Ch. 9 : The Cartan lemma.

Let $z_1, \dots, z_n$ be given (not necessarily distinct) points in $\mathbb{C}$ and $H > 0$.

Then, there exist closed disks $\Delta_1, \dots, \Delta_m$, $m \le n$ with radii r_j, $j = 1, \dots, m$: $\sum_{j=1}^{m} r_j \le 2H$ and

$$\prod_{j=1}^{n} |z - z_j| > \left(\frac{H}{e}\right)^n \quad \forall\, z \notin \bigcup_{j=1}^{m} \Delta_j .$$

$\equiv$ Let $P(z) = \prod_{i=1}^{n} (z - z_i)$. For every $H > 0$, the inequality $|P(z)| > \left(\frac{H}{e}\right)^n$ holds outside at most n circles, the sum of whose radii is at most $2H$.

Proof:- Note that we can assume, without loss of generality that all the z_i's are distinct: modiffy $z_i \longmapsto z_i + \varepsilon_i$ and take $H + a$ instead of H.

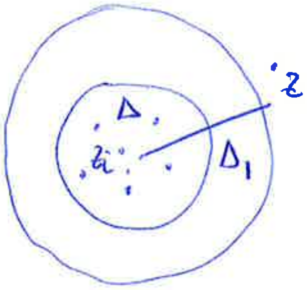
$$\Rightarrow \quad |\tilde{P}(z)| > \left(\frac{H+a}{e}\right)^n$$

$$\Rightarrow \quad |P(z)| \ge \left(\frac{H+a}{e}\right)^n > \left(\frac{H}{e}\right)^n$$

Though we won't need to do so!

<u>CASE 1.</u> $\exists$ a disk Δ of radius H that contains z_i, $i=1,\dots,n$. And let Δ_1 denote the disk of radius $2H$ and same center as Δ.

Then, $\forall z \notin \Delta_1$, $|z-z_i| > H$, $i=1,\dots,n$.



Hence,

$$\prod_{i=1}^{n} |z-z_j| > H^n > \left(\frac{H}{e}\right)^n .$$

<u>CASE 2.</u> No disk of radius H contains z_i, $i=1,\dots,n$.

For $1 \le k < n$, consider the radii $r_k = \frac{kH}{n}$.

and the property

$P_k = \{$ there's a closed disk of radius r_k such that at least k points of $\{z_i\}$ are contained in this disk $\}$.

Obviously, $k=1$ satisfies P_1, so that it makes sense to define k_1 as the greatest natural number such that P_k holds (and $1 \le k_1 < n$).

Hence, there is a closed disk Δ_1' of radius $\frac{k_1 H}{n}$ which contains exactly k_1 points z_i.
(By the def. of k_1!!)

Renumbering now, if needed, say that $z_1, \dots, z_{k_1} \in \Delta_1'$ while $z_{k_1+1}, \dots, z_n \notin \Delta_1'$.

We repeate the process now but we only consider $z_{k_1+1}, \dots, z_n$: Let k_2 be the greatest natural number such that there is a closed disk Δ_2' of radius $\frac{k_2 H}{n}$, k_2 points of $z_{k+1}, \dots, z_n$ are contained in Δ_2'.

Obviously, $k_2 \leq k_1$ (otherwise; $k_2 > k_1$ implies $k_1 = k_2$!)

After aplying the same approach m times $(m \leq n)$, we have all points $z_1, \dots, z_n$ contained in $\bigcup_{j=1}^{m} \Delta_j'$, Δ_j' = closed disks of radius $\frac{k_j H}{n}$ and $k_1 \geq k_2 \geq \dots \geq k_m$.

Moreover, since Δ_j' contains k_j points,

$$k_1 + k_2 + \dots + k_m = n.$$

and

$$\frac{k_1}{n} H + \frac{k_2}{n} H + \dots + \frac{k_m}{n} H = H.$$

Expand the disks Δ_j', $j = 1, \dots, m$, concentrically to Δ_j, of radius $\frac{2k_j}{n}$.

Obviously, the sum of the radius of Δ_j is $= 2H$.

Let now $z \notin \bigcup_{j=1}^{m} \Delta_j$.

We can assume (by renumbering the points $z_1, \dots, z_n$ again if needed):

$$|z - z_1| \le |z - z_2| \le \dots \le |z - z_n|.$$

Assume we can prove $|z - z_i| > \frac{i}{n} H, \; i = 1, \dots, n$, we obtain

$$\prod_{i=1}^{n} |z - z_i| > \prod_{i=1}^{n} \frac{i}{n} H = \frac{n!}{n^n} H^n \geq e^{-n} H^n = \left(\frac{H}{e}\right)^n$$

$$e^n = \sum_{i=0}^{\infty} \frac{1}{i!} n^i \geq \frac{n^n}{n!}$$

Hence, it remains to show

$$|z - z_i| > \frac{i}{n} H, \quad i = 1, \dots, n.$$

Suppose, to get a contradiction, that there is j : $|z - z_j| \le \frac{j}{n} H, \; j = 1, \dots, n$.

Recall that $k_1 \geq k_2 \geq \dots \geq k_m$ and note that the disk, centered at z and with radius $\frac{jH}{n}$ contains at least the points $z_1, z_2, \dots, z_j$. So that, by the definition of

K_1, $K_1 \geq j$.

Define p as the greatest natural number such that $K_p \geq j$* and consider the pairs of natural numbers (s,q) with $s \leq j$, $q \leq p$.

Claim. For these pairs (s,q), $z_s \notin \Delta_q'$.

To prove the claim, suppose that $z_s \in \Delta_q'$ for some $s \leq j$, $q \leq p$.

then, $K_q \geq j$ (by the definition of p) the radius of $\Delta_q' = \frac{K_q}{n} H$ and Δ_q' contains K_q points of $z_1, \dots, z_n$.

Let ζ be the center of Δ_q'. then,

$$|z-\zeta| \leq |z-z_s| + |\zeta - z_s| \underset{\substack{q \\ s \leq j!}}{\leq} |z-z_j| + |\zeta - z_s|$$

$$\leq \frac{jH}{n} + \frac{K_q}{n} H \leq \frac{2K_q}{n} H$$

$$\Rightarrow z \in \Delta_q \quad \rightarrow\leftarrow \left(z \notin \bigcup_{j=1}^{M} \Delta_j \right).$$

that is, $z_s \notin \Delta_q'$ $\forall (s,q)$: $s \leq j$, $q \leq p$ and,

therefore,

$$\{z_1, \dots, z_j\} \subset (\mathbb{C} \setminus \Delta_p') \cap \dots \cap (\mathbb{C} \setminus \Delta_1')$$

* Note that this means $K_p \geq j > K_{p+1}$!

But since

$$|z-z_1| \le |z-z_2| \le \dots \le |z-z_j| \le \frac{j}{n} H,$$

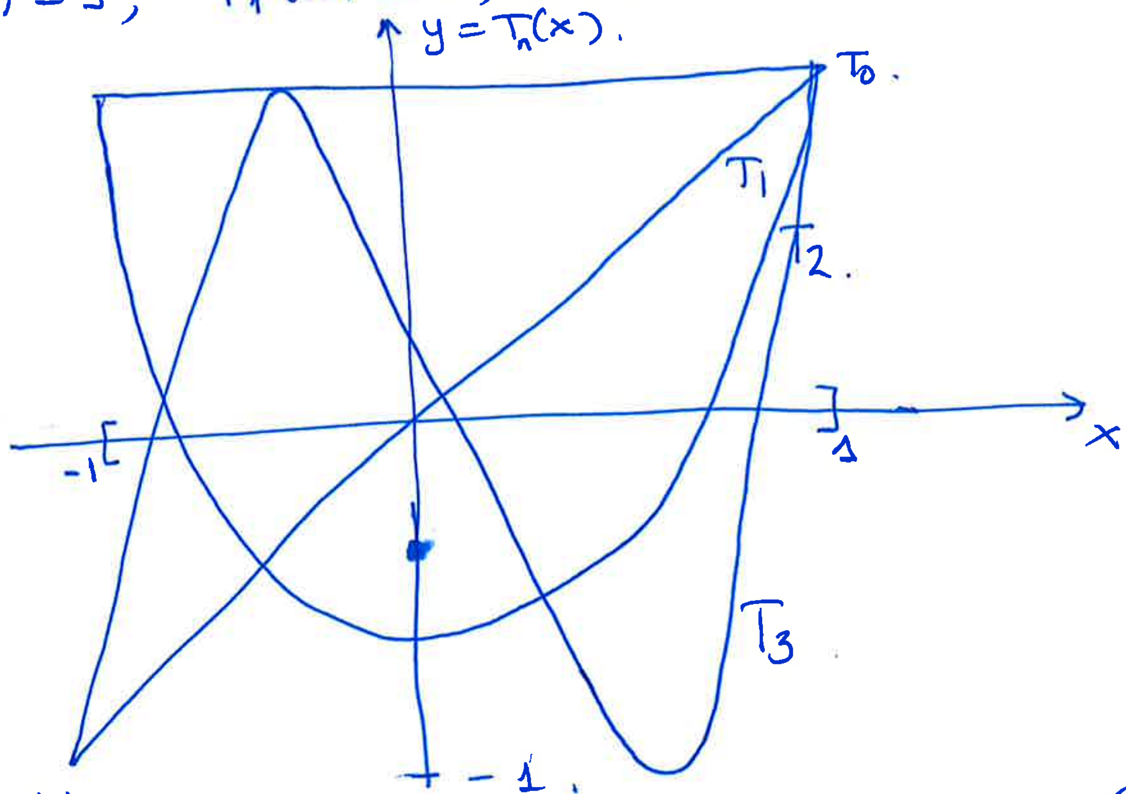
the disk of radius $\frac{j}{n} H$ and center z contains the points $z_1, \dots, z_j$.

that is, there are j points outside $\bigcup_{j=1}^{p} \Delta_j'$ contained on a disk of radius $j/_n H$. Hence, $k_{p+1} \ge j$. $\rightarrow\leftarrow$. ▨

Remark.-

Consider the chebyshev polinomials* of degree n defined recursively by

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_{n+1}(z) = 2z\, T_n(z) - T_{n-1}(z).$$



* of 1st kind!

Note that $|T_n(x)| \leq 1 \ \forall x \in [-1,1]$ for the cases considered in the previous example. This is, in fact, true $\forall n \in \mathbb{N}$.

Also, we have

$$T_n(z) = a_n z^n + \dots + a_1 z + a_0, \quad a_n = 2^{n-1}.$$

Now, define $f(z) = \rho^n \, T_n\left(\frac{z}{2^{\frac{n-1}{n}}\rho}\right)$, which is a monic polynomial of degree n.

Due to the previous remarks, it is clear that if $x \in \left[-2^{\frac{n-1}{n}}\rho, 2^{\frac{n-1}{n}}\rho\right]$, $|f(z)| \leq \rho^n$. Moreover, the length of this interval is $2 \cdot 2^{\frac{n-1}{n}}\rho$, so that it cannot be covered by disks whose sum of radii is less that $2^{\frac{n-1}{n}}\rho$.

So that if we choose $H = \rho e$ in Carlen's lemma, we get that the sum of the radii is at most $2e\rho$ and that such sum cannot be replaced by any number smaller than $2^{1-\frac{1}{n}}\rho$.