Complex Interpolation.

<u>thm</u>.- Let $\{z_n\}$ be a sequence of distinct points in $\mathbb{C}$ having <u>no</u> finite accumulation points & $\{\zeta_n\}$ a sequence of complex numbers, not necessarily $\neq$.

then $\exists\ f \in \mathcal{H}(\mathbb{C})$: $f(z_n) = \zeta_n \quad \forall n \in \mathbb{N}$.

Recall that if f is meromorphic, the Laurent expansion of f around a is

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-a)^j, \qquad m = m(a) \in \mathbb{Z}.$$

If $m>0$, the finite sum

$$\sum_{j=-m}^{-1} a_j (z-a)^j \equiv \text{singular part of } f \text{ at } z=a.$$

<u>thm</u> (Mittag-Leffler).

Let $\{z_n\}$ be as in the previous theorem. And let $\{P_n(z)\}$ be a sequence of polynomials with $P_n(0)=0$.

then, $\exists$ a meromorphic function f with singular part at $z=z_n$ equal to $P_n\left(\frac{1}{1-z_n}\right)$ and no other poles in $\mathbb{C}$.

Pf.- Assume for the moment that

$$0 < |z_1| \leq |z_2| \leq \cdots .$$

Let $\sum_{n=1}^{\infty} c_n$ be a convergent series of positive real numbers.

$P_n(z)$ is a polynomial $\Rightarrow P_n\left(\frac{1}{z-z_n}\right) \in H\{|z|<|z_n|\}$.

$$\Rightarrow P_n\left(\frac{1}{z-z_n}\right) = \sum_{j=0}^{\infty} a_j^{(n)} z^j \quad , \quad |z| < |z_n|$$

& $\sum_{j=0}^{\infty} a_j^{(n)} z^j$ converges absolutely & uniformly

in $\{|z|<\rho\}$, $\frac{|z_n|}{2} < \rho < |z_n|$.

Define

$$Q_n(z) = \sum_{j=0}^{K_n} a_j^{(n)} z^j ,$$

where K_n has been chosen to satisfy

$$\sup_{|z| < \frac{|z_n|}{2}} \left| P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right| < c_n$$

Let's now analyze the series

$$\sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z) \right) . \quad (*)$$

Take an arbitrary $R>0$ and notice that only those singular parts $P_n\left(\frac{1}{z-z_n}\right)$ with $|z_n|<R$ contribute poles to the sum (*).

Now, break * in 2 parts.

$$\underbrace{\sum_{|z_n|<2R}\left(P_n\left(\frac{1}{z-z_n}\right)-Q_n(z)\right)}_{} \quad \& \quad \underbrace{\sum_{|z_n|>2R}\left(P_n\left(\frac{1}{z-z_n}\right)-Q_n(z)\right)}_{}$$

Rational function with poles exactly at z_n, $|z_n|<R$. & singular part $= P_n\left(\frac{1}{1-z_n}\right)$.

no poles in $|z|<R$!!! Moreover, since $R<\frac{|z_n|}{2}$,

$\left|\sum\right| \leq \sum C_n < \infty$!

Hence, this part converges abs. & unif. in $|z|\leq R$ $\Rightarrow$ analytic in $|z|<R$.

But R was arbitrary! so that we are done with the exception of $z=0$.

Just consider

$$P_0\left(\frac{1}{z}\right)+\sum_{n=1}^{\infty}\left(P_n\left(\frac{1}{1-z_n}\right)-Q_n(z)\right)$$

to get the desired function f. $\square$.

Pf. (1st. theorem in this section).

By Weierstrass' thm, we can construct $g\in\mathcal{H}(\mathbb{C})$ with simple zeros at z_n.

$(\Rightarrow g'(z_n)\neq 0)$.

By the Mittag-Leffler thm, there exists a meromorphic function h with simple poles exactly at z_n and with residue $\frac{s_n}{g'(z_n)}$ at each z_n: $P_n(z) = \frac{s_n}{g'(z_n)} z$.

Define $f(z) = h(z) g(z)$

claim: $f \in \mathcal{H}(\mathbb{C})$. We just need to check f is analytic at $z = z_n$.

- $g \in \mathcal{H}(\mathbb{C})$ & $g(z_n) = 0$

$$\Rightarrow g(z) = g'(z_n)(z - z_n) + \frac{g''(z_n)}{2}(z - z_n)^2 + \cdots \qquad (|z - z_n| \ll)$$

$$= (z - z_n)\, g_n(z), \quad g_n(z_n) = g'(z_n), \; g_n \in \mathcal{H}(z_n)$$

- Also, near $z = z_n$,

$$h(z) = \frac{s_n}{g'(z_n)} \cdot \frac{1}{z - z_n} + A_0 + A_1 (z - z_n) + \cdots$$

$$= \frac{h_n(z)}{z - z_n}, \quad h_n(z_n) = \frac{s_n}{g'(z_n)}, \quad h_n \in \mathcal{H}(z_n)$$

then, $f(z) = g_n(z) h_n(z) \in \mathcal{H}(z_n)$!

Moreover, $f(z_n) = s_n$ $\square$