Growth of entire functions

Recall that if $f \in \mathcal{H}(\overline{D(0,R)})$ $(\equiv f \in \mathcal{H}(\Omega),$ $\overline{D(0,R)} \subset \Omega)$ and is not constant,

$$\max_{|z|\leq R} |f(z)| = \max_{|z|=R} |f(z)|.$$

For an entire function f, we denote the <u>maximum modulus</u>

$$M(r,f) = \max_{|z|\leq r} |f(z)| = \max_{|z|=r} |f(z)|.$$

The following lemma is obvious, but useful.

<u>Lemma</u>.- Let $P(z) = a_n z^n + \ldots + a_0$, $a_n \neq 0$, be a polynomial of degree n.

Then, $\forall \varepsilon > 0$, $\exists$ $R = R_\varepsilon$:

$$(1-\varepsilon)|a_n||z|^n \leq |P(z)| \leq (1+\varepsilon)|a_n||z|^n$$

$\forall$ $|z| > R_\varepsilon$.

<u>Pf</u>.- $\quad |P(z)| = |a_n||z|^n \left| 1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \ldots + \frac{a_0}{a_n}\cdot\frac{1}{z^n} \right|$

Since $\quad \varphi(z) = \frac{a_{n-1}}{a_n}\cdot\frac{1}{z} + \ldots + \frac{a_0}{a_n}\frac{1}{z^n} \xrightarrow[|z|\to\infty]{} 0,$

we have that $\exists$ $R_\varepsilon : |\varphi(z)| < \varepsilon$, $|z| \geq R_\varepsilon$.

Hence

$$(1-\varepsilon)|a_n||z|^n \le (1-|\varphi(z)|)|a_n||z|^n$$
$$\le |P(z)| \le (1+|\varphi(z)|)|a_n||z|^n$$

$\square$.

$\forall |z| > R_\varepsilon$.

<u>DEF</u>.- Let $f \in \mathcal{H}(\mathbb{C})$. The <u>order</u> of f is

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

the lower order,

$$\mu(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Notice that if f is a polynomial of degree n, $M(r,f) \sim c r^n$.

$$\Rightarrow \rho(f) \sim \frac{\log\log r^n}{\log r} \sim \frac{\log n + \log \log r}{\log r} \sim 0.$$

& same holds for $\mu(f)$!

On the other hand, $\rho(f)$ & $\mu(f) \ge 0$:

$$\frac{\log\log M(r_n,f)}{\log r_n} \sim \underset{(a>0)}{-a} \equiv \cancel{\log}\log M(r_n,f) \sim \cancel{\log}\, r_n^{-a}$$

$$\sim M(r_n,f) \sim e^{r_n^{-a}} \longrightarrow 1.$$

Hence f would be bounded, in $\mathbb{C}$, so that by the Liouville theorem, $f = \text{constant}$.

Remark.- the previous lemma shows that if f is a polynomial of degree n, then $M(r,f)$ grows no faster than r^n. In those cases when f is not a polynomial, this is no longer true.

Lemma.- Let $f \in \mathcal{H}(\mathbb{C})$ satisfy

$$|f(z)| \leq C\,|z|^K \quad , \quad |z| \geq R_0 .$$

for some $C>0$, $R_0>0$, $K \in \mathbb{N}$.
then f is a polynomial of degree at most K.

Pf.- By Cauchy's formula:

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta . \qquad (\forall r>0!)$$

Hence, for $n \geq K+1$, $r \geq R_0$

$$|f^{(n)}(0)| \leq \frac{1}{2\pi} \int_{|\zeta|=r} \frac{|f(\zeta)|}{r^{n+1}}\, |d\zeta|$$

$$\leq \frac{C}{\cancel{2\pi}} \frac{r^K}{r^{n+\cancel{1}}} \cancel{2\pi} r \xrightarrow[r\to\infty]{} 0 .$$

$$f \in \mathcal{H}(\mathbb{C}) \Rightarrow f(z) = \sum_{n=0}^{\infty} f^{(n)}(0) \cdot \frac{z^n}{n!} = \sum_{n=0}^{K} a_k z^k .$$

$\square$.

What if f is not a polynomial?
Maybe it makes sense to compare $M(r,f)$ with the exponential function. And this is the way the order comes up:

Suppose $f \in \mathcal{H}(\mathbb{D})$ has finite order ρ.

$$\equiv \quad \frac{\log \log M(r,f)}{\log r} < \rho + \varepsilon \quad \forall\, r \geq r_\varepsilon .$$

$$\equiv \quad \cancel{\log} \log M(r,f) < \cancel{\log}\, r^{\rho+\varepsilon}$$

$$\equiv \quad |f(z)| \leq M(r,f) \leq e^{r^{\rho+\varepsilon}} \quad \forall\, |z| \leq r .$$

<u>Lemma</u>:- Let

$$\alpha := \inf \{ \lambda > 0 : M(r,f) \leq e^{r^\lambda} \text{ for all } r \text{ large enough} \}.$$

then $\rho(f) = \alpha$.

<u>Pf</u>:- $M(r,f) \leq e^{r^\lambda} \Rightarrow \log \log M(r,f) \leq \lambda \log r$
$\Rightarrow \rho(f) \leq \lambda$ for all such λ. Hence, $\rho(f) \leq \alpha$. [This also shows that if $\rho(f) = \infty$, then $\alpha = \underline{\infty}$!].

Now, we have seen that if f has finite order ρ, then

$$M(r, f) \le e^{\rho+\varepsilon} \quad \forall \varepsilon > 0.$$

$\Rightarrow \alpha \le \rho + \varepsilon \quad \forall \varepsilon > 0 \Rightarrow \alpha \le \rho.$ $\square$

Examples.

(1) A direct application of the previous lemma shows that $\rho(e^z) = 1$. Another way to prove this (that also shows $\mu(e^z) = 1$) is:

$$|e^z| = e^{Re(z)} \Rightarrow \max_{|z|=r} |e^z| = e^r$$

$$\Rightarrow \frac{\log \log M(r, f)}{\log r} = \frac{\log r}{\log r} = 1.$$

Sometimes, it is useful to relate $\rho(f_1 + f_2)$ with $\rho(f_1)$ and $\rho(f_2)$, or similar.

$\left[\cos z = \dfrac{e^{zi} + e^{-zi}}{2} \right]$ The next theorem contains some information about this question.

Theorem.- Let $f_1, f_2 \in \mathcal{H}(\mathbb{D})$. Then,

(1) $\rho(f_1 + f_2) \le \max(\rho(f_1), \rho(f_2))$.

(2) $\rho(f_1 \cdot f_2) \le \max(\rho(f_1), \rho(f_2))$.

Moreover, if $\rho(f_1) < \rho(f_2)$, then

(3) $\rho(f_1 + f_2) = \rho(f_2)$.

Pf.– (1) Let $\rho_1 = \rho(f_1)$ & $\rho_2 = \rho(f_2)$.

$$\equiv \quad M(r, f_1) \le e^{r^{\rho_1+\varepsilon}}, \quad M(r, f_2) \le e^{r^{\rho_2+\varepsilon}}.$$

Now, $$M(r, f_1 + f_2) = \max_{|z|=r} |f_1(z) + f_2(z)|$$

$$\le M(r, f_1) + M(r, f_2) \le e^{r^{\rho_1+\varepsilon}} + e^{r^{\rho_2+\varepsilon}}$$

$$\le 2e^{r^{\max\{\rho_1+\rho_2\}+\varepsilon}}.$$

$$\Longrightarrow \quad \rho(f_1 + f_2) \le \max\{\rho_1 + \rho_2\}$$

inf of all powers

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(2) $$M(r, f_1 f_2) = \max_{|z|=r} |f_1(z) f_2(z)|$$

$$\le \left(\max_{|z|=r} |f_1(z)|\right) \cdot \left(\max_{|z|=r} |f_2(z)|\right)$$

$$\le e^{r^{\rho_1+\varepsilon}} \cdot e^{r^{\rho_2+\varepsilon}} \le \left(e^{r^{\max(\rho_1,\rho_2)+\varepsilon}}\right)^2$$

$$\xrightarrow[\varepsilon \to 0]{} e^{2r^{\max(\rho_1,\rho_2)}},$$

$$\log M(r, f_1 f_2) \le 2r^{\max(\rho_1, \rho_2)}$$

$$\log \log M(r, f_1 f_2) \le \log 2 + \max(\rho_1, \rho_2) \log r$$

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(3) Assume now that $\rho(f_1) < \rho(f_2) = \rho$.

then, by (1), $\rho(f_1, f_2) \le \rho$.

Let us prove that $\forall\, \varepsilon > 0$,

$$\rho(f_1, f_2) \geq \rho - \varepsilon.$$

We use again that for large r,

$M(r, f_1) \le e^{r^{\rho(f_1)+\varepsilon}}$. Also, that there is a sequence $\{r_n\}$, $r_n \xrightarrow[n\to\infty]{} \infty$:

$$M(r_n, f_2) \geq e^{r_n^{\rho-\varepsilon}}$$

$$\left[\limsup_{r\to\infty} \frac{\log\log M(r, f_2)}{\log r} = \rho \; ! \right]$$

Now, f_2 is continuous & $|z| = r_n$ compact.

Hence, $\exists\; |z_n| = r_n : |f(z_n)| = M(r_n, f_2) \geq e^{r_n^{\rho-\varepsilon}}$.

Hence:

$$|(f_1+f_2)(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \geq e^{r_n^{\rho-\varepsilon}} - e^{r_n^{\rho(f_1)+\varepsilon}}.$$

$$= e^{r_n^{\rho-\varepsilon}}\left(1 - e^{r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon}}\right) \geq \frac{1}{2} e^{r_n^{\rho-\varepsilon}}$$

because $\lim_{n\to\infty} e^{r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon}} \xrightarrow[n\to\infty]{} 0$ *.

$$\left[r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon} = r_n^{\rho-\varepsilon}\Big(\underbrace{r_n^{\rho(f_1)-\rho+2\varepsilon}}_{0} - 1\Big) = -\infty \,! \right]$$

□.

Remarks :-

① It is also true that if $\rho(f_1) < \rho(f_2)$ then $\rho(f_1 f_2) = \rho(f_2)$.

② A direct consequence is that if we sum (or multiply) a function f of order $\rho < \infty$ by a polynomial, the resulting function has the same order as f.

Example .- $f(z) = \cos z$. $\rho(f)$?.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz}}{2} + \frac{e^{-iz}}{2} = f_1(z) + f_2(z).$$

Exactly as before, (with e^z), $\rho(f_1) = \rho(f_2) = 1$

$\Rightarrow \rho(\cos z) \leq 1$.

Now, for $|z| = r$,

$$M(r, \cos z) \geq |\cos(-ir)| = \frac{e^{r} + e^{-r}}{2} = \frac{e^{r}(1+e^{-2r})}{2}$$

$$\Rightarrow \quad \log M(r, \cos z) \geq r + \log \frac{1+e^{-2r}}{2}$$

$$\Rightarrow \quad \rho(r, \cos z) \geq 1 + \lim_{r \to \infty} \frac{\log \log \frac{1+e^{-2r}}{2}}{\log r} = 1.$$

Define, for $f \in \mathcal{H}(\mathbb{C})$,

$$A(r,f) = \max_{|z|=r} \operatorname{Re}\{f(z)\}.$$

$f = u + iv \in \mathcal{H}(\Omega) \Rightarrow u, v \in h(\Omega)^{*}$.

Pf. $u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0$. $\square$.

Some properties of harmonic functions.

- MMP. $u \in h(D(z_0,r)) \cap \mathcal{C}(\overline{D(z_0,r)}) \Rightarrow \exists\, f \in \mathcal{H}(D(z_0,r)) : \operatorname{Re} f = u$.

Pf. $f = u + iv$, $\quad v_y = u_x \to v(x,y) = \int u_x \, dy$

$v_x = -u_y$

Now, $g(z) = e^{f(z)}$ is not constant if $u \neq$ const.

$\exists\, z_0 \in \Omega : \quad |\operatorname{Re} f(z)| \leq |\operatorname{Re} f(z_0)| \Rightarrow |g(z)| \leq |g(z_0)|$!

$\square$

- MVP: $u \in h(D(z_0,r)) \cap \mathcal{C}(D(z_0,r))$

$$\Rightarrow \quad u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Pf. For f analytic in $D = D(z_0,r)$ and continuous in $\overline{D}$, the Cauchy formula gives

$$f(z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\, d\theta$$

$\zeta = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$

$d\zeta = rie^{i\theta}\, d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

$*$ $h(\Omega) = \{ u \in \mathcal{C}^{\infty}(\Omega) : u_{xx} + u_{yy} = 0 \}$.

$$\Rightarrow \quad u(z_0) = \operatorname{Re} f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

$$\operatorname{Re}\left\{ \int (A + iB)\, d\theta \right\} = \int A\, d\theta. \qquad \square$$

<u>Theorem.</u>- Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{H}(\mathbb{C})$.

then, $\forall\ j \in \mathbb{N}$,

$$|a_j| r^j \leq \max\{0, 4A(r,f)\} - 2\operatorname{Re} f(0)$$

<u>Pf.</u>- $r = 0 \quad \equiv \quad 0 \leq \max\{0, 4\operatorname{Re} f(0)\} - 2\operatorname{Re} f(0).$

$$= \begin{cases} 2|\operatorname{Re} f(0)|, & \operatorname{Re} f(0) < 0. \\ 2\operatorname{Re} f(0), & \operatorname{Re} f(0) > 0. \end{cases}$$

For $r > 0$, write $z = re^{i\theta}$, $a_n = \alpha_n + i\beta_n$,

$$\operatorname{Re} f(re^{i\theta}) = \operatorname{Re}\left\{ \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j e^{ij\theta} \right\}$$

$$= \operatorname{Re}\left\{ \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j (\cos j\theta + i \sin j\theta) \right\}$$

$$= \sum_{j=0}^{\infty} (\alpha_j \cdot \cos j\theta - \beta_j \sin j\theta) r^j$$

$$\xrightarrow[\boxed{n>0}]{} \frac{1}{\pi} \int_0^{2\pi} (\operatorname{Re} f(re^{i\theta})) \cos n\theta\, d\theta = \frac{1}{\pi} \sum_{j=0}^{\infty} \int_0^{2\pi} \alpha_j \cos j\theta \cos n\theta\, r^j\, d\theta$$

$$- \frac{1}{\pi} \sum_{j=0}^{\infty} \int_0^{2\pi} \beta_j \sin j\theta \cos n\theta\, r^j\, d\theta$$

$$\int_0^{2\pi} \cos j\theta \cos n\theta \, d\theta = \frac{1}{2}\int_0^{2\pi}\left(\cos[(j+n)\theta] + \cos[(j-n)\theta]\right) d\theta$$

$$= \frac{1}{2}\left[\frac{\sin[(j+n)\theta]}{j+n} + \frac{\sin[(j-n)\theta]}{j-n}\right]\Bigg|_0^{2\pi} = 0.$$

$j \neq n$!

$$\int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} \frac{1+\cos 2n\theta}{2} \, d\theta = \pi, \quad n > 0.$$

and

$$\frac{1}{\pi}\int_0^{2\pi}\left(\operatorname{Re} f(re^{i\theta})\right)\cos n\theta \, d\theta = \alpha_n r^n$$

$$\frac{1}{\pi}\int_0^{2\pi}\left(\operatorname{Re} f(re^{i\theta})\right)\sin n\theta \, d\theta = -\beta_n r^n$$

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(\operatorname{Re} f(re^{i\theta})\right) d\theta, \quad \beta_0 = 0.$$

Hence,

$$a_n r^n = (\alpha_n + i\beta_n) r_n = \frac{1}{\pi}\int_0^{2\pi}\left(\operatorname{Re} f(re^{i\theta})\right).(\cos n\theta - i \sin n\theta) \, d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\operatorname{Re} f(re^{i\theta})\right) e^{-in\theta} \, d\theta$$

$$\Rightarrow |a_n| r^n \leq \frac{1}{\pi}\int_0^{2\pi} |\operatorname{Re} f(re^{i\theta})| \, d\theta$$

So that

$$|a_n|r^n + 2\alpha_0 \leq \frac{1}{\pi}\int_0^{2\pi} (|\mathrm{Re} f(re^{i\theta})| + \mathrm{Re} f(re^{i\theta}))\, d\theta .$$

Now: $A(r,f) < 0 \to \mathrm{Re} f(re^{i\theta}) < 0 \ \forall\ 0 \leq \theta \leq 2\pi$

Hence, the previous inequality becomes

$$|a_n|r^n + 2\alpha_0 \leq 0 \Rightarrow |a_n|r^n \leq -2\alpha_0 . \checkmark$$

$A(r,f) > 0$. gives

$$|a_n|r^n + 2\alpha_0 \leq \frac{1}{\pi}\int_0^{2\pi} 2A(r,f)\, d\theta = 4A(r,f) \qquad \square .$$

<u>Theorem</u>.- (Hadamard) If $f \in \mathcal{H}(\mathbb{C})$ and

$$L = \varliminf_{r\to\infty} \frac{A(r,f)}{r^s} < \infty$$

for some $s \geq 0, s \in \mathbb{N}$. then f is a polynomial of degree at most s.

<u>Pf</u>.- Choose a sequence $r_n \nearrow \infty$:

$$\frac{A(r_n,f)}{r_n^s} \leq L+1 .$$

So that $A(r_n,f) \leq (|L|+1)\, r_n^s$.

If $j > s$, $\quad |a_n| r_n^j \leq 4\,(|L|+1)\, r_n^s - 2\mathrm{Re} f(0)$

$$\Rightarrow |a_j| \leq \frac{4(|L|+1)}{r_n^{j-s}} - \frac{2\mathrm{Re} f(0)}{r_n^j} \xrightarrow[n\to\infty]{} 0 \qquad \square$$

Theorem – Let $f \in \mathcal{H}(\mathbb{C})$, with no zeros and such that $\mu(f) < \infty$. Then $f(z) = e^{P(z)}$ for a polynomial

$$P(z) = a_m z^m + \dots + a_0, \quad a_m \neq 0, \quad m = \mu(f) = \rho(f).$$

Corollary: $\mu(f) < \infty \iff \rho(f) < \infty$ for non-zero entire functions.

Pf. (theorem).

Since $f \in \mathcal{H}(\mathbb{C})$, $f \neq 0$ in $\mathbb{C}$, $f(z) = e^{g(z)}$, $g(z) \in \mathcal{H}(\mathbb{C})$.

$$e^{\operatorname{Re} g(z)} = |e^{g(z)}| = |f(z)| \leq M(r_n, f) \underset{\uparrow}{\leq} e^{r_n^{\mu(f)+\varepsilon}} \quad (*)$$

$$\liminf_{r \to \infty} \frac{\log \log M(r,f)}{\log r} = \mu(f)$$

$\Rightarrow \exists\, r_n \uparrow \infty : \forall \varepsilon > 0,$

$$\log \log M(r_n, f) \leq (\mu(f) + \varepsilon) \log r_n$$

$$\equiv \quad M(r_n, f) \leq e^{r_n^{\mu(f)+\varepsilon}}$$

Hence, $\operatorname{Re} g(z) \leq r_n^{\mu(f)+\varepsilon} \quad \forall\, |z| = r_n$

$\Rightarrow \quad A(r_n, g) \leq r_n^{\mu(f)+\varepsilon}.$

$\Rightarrow$ g is a polynomial of degree $\leq \mu(f) + \varepsilon$

hence $\leq \mu(f)$

Now, if $f(z) = e^{P(z)}$, $P(z) = a_m z^m + \dots + a_1 z + a_0$,

$$|f(z)| = e^{\operatorname{Re} P(z)} \le e^{|P(z)|} \le e^{A|z|^m}$$

$\Longrightarrow \rho(f) \le m$.

therefore, $\rho(f) \le m = \deg P \le \mu(f) \le \rho(f)$ $\square$.

<u>Theorem</u>. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Define

$$b_j = \begin{cases} 0, & a_j = 0 \\ \dfrac{j \log j}{\log \frac{1}{|a_j|}}, & a_j \ne 0 . \end{cases}$$

$$\rho(f) = \limsup_{j \to \infty} b_j .$$

<u>Pf.</u> Denote by $\mu = \overline{\lim_{j\to\infty}}\, b_j$.

- $\rho(f) \ge \mu$. (trivial if $\mu = 0$).

We will show that $\rho(f) \ge \sigma \;\; \forall\, 0 < \sigma < \mu$.
By the definition of μ (and σ), there exist infinitely many natural numbers j:

$$j \log j \ge \sigma \log \frac{1}{|a_j|} = -\sigma \log |a_j|$$

$$\Longrightarrow \log |a_j| \ge -\frac{1}{\sigma}\, j \log j .$$

Now, recall that by Cauchy's formula

$$|a_j| = \left|\frac{f^{(j)}(0)}{j!}\right| = \left|\frac{1}{2\pi i}\int_{|s|=r} \frac{f(s)}{s^{j+1}}\, ds\right|$$

$$\leq \frac{M(r,f)}{r^j}.$$

therefore,

$$\log M(r,f) \geq \log r^j |a_j| = j \log r + \log |a_j|$$

$$\geq j \log r - \frac{1}{\sigma} j \log j$$

Define, for the infinitely many values of j considered

$$r_j = (ej)^{1/\sigma} \equiv j = \frac{1}{e} r_j^{\sigma}$$

$$\Rightarrow \log M(r_j, f) \geq j \cdot \log (ej)^{1/\sigma} - \frac{1}{\sigma} j \log j$$

$$= \frac{1}{\sigma} j = \frac{1}{\sigma e} r_j^{\sigma}$$

$$\Rightarrow \log \log M(r_j, f) \geq \log \frac{1}{\sigma e} + \sigma \log r_j .$$

$$\Rightarrow \rho(f) \geq \sigma$$

• $\rho(f) \le \mu$. (trivial if $\mu = \infty$, so that suppose that $\mu < \infty$).

Fix $\varepsilon > 0$. Then $\exists N_0 : \forall j \geq N_0 : a_j \neq 0$,

$$0 \le \frac{j \log j}{\log \frac{1}{|a_j|}} \le \mu + \varepsilon. \quad \left(\text{otherwise, } \varlimsup_{j \to \infty} \frac{j \log j}{\frac{1}{|a_j|}} = \mu > \mu + \varepsilon ! \right)$$

$$\equiv \quad \frac{j \log j}{\mu + \varepsilon} \le \log \frac{1}{|a_j|} = - \log |a_j|$$

$$\Rightarrow \log |a_j| \le \frac{-j}{\mu + \varepsilon} \log j = \log \left(j^{\frac{-j}{\mu+\varepsilon}} \right).$$

$$\Rightarrow |a_j| \le j^{\frac{-j}{\mu+\varepsilon}}.$$

Now, let $g(z) = f(z) - \underbrace{\sum_{n=0}^{N_0} a_n z^n}_{\text{polynomial} = p(z)} = f(z) - p(z)$

then $\rho(f) \le \max\{\rho(g), \rho(p)\} = \rho(g)$.

But

$$M(r, g) = \max_{|z|=r} \left| \sum_{n=N_0+1}^{\infty} a_j z^j \right| \le \sum_{j=N_0+1}^{\infty} |a_j| r^j$$

$$\le \sum_{N_0+1 \le j < (2r)^{\mu+\varepsilon}} j^{\frac{-j}{\mu+\varepsilon}} r^j + \sum_{j \geq (2r)^{\mu+\varepsilon}} j^{\frac{-j}{\mu+\varepsilon}} r^j.$$

$$= S_1 + S_2$$

$\boxed{S_2}$ $\quad j \geq (2r)^{\mu+\varepsilon} \rightarrow 2r \leq j^{\frac{1}{\mu+\varepsilon}} \rightarrow r j^{-\frac{1}{\mu+\varepsilon}} \leq \frac{1}{2}$

$$\Rightarrow \quad S_2 \leq \sum_{j \geq (2r)^{\mu+\varepsilon}} \left(\frac{1}{2}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2.$$

$\boxed{S_1}$ $$\sum_{N_0+1 \leq j < (2r)^{\mu+\varepsilon}} j^{\frac{-j}{\mu+\varepsilon}} r^j \leq \sum_{N_0+1 \leq j < (2r)^{\mu+\varepsilon}} j^{\frac{-j}{\mu+\varepsilon}} r^{(2r)^{\mu+\varepsilon}}$$

$$\leq r^{(2r)^{\mu+\varepsilon}} \underbrace{\sum_{j=1}^{\infty} j^{\frac{-j}{\mu+\varepsilon}}}_{\text{converges!}^*} = K r^{(2r)^{\mu+\varepsilon}}, \quad K < \infty.$$

$$\therefore \quad M(r,g) \leq 2 + K r^{(2r)^{\mu+\varepsilon}} \asymp K r^{(2r)^{\mu+\varepsilon}}, \quad r \to \infty.$$

$$\Rightarrow \quad \log\log M(r,g) \leq \log\left(\log K + (2r)^{\mu+\varepsilon} \log r\right)$$

$$\asymp \log (2r)^{\mu+\varepsilon} + \log\log r = (\mu+\varepsilon)\log 2r + \log\log r$$

$$\Rightarrow \quad \frac{\log\log M(r,g)}{\log r} \asymp \mu+\varepsilon, \quad r \to \infty.$$

Hence, $\rho(f) \leq \rho(g) \leq \mu+\varepsilon \quad \forall \varepsilon > 0$ $\square$.

<u>Example</u>. • $f_1(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$, $\rho(f_1) = 1$:

$$b_j = \frac{j \cdot \log j}{\log j!} \underbrace{\quad}_{j! \sim \sqrt{2\pi j}\, e^{-j} j^j} \frac{j \log j}{j \log j - j + \log\sqrt{2\pi j}} \xrightarrow[j \to \infty]{} 1!$$

• $f_2(z) = \sum_{j=0}^{\infty} e^{-j^2} z^j$.

* $j^{\frac{-j}{j+\varepsilon}} \leq \frac{1}{j^2}$, $j >>$

$$b_j = \frac{j\log j}{\log e^{j^2}} = \frac{j\log j}{j^2 \cancel{\log j}} = \frac{\log j}{j} \xrightarrow[j\to\infty]{} 0 .$$

Hence, $\rho(f_2) = 0$ & f_2 is not a polynomial!!!!

- $f_3(z) = e^{e^z}$.

$$M(r, f_3) \geq f_3(r) = e^{e^r}$$

$$\Rightarrow \quad \frac{\log\log M(r, f_3)}{\log r} \geq \frac{\log e^r}{\log r} = \frac{r}{\log r} \xrightarrow[r\to\infty]{} \infty$$

$$\therefore \quad \rho(f_3) = \infty .$$

<u>DEF</u>:- For an entire function $f(z)$ of order $\rho : 0 < \rho < \infty$, its <u>type</u> τ is defined by

$$\tau = \tau(f) = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho} .$$

<u>Lemma</u>.

$$\beta := \inf\{k > 0 : M(r, f) \leq e^{kr^\rho}, \ r \text{ suff. large}\},$$

where $f \in \mathcal{H}(\mathbb{C})$ has order $\rho \in (0, \infty)$, then

$$\tau(f) = \beta .$$

Here, we understand $\inf \emptyset = +\infty$.

Pf.-

- If $\tau(f)=\infty$, $\forall K>0 \ \exists\ r_n \nearrow \infty$:

$$\log M(r_n,f) \geq K r_n^{\rho}$$

$$\Rightarrow \quad M(r_n,f) \geq e^{K r_n^{\rho}}$$

Hence, there is no $K_0>0$: $M(r,f) \leq e^{K_0 r^{\rho}}$

$\Rightarrow \quad \beta=+\infty$.

- If $\beta=\infty$, $\{K>0 / M(r,f) \leq e^{Kr^{\rho}}$ for r suff. large$\}$ $=\phi \Rightarrow \forall K>0, \exists r_n: M(r_n,f) > e^{K r_n^{\rho}} \Rightarrow \tau(f)=+\infty$.

- Let $\beta<\infty$. Take K: $M(r,f) \leq e^{Kr^{\rho}}$ (note that $K \geq \beta$) for r large. Then,

$$\frac{\log M(r,f)}{r^{\rho}} \leq K, \quad r>>$$

$$\Rightarrow \quad \tau(f) = \lim_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} \leq K$$

So that $\tau(f) \leq K \ \forall K \geq \beta \Rightarrow \tau(f) \leq \beta$.

- Finally, to show $\tau(f) \geq \beta$, note that

$\forall \varepsilon>0$, $\dfrac{\log M(r,f)}{r^{\rho}} \leq \tau(f)+\varepsilon$, $r>>$.

$$\Rightarrow \quad M(r,f) \leq e^{(\tau(f)+\varepsilon) r^{\rho}}$$

$$\Rightarrow \quad \beta \leq \tau(f)+\varepsilon \ \forall \varepsilon>0 \quad \square.$$

Lemma.- Let f be analytic in a neighborhood of $z=0$ with Taylor series expansion

$$f(z)=\sum_{j=0}^{\infty} a_j z^j$$

Suppose that $\exists\ \lambda>0,\ \mu>0$, and $N=N(\mu,\lambda)\in\mathbb{N}$

such that

$$|a_j| \le \left(\frac{e\mu\lambda}{j}\right)^{j/\mu} \quad \forall\ j>N.$$

then $f\in\mathcal{H}(\mathbb{C})$ and moreover, $\exists\ R=R(\varepsilon)$:

$\forall\ \varepsilon>0,$ $\quad M(r,f)\le e^{(\lambda+\varepsilon)r^{\mu}}$, $r>R$.

Pf.- Since $|a_j| \le \left(\frac{e\mu\lambda}{j}\right)^{j/\mu}$,

$$\sqrt[j]{|a_j|} \le \left(\frac{e\mu\lambda}{j}\right)^{1/\mu} \xrightarrow[j\to\infty]{} 0 .$$

Hence the radius of convergence of the series R, defined by

$$\frac{1}{R}=\overline{\lim_{j\to\infty}}\,|a_j|^{1/j}=0, \text{ so that } R=\infty$$

and $f\in\mathcal{H}(\mathbb{C})$.

To show the estimate $M(r,f) \leq e^{(\lambda+\varepsilon) r^\mu}$,

observe first that

$$\max_{x \geq 0} \left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x \overset{\boxed{\text{Exercise}}}{\overset{\downarrow}{\overset{*}{=}}} \left(\frac{e\cancel{\mu\lambda}}{\cancel{\mu\lambda} r^\mu}\right)^{\frac{\mu\lambda r^\mu}{\mu}} \cdot r^{\mu\lambda r^\mu}$$

$$= e^{\lambda r^\mu} \cdot \frac{r^{\mu\lambda r^\mu}}{r^{\mu\lambda r^\mu}} = e^{\lambda r^\mu}.$$

Moreover, if $j > N(r) = \max(N, 2^\mu e\mu\lambda r^\mu)$,

$$\sqrt[j]{|a_j| r^j} \leq \sqrt[j]{\left(\frac{e\mu\lambda}{j}\right)^{j/\mu} r^j}$$

$$= \left(\frac{e\mu\lambda}{j}\right)^{\frac{1}{\mu}} \cdot r < \frac{1}{2}$$

$$\Longrightarrow |a_j| r^j < \frac{1}{2^j}, \quad j > N(r)$$

$$\Longrightarrow M(r,f) \leq \sum_{j=0}^{\infty} |a_j| r^j = \sum_{j=0}^{N} |a_j| r^j + \sum_{j=N+1}^{N(r)} |a_j| r^j$$

$$+ \sum_{j=N(r)+1}^{\infty} |a_j| r^j \leq r^N \underbrace{\sum_{j=0}^{N} |a_j|}_{b} + (N(r)-N) \max_{N+1 \leq j \leq N(r)} |a_j| r^j$$

$$+ \sum_{j=1}^{\infty} \frac{1}{2^j} \leq b r^N + (N(r)-N) \max_{j \geq N} \left(\left(\frac{e\mu\lambda}{j}\right)^{j/\mu} r^j\right) + 1$$

$$\leq 1 + b r^N + \max\{0, 2^\mu e\mu\lambda r^\mu - N) e^{\lambda r^\mu} \leq K \cdot e^{(\lambda+\varepsilon) r^\mu}, \quad r \gg$$

Theorem.- Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \in \mathcal{H}(\mathbb{C})$.

Assume that $\rho(f)\in(0,\infty)$.

then $\tau = \frac{1}{e\rho} \limsup_{j\to\infty} \left(j|a_j|^{\rho/j}\right)$.

Pf.- Define $\nu = \limsup_{j\to\infty}\left(j|a_j|^{\rho/j}\right)$, so that we have to show $\tau=\frac{\nu}{e\rho}$.

$\boxed{\bullet \le}$. Trivial if $\nu=\infty$. Assume then that $0\le\nu<+\infty$. Take an arbitrary $k>\frac{\nu}{e\rho}$. By the definition of ν, for sufficiently large j,

$$j|a_j|^{\rho/j} < e\rho k. \equiv |a_j| < \left(\frac{e\rho k}{j}\right)^{j/\rho}.$$

By the previous lemma, $\forall\ \varepsilon>0$, $\exists\ R=R(\varepsilon)>0$:

$$M(r,f)\le e^{(k+\varepsilon)r^\rho}, \quad r>R(\varepsilon)$$

$\Rightarrow$ $\tau\le k+\varepsilon$ $\forall \varepsilon>0$. $\Rightarrow$ $\tau\le k$ $\forall\ k>\frac{\nu}{e\rho}$

$\Rightarrow$ $\tau\le\frac{\nu}{e\rho}$.

$\boxed{\bullet \ge}$ Note that if $\nu=0$, then $\tau=0$

For $0 < \nu \leq +\infty$, take β : $0 < \beta < \nu$.

Again, by the definition of ν,

$$j\,|a_j|^{\rho/j} \geq \beta, \quad j >>.$$

$$\Rightarrow \quad |a_j| \geq \left(\frac{\beta}{j}\right)^{j/\rho}.$$

Define (for these j's), $(r_j)^\rho = \frac{je}{\beta}$ $\left(\xrightarrow[j\to\infty]{}\infty\right)$

Using the Cauchy estimates

$$|a_j| \leq \frac{M(r,f)}{r^j}, \quad \text{we have}$$

$$M(r_j,f) \geq |a_j|\, r_j^{\,j} \geq \left(\frac{\beta}{j}\right)^{j/\rho} \cdot \left(\frac{je}{\beta}\right)^{j/\rho} = e^{j/\rho}$$

$$= e^{\frac{1}{\rho}\frac{\beta}{e}(r_j)^\rho}$$

this gives:

$$\tau \geq \limsup_{j\to\infty} \frac{\log M(r_j,f)}{r_j^{\,\rho}} \geq \frac{\beta}{\rho e}.$$

$$\Rightarrow \tau \geq \frac{\nu}{\rho e} \qquad \square.$$

Remarks

Please, remember what we have learned:

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{\log r} = \text{order. } (\geq 0).$$

$$\mu(f) = \underline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{\log r} = \text{lower order.}$$

- $p \in \mathcal{P}_n \Rightarrow \rho(f) = 0$
- $\rho(f_1 + f_2) \leq \max\{\rho(f_1), \rho(f_2)\}$ $\left(= \text{ if } \rho(f_1) < \rho(f_2) \right)$
- $\rho(f_1 \cdot f_2) \leq$ —— " ——
- $\rho(f) = \rho < \infty \Rightarrow |f(z)| \leq M(r,f) \leq e^{r^{\rho+\varepsilon}}, \ |z| \leq r.$
- $\rho(f) = \inf\{\lambda > 0 / M(r,f) \leq e^{r^\lambda} \text{ for } r >>\}.$
- $f \in \mathcal{H}(\mathbb{C})$ with <u>no</u> ceros and $\mu(f) < \infty$ $\Rightarrow f = e^P$, P polynomial of degree $m = \mu(f) = \rho(f)$.

Consequence of Hadamard's thm.

$$\liminf_{r\to\infty} \frac{A(r,f)}{r^s} < \infty \Rightarrow f \text{ is a polynomial of degree} \leq s.$$

$$A(r,f) = \max_{|z|=r} [\operatorname{Re} f(z)]$$

- $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $z\in\mathbb{C}$ & $b_j=\begin{cases} 0, & a_j=0 \\ \dfrac{j\log j}{\log\frac{1}{|a_j|}}, & a_j\neq 0\end{cases}$

$\Rightarrow \rho(f)=\limsup_{j\to\infty} b_j$.

- $\tau=\tau(f)=\limsup_{r\to\infty}\dfrac{\log M(r,f)}{r^\rho}$, τ = type

(defined for entire functions with order ρ, $0<\rho<\infty$).

- $\tau=\inf\{k>0 : M(r,f)\le e^{kr^\rho}, r>>\}$.
- $\tau=\dfrac{1}{e\rho}\overline{\lim_{j\to\infty}}\left(j|a_j|^{\rho/j}\right)$.

And the examples!!!

(1) $p\in\mathcal{P}_n \rightarrow \rho(f)=0$.

(2) $f(z)=e^z \rightarrow \rho(f)=1$.

(3) $f(z)=\sum_{n=0}^{\infty} e^{-n^2} z^n \rightarrow \rho(f)=0$.

(4) $f(z)=e^{e^z} \longrightarrow \rho(f)=\infty$.

(5) $f(z)=\cos(\sqrt{z}) \rightarrow \rho(f)=\frac{1}{2}$.

Type of :

- $f_1(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j.$

$$\overline{\lim_{j\to\infty}}\; j \cdot \left(\frac{1}{j!}\right)^{\frac{1}{j}} = \overline{\lim_{j\to\infty}}\; j \left(\frac{1}{\sqrt{2\pi j}\, e^{-j} j^j}\right)^{1/j}$$

$$= \overline{\lim_{j\to\infty}}\; j \cdot \frac{1}{(2\pi j)^{\frac{1}{2j}} e^{-1} j} = e.$$

$\therefore \ \tau(f_1) = 1.$

- $f_2(z) = e^{az} = \sum_{j=0}^{\infty} \frac{a^j}{j!} z^j \, ; \quad a \neq 0.$

$$\cancel{\tau(f_2)} \ \overline{\lim_{j\to\infty}}\; j \cdot \left(\frac{1}{j!}\right)^{1/j} \cdot \left(a^j\right)^{1/j} = ae$$

$\Big[\ |f_2(z)| \leq e^{\operatorname{Re} az} \leq e^{|a| r} \implies \rho \leq 1.$

$\ \ \&\ \ \rho \geq 1$ since $|M(r, f_2)| = e^{|a| r} \Big]$

Hence $\tau(f_2) = a$.

- $\sum_{n=2}^{\infty} \frac{z^n}{(n \log n)^n}$ has order 1 and type 0.

- $z \prod_{n=1}^{\infty} \frac{1 + \frac{z}{n}}{\left(1 + \frac{1}{n}\right)^z} \left(= \frac{1}{\Gamma(z)} \right)$ has order 1 and type ∞.