Weierstrass factorization theorem

Let $P = P(z)$ be a polynomial with zeros $z_1, z_2, \dots, z_n$ (here, we assume $z_j \neq 0$).

$$P(z) = C \cdot z^m \cdot (z_1 - z) \cdot \ldots \cdot (z_n - z) \qquad , m \geq 0.$$

$$= C \cdot z^m \cdot z_1 \cdot \ldots \cdot z_n \cdot \left(1 - \frac{z}{z_1}\right) \cdot \ldots \cdot \left(1 - \frac{z}{z_n}\right)$$

$$= P(0) \left(1 - \frac{z}{z_1}\right) \cdot \ldots \cdot \left(1 - \frac{z}{z_n}\right) !$$

Consider now a more general entire function f with zeros $z_1, z_2, \dots, z_n, \dots$ (all $\neq 0$).

then, by the uniqueness principle for analytic functions, $\lim_{n\to\infty} |z_n| = \infty$.

Arrange the "non-zero" zeroes by increasing moduli:

$$0 < |z_1| \leq |z_2| \leq \dots$$

Question! Can we write

$$f(z) = f(0) \cdot e^{h(z)} \cdot \underbrace{\prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right)}_{} ?$$

Problem!!! what if this infinite product diverges???

<u>DEF</u>.– Let $\nu \in \mathbb{N}\cup\{0\}$. Define the <u>Weierstrass factor</u>

$$E_\nu(z) = \begin{cases} (1-z)\, e^{Q_\nu(z)}, & \nu \geq 1 \\ 1-z, & \nu = 0 \end{cases},$$

where $Q_\nu(z) = \underbrace{z + \frac{1}{2} z^2 + \dots + \frac{1}{\nu} z^\nu}_{\mathrm{Log}\frac{1}{1-z}}$, $\nu \geq 1$.

[In what follows, we understand that $\underline{Q_0 \equiv 0}$].

<u>Properties</u>

(0) E_ν, $\nu \geq 0$ is an entire function.

(1) $\forall\ \nu \geq 0$, $E_\nu'(z) = -z^\nu e^{Q_\nu(z)}$

<u>Pf</u>.– $\nu = 0$ $\quad E_0'(z) = -1 = -z^0 \cdot e^{Q_0(z)}$ ✓.

$\nu \geq 1$:

$$E_\nu(z) = -e^{Q_\nu(z)} + (1-z)\cdot Q_\nu'(z)\, e^{Q_\nu(z)}$$

$$= e^{Q_\nu(z)}\left[-1 + (1-z)\left(1 + z + \dots + z^{\nu-1}\right)\right]$$

$$= e^{Q_\nu(z)}\left[\cancel{-1} + \cancel{1} + \cancel{z} + \dots + z^{\nu-1} - \cancel{z} - z^2 - \dots - z^\nu\right]$$

$$= -z^\nu e^{Q_\nu(z)}$$

(2) $\forall\ \nu \geq 0, \quad E_\nu(z) = 1 + \sum_{j>\nu} a_j z^j, \quad \sum_{j>\nu} |a_j| = 1.$

<u>Pf.</u>— $\nu = 0$. $\quad E_0(z) = 1 - z$ ✓.

$\nu \geq 1$. $\quad E_\nu$ is entire.

$$E_\nu(z) = \sum_{j=0}^{\infty} a_j z^j \qquad , z \in \mathbb{C}.$$

differentiating
$\Longrightarrow$
$$\sum_{j=1}^{\infty} j a_j z^{j-1} = E_\nu'(z) = -z^\nu e^{Q_\nu(z)}$$
$$= -z^\nu \cdot \sum_{j=0}^{\infty} \beta_j z^j.$$

$\Rightarrow$. $a_j = 0 \quad \forall j \leq \nu.$

& $a_j \leq 0 \quad \forall j > \nu$! (Taylor coefficients of e^z & Q_ν are positive!)

Hence, $|a_j| = -a_j \quad \forall j > \nu.$

then, $E_\nu(z) = \underbrace{1}_{E_\nu(0)} + \sum_{j>\nu} a_j z^j$

But $\underset{\overset{\parallel}{0}}{E_\nu(1)} = 1 + \sum_{j=\nu+1}^{\infty} a_j \Rightarrow \sum_{j=\nu+1}^{\infty} a_j = -1$

$$= -\sum_{j=\nu+1}^{\infty} |a_j| \qquad \#.$$

(3) If $|z| \leq 1$, then $|E_\nu(z) - 1| \leq |z|^{\nu+1}, \ \nu \geq 0.$

<u>Pf.</u>— $\nu = 0 \to |E_\nu(0) - 1| = |z|.$

• $\nu \geq 1$.

$$|E_\nu(z)-1| = \left|\sum_{j=\nu+1}^{\infty} a_j z^j\right| \leq \sum_{j=\nu+1}^{\infty} |a_j|\,|z^j|$$

$$= |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j|\,|z|^{j-(\nu+1)}$$

$$\leq |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j| = |z|^{\nu+1} \quad \#.$$

<u>Theorem</u> (Weierstrass).

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers, arranged in increasing moduli and such that $\lim_{n\to\infty} |z_n| = +\infty$.

Let $m \in \mathbb{N}\cup\{0\}$.

Every entire function with zeros z_n and no other zero in $\mathbb{C}\setminus\{0\}$ and with a zero of multiplicity m at $z=0$ can be written in the form

$$G(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} \left(1-\frac{z}{z_j}\right) e^{Q_\nu\left(\frac{z}{z_j}\right)}$$

for some $\nu \in \mathbb{N}\cup\{0\}$ and some entire function g.

<u>RK</u>. – Note that $\{z_n\}$ is not necessarily formed by distinct points. A repeated z_n represents a multiple zero of G!

Pf.- Let's determine ν so that

$$\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right) \text{ converges absolutely}$$

and uniformly for $|z| \leq R$, R large enough:

Fix $R > 1$ & $0 < \alpha < 1$.

Since $\lim_{n\to\infty} |z_n| = \infty$, $\exists\, q$:

$$|z_q| \leq \frac{R}{\alpha} \quad \& \quad |z_{q+1}| > \frac{R}{\alpha}.$$

We have:

$\rightarrow$ $\prod_{j=1}^{q} E_\nu\left(\frac{z}{z_j}\right)$ is entire (finite product of entire fcns).

$\rightarrow$ $\prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$?.

Fix $z \in \{|z| \leq R\}$. Since $j > q$, $|z_j| > \frac{R}{\alpha}$

$$\Longrightarrow \left|\frac{z}{z_j}\right| < \alpha < 1.$$

Write

$$E_\nu\left(\frac{z}{z_j}\right) = \left(1 - \frac{z}{z_j}\right) e^{Q_\nu\left(\frac{z}{z_j}\right)} = 1 + U_j(z),$$

$$U_j(z) = E_\nu\left(\frac{z}{z_j}\right) - 1.$$

By (3), $|U_j(z)| = \left|E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1}$

2 options.

(1) $\exists\ p \in \mathbb{N}: \sum_{j=1}^{\infty} |z_j|^{-p} < \infty.$ [the infimum of such $p \equiv$ exponent of convergence of the zeros]

then, define $\nu := p-1$ to get

$$\sum_{j=q+1}^{\infty} |U_j(z)| \leq \sum_{j=q+1}^{\infty} \left|\frac{z}{z_j}\right|^p \leq R^p \sum_{j=q+1}^{\infty} |z_j|^{-p} < \infty.$$

$\forall\ |z| \leq R.$

$$\Longrightarrow \prod_{j=q+1}^{\infty} (1+U_j(z)) = \prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely & uniformly in $|z| \leq R$.

(2) $\forall\ p \in \mathbb{N}, \sum_{j=1}^{\infty} |z_j|^{-p} = \infty.$

Set $\nu = j-1$. (Note that $\nu = \nu(j)$!)

then, $|U_j(z)| \leq \left|\frac{z}{z_j}\right|^j, \quad j > q.$

But $\overline{\lim_{j\to\infty}} \sqrt[j]{\left|\frac{z}{z_j}\right|^j} = \overline{\lim_{j\to\infty}} \left|\frac{z}{z_j}\right| \leq \alpha < 1$

$$\Longrightarrow \sum_{j=q+1}^{\infty} |U_j(z)| \text{ converges.}$$

that is, in either case,

$$\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely & uniformly for $|z| \leq R$.

Assume we can prove that $\prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$ is entire. Then, G is entire too and has the prescribed zeroes.

Moreover, if $\tilde{G}$ is another such function, then $\frac{G}{\tilde{G}}$ is an entire function with no zeros and the result follows.

Therefore, it remains to prove the following result.

<u>Theorem</u>. Let $\{f_n\}$ be a sequence of analytic functions in a domain G.

If $\exists \lim_{n\to\infty} f_n(z) = f(z)$

uniformly in closed subdomains of G.

Then $f \in \mathcal{H}(G)$ & $f'(z) = \lim_{n\to\infty} f_n'(z)$, $z \in \mathbb{D}$.

Pf:- Fix $z_0 \in G$ & r: $\overline{B(z_0, r)} \subset G$.

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \,, \quad z \in D(z_0, r)$$
$$(\Rightarrow z \in D(z_0, \tfrac{r}{2}))$$

Now, ∂D is compact

$\Rightarrow \quad |f_n(\zeta) - f(\zeta)| < \varepsilon \quad \forall\, n \geq N_0$, say &

all $\zeta \in \partial D$. Also, for $\zeta \in \partial D(z_0, r)$ & $z \in D(z_0, \frac{r}{2})$,

$|\zeta - z| > \frac{r}{2}$.

Now: $z \in D(z_0, \frac{r}{2})$:

$$\left| \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z} - \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right|$$

$$\leq \frac{1}{2\pi} \int_{\partial D} \frac{|f_n(\zeta) - f(\zeta)|}{|z - \zeta|} |d\zeta| \leq \frac{\varepsilon \cdot 2\pi r}{2\pi \cdot \frac{r}{2}} = 2\varepsilon .$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Hence,

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h} \int_{\partial D} \left(\frac{f(\zeta)}{\zeta-(z+h)} - \frac{f(\zeta)}{\zeta - z} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)(\zeta-(z+h))}$$

$$\xrightarrow[\text{uniform convergence}]{h\to 0} \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta-z)^2} d\zeta .$$

It is now trivial to show

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\lim_{n\to\infty} f_n(z)}{(\zeta-z)^2} d\zeta = \lim_{n\to\infty} f_n'(z) \quad \square .$$

the proof of Weierstrass' thm is complete. $\square$

<u>Example</u> .- Let $f(z) = \sin \pi z$.

ceros: $z=n,\ n\in\mathbb{Z}$.

$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \implies$ take $\nu = 1$ to

get

$$f(z) = z e^{\tilde{g}(z)} \cdot \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{n}\right) \cdot e^{\frac{z}{n}}$$

$$= z e^{\tilde{g}(z)} \cdot \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right), \quad \tilde{g}\in\mathcal{H}(\mathbb{C}).$$

$$\equiv \quad \sin \pi z = \pi z e^{g(z)} \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right), \quad g\in\mathcal{H}(\mathbb{C}).$$

g?

$$\frac{f'(z)}{f(z)} = \pi \cot(\pi z) = \frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z)$$

$$+ \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2} .$$

Define $h(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2-j^2}$.

→ h is meromorphic: Poles $j : j \in \mathbb{Z}$.

→ $h(z) = \lim_{n \to \infty} \sum_{j=-n}^{n} \frac{1}{z+j}$. $\left[\frac{1}{z-j} + \frac{1}{z+j} = \frac{2z}{z^2-j^2} \checkmark \right]$.

Exercises.–

(1) h has simple poles at $z = n$, $n \in \mathbb{Z}$ with residue $= 1$. just like $\pi \cdot \cot \pi z$.

(2) $2\pi \cot 2\pi z = \pi \cot \pi z + \pi \cot\left(\pi\left(z + \frac{1}{2}\right)\right)$.

Lemma.– Let $g \in \mathcal{H}(\mathbb{C} \setminus \mathbb{Z})$ with simple poles w/. residue $= 1$ at $z = n$, $n \in \mathbb{Z}$.

Suppose that $g(-z) = -g(z)$ and

$$2\, g(2z) = g(z) + g\left(z + \tfrac{1}{2}\right).$$

then $g(z) = \pi \cot \pi z$.

Pf.– $H(z) = g(z) - \pi \cot \pi z$ is entire*, odd**, $H(0) = 0$*** & $2H(2z) = H(z) + H\left(z + \frac{1}{2}\right)$

[* Laurent series : $\frac{1}{z-j} + \sum_{n \geq 0} a_n (z-j)^n$!] [*** ~~$g(z) = \frac{1}{z} + \dots$~~ $H(0) = -H(0)$].

[** $H(-z) = -g(z) + \pi \cot \pi z$]

$$2H(2z) = 2g(2z) - 2\cot 2\pi z$$
$$= g(z) + g\left(z+\tfrac{1}{2}\right) - \pi\cot\pi z - \pi\cot\left(\pi\left(z+\tfrac{1}{2}\right)\right)$$
$$= H(z) + H\left(z+\tfrac{1}{2}\right). \quad (*)$$

(Sup) $H(z) \not\equiv 0$. Consider $\overline{D} = \overline{D(0,2)}$.

$\exists\, c \in \partial D: \quad |H(z)| < |H(c)| \quad \forall z \in D$.

Now, $\frac{c}{2}$ & $\frac{c+1}{2} \in D$

$$\Rightarrow \left|H\left(\tfrac{c}{2}\right) + H\left(\tfrac{c}{2}+\tfrac{1}{2}\right)\right| \le \left|H\left(\tfrac{c}{2}\right)\right| + \left|H\left(\tfrac{c}{2}+\tfrac{1}{2}\right)\right|$$
$$< 2|H(c)|$$
$$\rightarrow\leftarrow \quad (*)$$

$\therefore H(z) \equiv 0$ $\square$.

Now, h is odd ✓

$$2h(2z) \stackrel{?}{=} h(z) + h\left(z+\tfrac{1}{2}\right)$$

Write $\quad S_n(z) = \frac{1}{z} + \sum_{j=1}^{n}\left(\frac{1}{z+j} + \frac{1}{z-j}\right).$

$$2S_{2n}(2z) - S_n(z) - S_n\left(z+\tfrac{1}{2}\right)$$
$$= \sum_{j=1}^{2n}\left(\frac{2}{2z+j} + \frac{2}{2z-j}\right) - \sum_{j=1}^{n}\left(\frac{1}{z+j} + \frac{1}{z-j}\right)$$
$$- \sum_{j=1}^{n}\left(\frac{1}{z+\frac{1}{2}+j} - \frac{1}{z+\frac{1}{2}-j}\right) + \frac{2}{2z} - \frac{1}{z} - \frac{1}{z+1/2}$$
$$= \frac{-2}{2z+1} + \sum_{j=1}^{2n}\frac{2}{2z+j} - \sum_{j=1}^{n}\frac{1}{z+j} - \sum_{j=1}^{n}\frac{1}{z-j}$$

$$+\sum_{j=1}^{2n}\frac{2}{2z-j}-\sum_{j=1}^{n}\frac{2}{\underbrace{2z+1+2j}}-\sum_{j=1}^{n}\frac{2}{\underbrace{2z+1-2j}}$$

$$=\frac{-2}{2z+1}+\sum_{j=1}^{2n}\frac{2}{2z+j}-\sum_{j=1}^{n}\frac{2}{2z+2j+1}$$

$$+\sum_{j=1}^{2n}\frac{2}{2j-z}-\sum_{j=1}^{n}\frac{2}{2z+1-2j}-\sum_{j=1}^{n}\frac{1}{z+j}$$

$$-\sum_{j=1}^{n}\frac{1}{z-j}$$

$$=\frac{-2}{2z+1}+\frac{2}{2z+1}+\cancel{\sum_{j=1}^{n}\frac{1}{z+j}}+\cancel{\sum_{j=1}^{n}\frac{1}{z-j}}$$

$$-\frac{2}{2z+2n+1}=-\frac{2}{2z+2n+1}\xrightarrow[n\to\infty]{}0.$$

$$\Rightarrow\quad h(z)=\pi\cot\pi z.$$

$$\Rightarrow\quad g'(z)=0\ \rightarrow\ g\ \text{constant}.$$

$$\sin\pi z=\pi z\, e^{g(z)}\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$$

$$\Rightarrow\quad e^{g(0)}=\lim_{z\to0}\frac{\sin\pi z}{\pi z}=1\ \rightarrow\ g(z)\equiv 0.$$

i.e., $\sin\pi z=\pi z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)$!!!!!

$$\boxed{\prod_{j=1}^{\infty}\left(1+\frac{1}{j^2}\right)=\prod_{j=1}^{\infty}\left(1-\frac{i^2}{j^2}\right)=\frac{\sin\pi i}{\pi i}}.$$

Some remarks

(1) On the definition of convergence.

DEF.- $\prod_{j=1}^{\infty} p_j$ converges if $\{P_n\}$, $P_n = \prod_{j=1}^{n} p_j$ converges to a non-zero value.

Ahlfors: "there are good reasons for excluding the value zero: If $P = \lim_{n\to\infty} P_n$ were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors.

On the other hand, in certain connections this convention is too radical: we wish to express a function as an infinite product and this must be possible even if the function has zeros. For this reason, we make the following agreement: An infinit product converges $\Leftrightarrow$ at most a finite number of factors are zero and if the partial products formed by the non-vanishing factors tend to a non-zero limit!

Notice that this is exactly what is used in the proof of Weierstrass' thm!

② Weierstrass' theorem provides a representation of the form

$$f(z) = z^m e^{g(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) e^{Q_{\nu_j}\left(\frac{z}{a_j}\right)}$$

If $g(z)$ reduces to a polynomial, then f is said to be of <u>finite genus</u> and the genus of f is by definition equal to the degree of this polynomial or to the genus of the cannonical product*, of whichever is larger.

* the genus of the cannonical product = exponent of convergence of the zeros $- 1 \equiv \nu$ $(= (p-1)$!!!! see p.22)

<u>Examples</u>

- An entire function of genus zero is of the form

$$Cz^m \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right), \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|} < \infty .$$

- Genus 1:

$$\rightarrow Cz^m e^{\alpha z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}, \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|^2} < \infty,$$

and $\displaystyle\sum_{j=1}^{\infty} \frac{1}{|a_j|} = \infty$, $\alpha \in \mathbb{C}$, or

$$\rightarrow Cz^m e^{\alpha z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_n}\right), \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|} < \infty, \ \alpha \neq 0.$$

$\sin \pi z = \pi z \prod_{j \neq 0} \left(1 - \frac{z}{j}\right) e^{z/j}$ has genus 1.