Ch. 8 : Zeros of entire functions.

Let $f \in \mathcal{H}(\mathbb{C})$ and consider a disk $|z| \leq r$ centered at $z=0$. If r is large enough and f is a polynomial of degree n, then $f(z)=\alpha$ has $\underline{n}$ roots in $|z| \leq r$. Moreover, $M(r,f) \sim r^{\underline{n}}$ on the boundary of the disk.

this connection between the number of α-points $(z : f(z)=\alpha)$ and the maximum modulus carries over to trascendental entire functions (i.e., entire functions which are not polynomials).

<u>DEF</u>.- Let $\{r_j\}_{j\geq 1}$ be a sequence of real numbers such that $0<r_1 \leq r_2 \leq \cdots$ the <u>convergence exponent for $\{r_j\}$</u> is $\lambda = \inf\left\{ \alpha>0 : \sum_{j=1}^{\infty} (r_j)^{-\alpha} \text{ converges} \right\}$.

If $\sum_{j=1}^{\infty} (r_j)^{-\alpha}$ diverges for all $\alpha>0$, $\lambda=+\infty$ (as the infimum of an empty set).

<u>DEF</u>.- Let f be entire and let $\{z_j\}_{j\geq 1}$ be the zero-sequence of f, deleting the possible zero at the origin, every zero $\neq 0$ repeated according to its multiplicity, and arranged according to increasing moduli, i.e., $0<|z_1| \leq |z_2| \leq \cdots$

the convergence exponent $\lambda(f)$ (for the zero-sequence of f) is

$$\lambda(f) = \inf \{ \alpha > 0 : \sum_{j=1}^{\infty} |z_j|^{-\alpha} \text{ converges} \}.$$

DEF. – $n(t) = n(t, \frac{1}{f}) = \#$ ceros of f in $|z| \leq t$, each zero counted according to its multiplicity.

Remark. – We will assume $f(0) \neq 0$. Notice that this is not an essential restriction, since we may always replace $n(t)$ by $n(t) - n(0)$ below if $f(0) = 0$.

Lemma. $\sum_{j=1}^{\infty} |z_j|^{-\alpha}$ converges $\Leftrightarrow \int_0^{\infty} n(t) t^{-(\alpha+1)} dt$ converges.

Proof. Note that $n(t)$ is a step function: zeros of f are situated on countable many circles centered at $z=0$. Between these radii, $n(t)$ is constant and, hence, $dn(t) = 0$ for these intervals. Passing over these radii, $dn(t)$ jumps by an integer equal to the number of zeros on the circle. Then,

$$\sum_{j=1}^{N} |z_j|^{-\alpha} = \int_0^T \frac{dn(t)}{t^\alpha}, \quad T = |z_N|.$$

Now,
$$\int_0^T \frac{dn(t)}{t^\alpha} = \frac{n(t)}{t^\alpha}\Big|_0^T + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt$$

$$= \frac{n(T)}{T^\alpha} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt.$$

then,
$$\alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt \underset{\uparrow}{\leq} \int_0^T \frac{dn(t)}{t^\alpha}$$

$$\frac{n(T)}{T^\alpha} \geq 0$$

$$= \sum_{j=1}^{N} |z_j|^{-\alpha} \leq \sum_{j=1}^{\infty} |z_j|^{-\alpha} < \infty.$$

$$\Rightarrow \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \quad \text{converges.}$$

Conversely, suppose that $K := \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt < \infty$.

$$\frac{n(T)}{T^\alpha}(1-2^{-\alpha})\cdot\frac{1}{\alpha} \underset{\uparrow}{=} n(T)\int_T^{2T} \frac{dt}{t^{\alpha+1}} \underset{\uparrow}{\leq} \int_T^{2T} \frac{n(t)\,dt}{t^{\alpha+1}}$$

just make the $\int$! $\qquad$ $n(t) \geq n(T)\ \forall\, t \geq T$!

$$\leq \int_0^\infty \frac{n(t)\,dt}{t^{\alpha+1}} = K.$$

therefore,

$$\sum_{j=1}^{N} |z_j|^{-\alpha} = \frac{n(T)}{T^\alpha} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt \leq \frac{K\alpha}{1-2^{-\alpha}} + \alpha K.$$

$$\Rightarrow \sum_{j=1}^{\infty} |z_j|^{-\alpha} \text{ converges.}$$

<u>Corollary</u>. $f \in \mathcal{H}(\mathbb{C}),\ f(0) \neq 0.$

$$\lambda(f) = \inf \left\{ \alpha > 0 : \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges} \right\}.$$

<u>Theorem</u>. $\lambda(f) = \limsup_{r\to\infty} \frac{\log n(r)}{\log r}$

<u>Pf</u>: Denote $\sigma = \limsup_{r\to\infty} \frac{\log n(r)}{\log r}$.

- $\lambda(f) \leq \sigma$ (for $\sigma < \infty \ldots$)

the definition of σ show that given $\varepsilon > 0$, $\exists\ r = r_\varepsilon$:

$$n(r) \leq r^{\sigma+\varepsilon} \quad \forall r \geq r_\varepsilon$$

then,

$$\int_0^M \frac{n(t)}{t^{\alpha+1}}\,dt = \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^{M} \frac{n(t)}{t^{\alpha+1}}\,dt$$

$$\leq \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^{M} t^{\sigma+\varepsilon-\alpha-1}\,dt.$$

$\int_A^\infty t^\gamma\,dt$ converges if $\gamma < -1$.

But $\sigma+\varepsilon-\alpha-1 < -1 \iff \alpha > \sigma+\varepsilon$.

Hence, $\inf\left\{\alpha>0 : \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges}\right\} \leq \sigma+\varepsilon$.

$\Longrightarrow \lambda(f) \leq \sigma+\varepsilon \quad \forall\, \varepsilon>0$, so that $\lambda(f) \leq \sigma$.

- $\sigma \leq \lambda(f)$. $(\sigma>0!)$

Take $\varepsilon>0$ such that $\varepsilon<\sigma$. then, $\exists$ a sequence $r_j \to \infty$:

$$\frac{\log n(r_j)}{\log r_j} \geq \sigma - \varepsilon.$$

$$\equiv \quad n(r_j) \geq r_j^{\sigma-\varepsilon}.$$

Take now $\alpha > 0$: $0 < \alpha < \sigma - \varepsilon$ and, for each j, select $s_j \geq 2^{\frac{1}{\alpha}} r_j$.

We get: $\boxed{n(r) \text{ is increasing!}}$

$$\int_{r_j}^{s_j} \frac{n(t)dt}{t^{\alpha+1}} \geq n(r_j) \int_{r_j}^{s_j} \frac{dt}{t^{\alpha+1}} \geq r_j^{\sigma-\varepsilon} \cdot \frac{1}{\alpha} \left(\frac{1}{r_j^{\alpha}} - \frac{1}{s_j^{\alpha}} \right)$$

$$\geq \frac{1}{\alpha} r_j^{\sigma-\varepsilon} \left(\frac{1}{r_j^{\alpha}} - \frac{1}{2 r_j^{\alpha}} \right) = \frac{1}{2\alpha} r_j^{\sigma-\varepsilon-\alpha} \xrightarrow{(\sigma-\varepsilon>\alpha)} \infty .$$

$$\therefore \int_{r_j}^{s_j} \frac{n(t)}{t^{\alpha+1}} dt \xrightarrow[j \to \infty]{} \infty .$$

$\Rightarrow \displaystyle\int_0^{\infty} \frac{n(t)}{t^{\alpha+1}} dt$ diverges $\forall \alpha$, $0 < \alpha < \sigma - \varepsilon$.

$\Rightarrow \inf \left\{ \alpha > 0 : \displaystyle\int_0^{\infty} \frac{n(t)}{t^{\alpha+1}} dt \text{ converges} \right\} \geq \sigma - \varepsilon$

$\Rightarrow \lambda(f) \geq \sigma$ $\square$.

Theorem (Jensen).

Let $f \in \mathcal{H}(\mathbb{C})$ and $f(0) \neq 0$.

Assume that f has no ceros on $|z| = r > 0$.

then $N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| \, d\varphi - \log |f(0)|$,

$N(r) = \int_0^r \frac{n(t)}{t} dt$

Pf: Let $a_1, a_2, \dots, a_n$ be the zeros of f in $|z| \leqslant r$. Consider:

$$g(z) = f(z) \prod_{j=1}^{n} \frac{r^2 - \overline{a_j} z}{r(z - a_j)}.$$

Note that if $|z| \leq \rho < r$, $g(z) \neq 0$ (ceros of f ere cancel out by the factors in the numerator).

If $|z| = r$, we have

$$\left| \frac{r^2 - \overline{a_j} re^{i\theta}}{r(re^{i\theta} - a_j)} \right| = \left| \frac{r - \overline{a_j} e^{i\theta}}{re^{i\theta} - a_j} \right| = \left| \frac{r - \overline{a_j} e^{i\theta}}{r - a_j e^{-i\theta}} \right| = \left| \frac{w}{\overline{w}} \right| = 1$$

[$w \neq 0$ since $|a_j| \neq r$].

Hence, $g(z) \neq 0$ $\forall |z| \leq r$. Hence, for $|z| < R$ for some $R > r$.

Now, g is analytic in $|z| \le R$ and different from 0. therefore, we can define a branch of $\operatorname{Log} g(z)$ in such a way that this function is analytic. therefore, $\operatorname{Log}|g(z)| = \operatorname{Re}\{\operatorname{Log} g(z)\}$ is harmonic and

$$\operatorname{Log}|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\, d\theta \quad \lceil r<R! \rfloor$$

$$\equiv \operatorname{Log}|f(0)| \cdot \prod_{j=1}^{n} \frac{r}{|a_j|} = \log|f(0)| + \sum_{j=1}^{n} \log\frac{r}{|a_j|}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{r}{|a_j|}.$$

$$= \log \prod_{j=1}^{n} \frac{r}{|a_j|} = \log\frac{r^n}{|a_1| \cdots |a_n|} = n\log r - \sum_{j=1}^{n} \log|a_j|$$

$$= \sum_{j=1}^{n-1} j\left(\log|a_{j+1}| - \log|a_j|\right) + n\left(\log r - \log|a_n|\right)$$

$$= \sum_{j=1}^{n-1} j\int_{|a_j|}^{|a_{j+1}|} \frac{dt}{t} + n\int_{|a_n|}^{r} \frac{dt}{t} \overset{(*)}{=} \int_0^r \frac{n(t)}{t}\, dt.$$

(*) Note that $j = n(t)$ for $|a_j| \le t < |a_{j+1}|$ and $n = n(t)$ for $|a_n| \le t < r$!!! □.

Theorem.– Let $f \in \mathcal{H}(\mathbb{C})$, $\rho(f) = \rho$. Then, for each $\varepsilon > 0$, $n(r) = O(r^{\rho+\varepsilon})$ as $r \to \infty$.

$$\equiv \lim_{r\to\infty} \frac{n(r)}{r^{\rho+\varepsilon}} \le C.$$ Also, $f(0) \neq 0$!!! .

Pf.– Change $f(z)$ by $Cf(z)$, $C \neq 0$, so that $|f(0)| > 1$

then

$$N(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \log M(r,f)\,d\theta$$

$$= \log M(r,f) \le r^{\rho+\varepsilon} \text{ for } r>>.$$

But $n(t)$ is non-decreasing,

$$n(r)\log 2 = n(r)\int_r^{2r}\frac{dt}{t} \le \int_r^{2r}\frac{n(t)\,dt}{t} \le \int_0^{2r}\frac{n(t)}{t}\,dt$$

$$= N(2r) \le \log M(2r,f) \le (2r)^{\rho+\varepsilon}$$

$$\Rightarrow n(r) \le \left(\frac{1}{\log 2}\, 2^{\rho+\varepsilon}\right) r^{\rho+\varepsilon}, \ r>> \qquad \square$$

<u>Theorem</u>.— $f \in \mathcal{H}(\mathbb{C})$, $\lambda(f) \leq \rho(f)$.

<u>Pf</u>.— By the previous theorem,

$$n(r) \leq K r^{\rho+\varepsilon}, \quad \rho = \rho(f), \; r \geq r_0.$$

then

$$\int_0^M \frac{n(t)}{t^{\alpha+1}}\,dt = \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\,dt + \int_{r_0}^M \frac{n(t)}{t^{\alpha+1}}\,dt$$

$$\leq \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\,dt + K \int_{r_0}^M r^{\rho+\varepsilon-\alpha-1}\,dt$$

If $\alpha > \rho + \varepsilon$, $\rho + \varepsilon - \alpha - 1 < -1$, the last integral converges, hence

$$\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges.}$$

$\Rightarrow \lambda(f) \leq \rho + \varepsilon$, hence $\lambda(f) \leq \rho$. $\square$.