}{\Rightarrow} \sum_{j=1}^{\infty} \frac{a_j}{1+a_j}$ converges

and $\prod_{j=1}^{\infty} \left(1 - \frac{a_j}{1+a_j}\right)$ is convergent.

But $\prod_{j=1}^{n} \left(1 - \frac{a_j}{1+a_j}\right) = \prod_{j=1}^{n} \frac{1}{1+a_j} = \frac{1}{P_n}$

Hence, $P_n \neq 0$.

$**$ $\sum_{k=n+1}^{\infty} |a_k| = \left| \sum_{k=1}^{m} |a_k| - \sum_{k=1}^{n} |a_k| \right| < \varepsilon$ if $n, m \geq N$
(conv $\Rightarrow$ Cauchy)

& $\left| \sum_{k=1}^{M} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|$

$\Rightarrow \sum_{k=1}^{n} a_k$ is a Cauchy seq $\Rightarrow$ converges

Let now $\{f_n(z)\}_{n\in\mathbb{N}}$ (or $\{f_n\}_{n\in\mathbb{N}}$) be a sequence of analytic functions in a domain $\Omega \subset \mathbb{C}$.

<u>DEF</u>:– $\prod_{j=1}^{\infty}(1+f_j(z))$ converges in Ω if $\forall z\in\Omega$, $\exists \lim_{n\to\infty}\prod_{j=1}^{n}(1+f_j(z)) \neq 0$.

<u>thm</u>:- An infinite product $\prod_{j=1}^{\infty}(1+f_j(z))$ is (locally) uniformly convergent in Ω if the series $\sum_{j=1}^{\infty}|f_j(z)|$ converges (locally) uniformly in Ω, where local uniform convergence means uniform convergence in every compact subset of Ω.

* Please, from now on, consider what is written on p. 29! Also, that zeros of analytic functions are isolated!

<u>Pf</u>:– Let $K \subset\subset \Omega$.

then, $\forall z\in K$, $\sum_{j=1}^{n}|f_j(z)| \underset{n\to\infty}{\rightrightarrows} |f_0(z)|$

$$\equiv \forall \varepsilon>0,\ \forall z\in K,\ \exists N:\ \left|\sum_{j=1}^{n}|f_j(z)| - |f_0(z)|\right| < 1.$$

$\forall n \geq N$.

$$\Rightarrow \sum_{j=1}^{n}|f_j(z)| \leq 1 + |f_0(z)| \leq 1+\max_{z\in K}|f_0(z)|.$$

that is,

$$\sum_{j=1}^{\infty} |f_j(z)| \leq M \qquad \forall\, z \in K.$$

then, as before,

$$(1+|f_1(z)|)(1+|f_2(z)|)\cdots(1+|f_n(z)|)$$
$$\leq e^{|f_1(z)|+\dots+|f_n(z)|} \leq e^M, \quad z \in K$$

then, for $\quad P_n(z) = \prod_{j=1}^{n} (1+|f_j(z)|)$,

we have

$$P_n(z) - P_{n-1}(z) = |f_n(z)| \, (1+|f_1(z)|)\cdots(1+|f_{n-1}(z)|)$$
$$\leq e^M \, |f_n(z)|.$$

And hence,

$$P_m(z) - P_{n-1}(z) = \sum_{j=n}^{m} (P_j(z) - P_{j-1}(z)) \leq e^M \sum_{j=n}^{m} |f_j(z)|$$
$$= e^M \left[\sum_{j=1}^{m} |f_j(z)| - \sum_{j=1}^{n-1} |f_j(z)| \right] < \varepsilon$$

uniformly $\forall\, z \in K$!

$\Rightarrow$ $\{P_n(z)\}$ is a Cauchy uniform sequence in K

$\Rightarrow$ $P_n(z) \underset{K \subset\subset \Omega}{\rightrightarrows}$. (and the limit cannot be 0!)

$\square$.

I believe it could be important to stress the following remark explicitly (though we used the result -implicitly- before).

<u>Remark</u>. By the definition of convergent infinite product, we have that $\exists\, N_1$:

$$n \geq N_1, \quad \left| \prod_{j=1}^{n} p_j - p \right| < 1.$$

$$\Rightarrow \left| \prod_{j=1}^{n} p_j \right| < 1 + |p| \;, \; p = \prod_{j=1}^{\infty} p_j \quad , \quad n \geq N_1 .$$

Now, on the other hand, any convergent sequence is Cauchy. Hence, $\forall\, \varepsilon > 0$, $\exists\, N_2$: $n, m \geq N_2$,

$$\left| \prod_{j=1}^{n} p_j - \prod_{j=1}^{m} p_j \right| < \varepsilon . \quad (*)$$

Suppose $m = n + k$. & choose $n \geq \max\{N_1, N_2\} \doteq N$.

Then, if $n \geq N$, $m \geq N$,

$$\left| \prod_{j=1}^{n} p_j - \prod_{j=1}^{m} p_j \right| < \varepsilon$$

$$\shortparallel$$

$$\prod_{j=1}^{n} |p_j| \left| \prod_{j=n+1}^{m} p_j - 1 \right| < \varepsilon .$$

$$\Rightarrow \left| \prod_{j=n+1}^{\infty} p_j - 1 \right| < \frac{\varepsilon}{1+|p|}.$$

Moreover, you can "undo" this argument if, in addition you have proved that $p \neq 0$:

$$\left| \prod_{j=n+1}^{m} p_j - 1 \right| < \varepsilon$$

$$\Rightarrow \prod_{j=1}^{n} |p_j| \left| \prod_{j=n+1}^{m} p_j - 1 \right| < \varepsilon \prod_{j=1}^{n} |p_j| < \varepsilon(1+|p|)$$

$$\Rightarrow \left\{ \prod_{j=1}^{n} p_j \right\} \text{ is Cauchy} \Rightarrow \text{converges!}.$$

Another important result. [Ahlfors]

Thm.- the product $\prod_{j=1}^{\infty}(1+a_j)$, $1+a_j \neq 0$, converges simultaneously with $\sum_{j=1}^{\infty} \mathrm{Log}(1+a_n)$.

Pf.- $\prod_{j=1}^{\infty}(1+a_j) = p \neq 0$

$$\Rightarrow \log \prod_{j=1}^{\infty}(1+a_j) = \log p$$

$$\qquad \parallel$$

$$\sum_{j=1}^{\infty} \log(1+a_j) \qquad (\mathrm{mod}\ 2\pi i).$$

Now, use the appropriate branch of the argument so that

$$\arg P - \pi < \arg \underbrace{\prod_{j=1}^{n}(1+a_j)}_{P_n} < \arg P + \pi$$

then, $$s_n = \sum_{j=1}^{n} \log(1+a_j) = \operatorname{Log} P_n + h_n \cdot 2\pi i, \quad h_n \in \mathbb{Z}.$$

$$\Rightarrow (h_{n+1} - h_n) 2\pi i = s_{n+1} - s_n + \log P_n - \log P_{n+1}$$

$$= \log(1+a_{n+1}) + \log P_n - \log P_{n+1}$$

Now: $a_{n+1} \to 0$. Hence $|\operatorname{Arg}(1+a_{n+1})| < \frac{2\pi}{3}$.

& $|\arg P_n - \arg P|$, $|\arg P_{n+1} - \arg P| < \frac{2\pi}{3}$.

$$\Rightarrow |h_{n+1} - h_n| < 1$$

$$\Rightarrow \sum_{j=1}^{\infty} \log(1+a_j) = \log P.$$

Conversely, $$\sum_{j=1}^{\infty} \log(1+a_j) = s$$

$$\| $$

$$\operatorname{Log} \prod_{j=1}^{\infty}(1+a_j) \quad (2\pi i)$$

$$\Rightarrow \prod_{j=1}^{\infty}(1+a_j) = e^s \qquad \square$$

e^1

Corollary - the series $\prod_{j=1}^{\infty}(1+a_j)$ is absolutely convergent if and only if $\sum_{j=1}^{\infty} a_j$ is absolutely convergent.

Pf. $\prod_{j=1}^{\infty}(1+a_j)$ abs. conv $\equiv$ $\prod_{j=1}^{\infty}(1+|a_j|)$ converges

$\equiv$ $\sum_{j=1}^{\infty} \operatorname{Log}(1+|a_j|)$ converges $\equiv$ $\sum_{j=1}^{\infty}|a_j|$ converges.

$$\lim_{j\to\infty}\frac{\operatorname{Log}(1+|a_j|)}{|a_j|} = 1$$

Summary

* If $\lim_{j\to\infty} b_j \neq 1 \Longrightarrow \prod_{j=1}^{\infty} b_j$ diverges.
* Absolute convergence $\Longrightarrow$ convergence.
* $\prod_{j=1}^{\infty}(1+a_j)$ abs. conv. $\Longleftrightarrow$ $\sum_{j=1}^{\infty}\operatorname{Log}(1+|a_j|)$ converges

  $\Longleftrightarrow$ $\sum_{j=1}^{\infty}|a_j|$ converges

On analytic functions without zeros.

Let $f \in \mathcal{H}(\mathbb{C}) \equiv$ entire function

Theorem.- Suppose that the entire function f has no zeros. Then, there is an entire function g:

$$f(z) = e^{g(z)}.$$

Pf.- Since $f \neq 0$, the function $\frac{f'}{f} \in \mathcal{H}(\mathbb{C})$.

$$\Rightarrow \frac{f'(z)}{f(z)} = \sum_{j=0}^{\infty} a_j z^j \quad , z \in \mathbb{C}. \ \& \ \overline{\lim_{j\to\infty}} \sqrt[j]{|a_j|} = 0$$

Define

$$h(z) = a_0 z + \frac{1}{2} a_1 z^2 + \frac{1}{3} a_2 z^3 + \dots = z\left(a_0 + \frac{1}{2} a_1 z + \frac{1}{3} a_2 z^2 + \dots\right)$$

Note that $\overline{\lim_{j\to\infty}} \sqrt[j]{\frac{1}{j+1} |a_j|} = \overline{\lim_{j\to\infty}} \frac{1}{\sqrt[j]{j+1}} \sqrt[j]{|a_j|} = 1 \cdot 0.$

hence, $h \in \mathcal{H}(\mathbb{C})$. In fact,

$$h'(z) = a_0 + a_1 z + a_2 z^2 + \dots = \frac{f'(z)}{f(z)}.$$

Now, $\varphi(z) := f(z) e^{-h(z)}$. then,

$$\varphi'(z) = f'(z) e^{-h(z)} - f(z) h'(z) e^{-h(z)} \equiv 0.$$

$$\Rightarrow f(z) = K e^{h(z)} \underset{\substack{\uparrow \\ K \neq 0}}{=} e^{a + h(z)}$$

$\square$