Ch. 7: Phragmén-Lindelöf theorems.

Let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, where Ω is a bounded domain in $\mathbb{C}$.

If $|f(z)| \leq M$ on $\partial\Omega$ then $|f(z)| \leq M$ in $\overline{\Omega}$.

What if Ω is not bounded?

Example.

$\Omega = \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

$f(z) = e^z$. ($f \in \mathcal{H}(\mathbb{C})$, hence $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$).

$\partial\Omega = \{z \in \mathbb{C} : \operatorname{Re} z = 0\}$.

If $z \in \partial\Omega$, $|e^z| = e^{\operatorname{Re} z} = 1$.

BUT $\lim\limits_{\substack{r \to \infty \\ r \in \mathbb{R}}} e^z = +\infty$!!!!!

Theorem.- (Phragmen-Lindelöf theorem for the half-plane).

Let $f \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\mathbb{H})$. Assume that $|f(z)| \leq M$ on $\partial\mathbb{H}$ and $|f(z)| \leq C e^{|z|^\alpha}$ for some $\alpha < 1$, $C \in \mathbb{R}$.

Then $|f(z)| \leq M$ $\forall z \in \mathbb{H}$.

Remarks.

(1) W.L.O.G, $M = 1$ (take $\frac{f}{M}$!).

(2) $|f(z)| \le Ce^{|z|^\alpha} \Rightarrow M(r,f) \le Ce^{|z|^\alpha}$

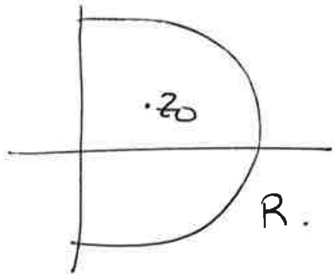
$$\Rightarrow \frac{\log M(r,f)}{r} = \frac{\log C + r^\alpha}{r} \xrightarrow[r=|z|\to\infty]{} 0.$$

(3) Obviously, the interesting case is $\alpha \in [0,1)$.

Proof. Fix $z_0 \in \mathbb{H}$. Choose $\beta \in (\alpha, 1)$ and, for $\varepsilon > 0$, define $g_\varepsilon(z) = e^{-\varepsilon z^\beta}$.

Consider the semicircle

$$S = \{z : \operatorname{Re} z > 0, |z| \le R\}.$$



If $z \in S$, $z = re^{i\theta}$

$$|g_\varepsilon(z)| = e^{-\varepsilon \operatorname{Re}\{z^\beta\}} = e^{-\varepsilon r^\beta \cos(\beta\theta)}. \quad (**)$$

Moreover, $\forall z \in \mathbb{H}$,

$$C \cdot e^{|z|^\alpha} |g_\varepsilon(z)| = C \cdot e^{r^\alpha - \varepsilon r^\beta \cos(\beta\theta)} \xrightarrow[r\to\infty]{} 0,$$

since $\cos\beta\theta \ge \cos\beta\frac{\pi}{2} > 0$, $\beta > \alpha$.

Consider $h_\varepsilon(z) = f(z) g_\varepsilon(z)$.

For large R, $z_0 \in S$.

Also, on ∂S, $|h_\varepsilon(z)| \le M^*$. Hence

$*$ follows from "↙" and also from $(**)$ which gives

$$|g_\varepsilon(i\mathbb{R})| = e^{-\varepsilon r^\beta \underbrace{\cos(\beta\frac{\pi}{2})}_{>0!!!}} \le 1 \,!$$

$|h_\varepsilon(z)| \le M$ on ∂S bounded

$$\Longrightarrow \quad \underset{\substack{\uparrow \\ \text{independent of } \varepsilon}}{|f(z_0)|} \le \frac{M}{|g_\varepsilon(z_0)|} \xrightarrow[\varepsilon \to 0]{} M \qquad \square .$$

<u>Corollary</u>.- the same conclusion holds if we assume that $f \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\mathbb{H})$, $|f(z)| \le M$ on $\partial\mathbb{H}$ and $|f(z)| \le C e^{\delta |z|^\alpha} = C(\delta) e^{\delta|z|^\alpha}$ $\forall \delta > 0$.
$(\alpha < 1)$!

<u>Pf.</u> Given $\varepsilon > 0$, define $F(z) = e^{-\varepsilon z^\alpha} f(z)$.

$$|F(z)| \le C(\delta) \cdot e^{\delta r^\alpha} e^{-\varepsilon r^\alpha \cos(\alpha\theta)}$$

$$z = re^{i\theta} \quad = C(\delta)\, e^{r^\alpha(\delta - \varepsilon\cos(\alpha\theta))}$$

$$\le C(\delta)\, e^{(\delta+\varepsilon) r^\alpha} \le K(\delta)\, e^{r^\beta},$$

where $\beta \in (\alpha, 1)$, $|z| = r \to \infty$ & argue as in the proof of the previous theorem $\square$

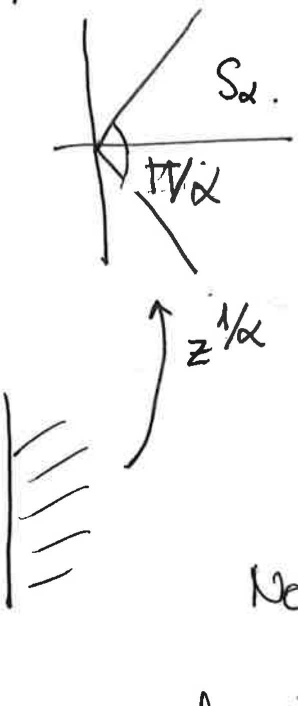
<u>Theorem</u>.- (Phragmen-Lindelöf theorem in a sector). Suppose f is analytic inside a sectorial domain centered at the origin of opening $\frac{\pi}{\alpha}$, $\alpha > 1$. Assume that f is continuous on the closure of the sectorial domain.

If $|f(z)| \leq M$ on the boundary of the domain and

$$|f(z)| \leq C e^{|z|^\beta}$$

inside the domain for some $\beta < \alpha$, then $|f(z)| \leq M$ inside the domain.

<u>Proof</u>:- By applying a rotation if needed, we may assume that the sector is contained in $\mathbb{H}$ and is symmetric with respect to the real axes.



Now, $0 \notin \mathbb{H}$, hence, we can define analyticaly the branch of $z^{1/\alpha}$ on $\mathbb{H}$ that maps $\mathbb{H}$ onto S_α and define

$$F(z) = f(z^{1/\alpha}) \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\overline{\mathbb{H}}).$$

Now: $|F(z)| \leq M$ on $\partial\mathbb{H}$.

and $|F(z)| = |f(z^{1/\alpha})| \leq C e^{|z|^{(\beta/\alpha)}}$ $\quad (\beta/\alpha < 1)$

$\Rightarrow |F(z)| \leq M$ on $\mathbb{H}$. Hence

$$|f(z_0)| = |F(z_0^\alpha)| \leq M \quad \forall z_0 \in S_\alpha \qquad \square.$$

theorem – (Phragmen-Lindelöf theorem in a strip).

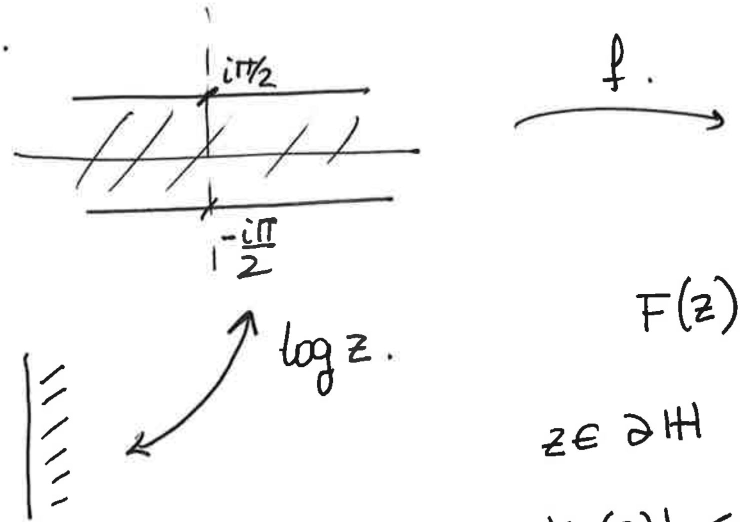
$S = \{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{2} \}$.

→ $f \in \mathcal{H}(S) \cap \mathcal{C}(\bar{S})$. → $|f(z)| \le e^{A e^{c|\operatorname{Re} z|}}$,

$c < 1$, $A \in \mathbb{R}$. → $|f(z)| \le M \quad \forall z \in \partial S$.

then $|f(z)| \le M \quad \forall z \in \bar{S}$.

Pf.



$F(z) = f(\log z) \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\mathbb{H})$.

$z \in \partial \mathbb{H} \rightarrow \log z \in \partial S$.

$\Rightarrow |F(z)| \le M \quad \forall z \in \partial \mathbb{H}$.

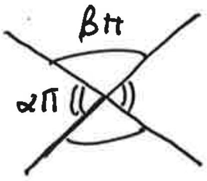
For $z \in \mathbb{H}$,

$$|F(z)| \le e^{A e^{c \log|z|}} = e^{A|z|^c}$$ ▨

<u>Theorem</u>:- Suppose that $f(z) \to a$ as $z \to \infty$ along two half-lines starting from the origin and assume that f is analytic and bounded in one S of the sectors between these two half-lines. and continuous in $\overline{S}$. Then,

$f(z) \rightrightarrows a$ as $r \to \infty$ in S.

<u>Pf</u>:- Considering $f(z) - a$ if needed, we can assume $a = 0$.



$(\beta + \alpha)\pi = 2\pi$, so that either α or $\beta < 1$.

If $f \in \mathcal{H}(S_\alpha)$, we just rotate f to consider a function analytic in a sector contained in $\mathbb{H}$.

and define $F(z) = f(z^{1/\alpha})$ to produce a function $F \in \mathcal{H}(\mathbb{H})$.

If $f \in \mathcal{H}(S_\beta)$, we just do the same:

$$F(z) = f(z^{1/\beta}) \in \mathcal{H}(\mathbb{H}).$$

Properties of F:

→ $F \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\overline{\mathbb{H}})$.

→ F is bounded on $\overline{\mathbb{H}}$ ($|F(z)| \leq M$, say, $\forall z \in \overline{\mathbb{H}}$).

→ $\lim_{x \to \pm\infty} F(ix) = 0$. *

We want to show that $\forall \varepsilon > 0, \exists R: |z| > R, z \in \mathbb{H}$
$\Rightarrow |F(z)| < \varepsilon$.

Take $\varepsilon > 0$. then, from ⊛, $\exists r_1 : \forall r > r_1$,

$$|F(z)| < \varepsilon \quad , \; z = ir.$$

Define $\lambda = \dfrac{r_1 M}{\varepsilon}$, $g(z) = \dfrac{z}{z+\lambda} F(z) \in \mathcal{H}(\mathbb{H}) \cap \mathcal{C}(\overline{\mathbb{H}})$.

then:

(1) $r = |z| \leq r_1$,

$$|g(z)| = \frac{r}{\sqrt{r^2+\lambda^2+2\lambda \operatorname{Re} z}} |F(z)| \leq \frac{rM}{\sqrt{\lambda^2+r^2}} \leq \frac{rM}{\lambda}$$

$$\leq \frac{r_1 M}{\lambda} = \varepsilon.$$

(2) $z = \pm ir$, $r > r_1$,

$$|g(z)| \leq \frac{r}{\sqrt{\lambda^2+r^2}} |F(z)| \leq |F(z)| < \varepsilon.$$

$\therefore \; |g(ir)| < \varepsilon \quad \forall \; r \in \mathbb{R}$.

Moreover, we also have for $z \in \mathbb{H}$

→ $|g(z)| \leq \varepsilon$, $|z| \leq r_1$.

→ $|g(z)| \leq M$, $|z| \geq r_1$.

that is $|g(z)| \leq m \leq me^{|z|^\alpha}$ for $\alpha \hat{<} 1$.

So that we can apply the Phragmen-Lindelöf theorem for the half-plane to get that $|g(z)| \leq \varepsilon \quad \forall z \in \overline{\mathbb{H}}$.

And finally, take $r > \lambda$ to obtain

$$|F(z)| = \left|1 + \frac{\lambda}{z}\right| |g(z)| \leq \left(1 + \frac{\lambda}{r}\right) |g(z)| \leq 2\varepsilon$$

$\forall |z| \geq r$. $\square$

<u>Theorem</u>.- Suppose $f(z) \to a$ along a half-line starting from the origin and $f(z) \to b$ along a second half-line, again starting from the origin. Suppose that f is analytic and bounded in one of the two sectors between these half-lines (and continuous in the closure). Then, $a = b$ and $f(z) \to a$ uniformly in that sector as $r \to \infty$.

Pf.- We just need to show that $a=b$.

As before, we can assume $a=0$ and that the sector is $\mathbb{H}$.

Suppose $\lim_{x\to+\infty} f(ix) = 0$, $\lim_{x\to-\infty} f(ix) = b \neq 0$.

[Take $\overline{f(\overline{z})}$ instead of f if needed]

~~Moreover, we can assume $b>0$ [Take $-f$, if needed]~~

↑ No need to do this!

Define $$g(z) = \left(f(z) - \frac{b}{2}\right)^2.$$

Notice that

$$\lim_{x\to+\infty} g(ix) = \frac{b^2}{4} = \lim_{x\to-\infty} g(ix).$$

and g satisfies the hypotheses in the previous theorem (with $S=\mathbb{H}$). Then, $g(z) \xrightarrow[|z|\to\infty]{} \frac{b^2}{4}$ uniformly in $\mathbb{H}$. Therefore, for $|z|\to\infty$,

$$g(z) - \frac{b^2}{4} = \left(f(z) - \frac{b}{2}\right)^2 - \frac{b^2}{4} = f(z)\left(f(z) - b\right) \to 0.$$

Take now a circular arc C centered at $z=0$ such that

$$|f(z)|\,|f(z)-b| \le \varepsilon \quad \forall z \in C.$$

Hence, at every point on C, either $|f(z)| \le \sqrt{\varepsilon}$ or $|f(z)-b| \le \sqrt{\varepsilon}$

If one of these inequalities hold $\forall z \in C$, for instance $|f(z) - b| \leq \sqrt{\varepsilon}$ $\forall z \in C$, and that the circular arc has a radius R large enough, we would have

$$|b| \leq |f(z) - b| + |f(z)| \leq \sqrt{\varepsilon} + |f(z)| \, , z \in C.$$

in particular, if $z = iR$, we get $|b| \leq 2\sqrt{\varepsilon}$, which is not true for $2\sqrt{\varepsilon} < |b|$.

$|f(z)| < \sqrt{\varepsilon}$ $\forall z \in C$

$$\Rightarrow |b| \leq |f(z) - b| + |f(z)| \leq |f(z) - b| + \sqrt{\varepsilon}$$

$\leadsto$ $z = -iR$. $\leq 2\sqrt{\varepsilon}$

If this is not the case, denote by $\Gamma_0 = \{z \in C : |f(z)| \leq \sqrt{\varepsilon}\}$, $\Gamma_b = \{z \in C : |f(z) - b| \leq \sqrt{\varepsilon}\}$.

Note that $iR \in \Gamma_0$ & $-iR \in \Gamma_b \Rightarrow \Gamma_0 \neq \emptyset$, $\Gamma_b \neq \emptyset$.

Also, Γ_0, Γ_b are closed & $\Gamma_0 \cup \Gamma_b = C$.

If $\Gamma_0 \cap \Gamma_1 = \emptyset$ then $C = \Gamma_0 \mathring{\cup} \Gamma_b$. But C is compact and connected $\Rightarrow$ $\begin{cases} \Gamma_0 = \emptyset \\ \Gamma_1 = \emptyset \end{cases}$ and we reduce to the previous case. Hence, $\exists z_0 \in \Gamma_0 \cap \Gamma_b$ and we have $|b| \leq |f(z_0)| + |f(z_0) - b| \leq 2\sqrt{\varepsilon}$. Let $\varepsilon \to 0$ to get $b = 0$, q.e.d.

$\square$.

Remark.-

the paper that gave name to this chapter is:

E. Phragmén and E. Lindelöf, Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier, Acta Math. 31 (1908), 381-406.