Ch. 10. The Hadamard theorem.

Recall the definition of the Weierstrass factors used in "his" theorem:

$$E_0(z) = (1-z),$$
$$E_\nu(z) = (1-z)\, e^{Q_\nu(z)}, \quad Q_\nu(z) = z + \frac{1}{2}z^2 + \dots + \frac{1}{\nu} z^\nu,$$

$\nu \geq 1$*.

Also, the convergence exponent of zeroes of entire functions f:

$$\lambda(f) = \inf\left\{\alpha > 0 : \sum_{j=1}^{\infty} |z_j|^{-\alpha} \text{ converges}\right\}$$
$$= \inf\left\{\alpha > 0 : \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\, dt \text{ converges}\right\}.$$

Finally, the order

$$\rho = \rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

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* Recall that in the proof of Weierstrass' theorem, $\nu = p-1$, where $p \in \mathbb{N}$ satisfied $\sum |z_j|^{-p} < \infty$!.

Let $f \in \mathcal{H}(\mathbb{C})$ have <u>finite</u> order ρ, and let $\{z_n\}_{n\in\mathbb{N}}$ be the sequence of its non-zero zeros, arranged according to increasing moduli. Let λ be the convergence exponent of the zeros of f and define

$$\nu = \begin{cases} [\lambda] \text{ (integer part of } \lambda), & \lambda \notin \mathbb{N}. \\ \lambda - 1, \text{ if } \lambda \in \mathbb{N} \text{ and } \sum_{j=1}^{\infty} |z_j|^{-\lambda} \text{ converges} \\ \lambda, \text{ otherwise.} \end{cases}$$

Note that $\sum_{j=1}^{\infty} |z_j|^{-(\nu+1)}$ converges. Also, that

$$Q(z) = \prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)^{*}$$

is an entire function with zeroes exactly at z_j. Hence $\lambda(Q) = \lambda$.

And $\lambda \leq \rho(Q)$.

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* $Q \equiv$ canonical product determined by the non-zero zeroes of f.

Theorem. For a cannonical product Q,

$$\lambda = \lambda(Q) = \rho(Q).$$

Pf.- We just need to show $\rho(Q) \le \lambda$.

For $z \in \mathbb{C}$, $|z| = r$ and $\varepsilon > 0$,

$$\log M(r, Q) = \log \max_{|z|=r} |Q(z)| = \max_{|z|=r} \log |Q(z)|.$$

Now,

$$\log |Q(z)| = \log \prod_{j=1}^{\infty} \left| E_\nu\left(\frac{z}{z_j}\right) \right|$$

$$= \sum_{j : \left|\frac{z}{z_j}\right| \ge 1/2} \log \left| E_\nu\left(\frac{z}{z_j}\right) \right| + \sum_{j : \left|\frac{z}{z_j}\right| < \frac{1}{2}} \log \left| E_\nu\left(\frac{z}{z_j}\right) \right|.$$

$$:= S_1^{*} + S_2.$$

Let's estimate S_2, where $\left|\frac{z}{z_j}\right| < \frac{1}{2}$.

Hence $\left| E_\nu\left(\frac{z}{z_j}\right) - 1 \right| \le \left|\frac{z}{z_j}\right|^{\nu+1}$ **

$$\Rightarrow \left| E_\nu\left(\frac{z}{z_j}\right) \right| \le 1 + \left|\frac{z}{z_j}\right|^{\nu+1}$$

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* S_1 is a finite sum! $\#j : |z_j| \le 2|z|$ for a given (fixed) z is finite if the function is not $\equiv 0$!

** Recall: $|z| \le 1$, $|E_\nu(z) - 1| \le |z|^{\nu+1}$, $\nu \ge 0$!!!

therefore,

$$S_2 \le \sum_{j:\left|\frac{z}{z_j}\right|<\frac{1}{2}} \log\left(1+\left|\frac{z}{z_j}\right|^{\nu+1}\right) \overset{\text{obvious!}}{\le} \sum_{j:\left|\frac{z}{z_j}\right|<\frac{1}{2}} \left|\frac{z}{z_j}\right|^{\nu+1} \quad (*).$$

<u>CASE 1.</u> $\lambda \in \mathbb{N}$ & $\sum_{j=1}^{\infty} |z_j|^{-\lambda}$ converges.

$$\sum_{\left|\frac{z}{z_j}\right|<\frac{1}{2}} \left|\frac{z}{z_j}\right|^{\lambda} = |z|^{\lambda} \sum_{\left|\frac{z}{z_j}\right|<\frac{1}{2}} |z_j|^{-\lambda} = O(r^{\lambda}).$$

<u>CASE 2.</u> Otherwise.

then, $\nu+1 > \lambda+\varepsilon$ for some small enough ε. Hence,

$$\left|\frac{z}{z_j}\right|^{\nu+1} = |z|^{\lambda+\varepsilon}\left|\frac{z}{z_j}\right|^{\nu+1-\lambda-\varepsilon} |z_j|^{-(\lambda+\varepsilon)} \underset{\left|\frac{z}{z_j}\right|<\frac{1}{2}!}{\le} |z|^{\lambda+\varepsilon}|z_j|^{-(\lambda+\varepsilon)}$$

and $$\sum_{\left|\frac{z}{z_j}\right|<\frac{1}{2}} \left|\frac{z}{z_j}\right|^{\nu+1} = O(r^{\lambda+\varepsilon}). \quad \left(\sum |z_j|^{-(\lambda+\varepsilon)} \text{ converges!}\right)$$

To estimate S_1, we first consider the case $\nu=0$. then,

$$S_1 = \sum_{\left|\frac{z}{z_j}\right|\ge\frac{1}{2}} \log\left|E_0\left(\frac{z}{z_j}\right)\right| = \sum_{\left|\frac{z}{z_j}\right|\ge\frac{1}{2}} \log\left|1-\frac{z}{z_j}\right|$$

$$\leq \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log\left(1+\left|\frac{z}{z_j}\right|\right) \leq A \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \left|\frac{z}{z_j}\right|^{\varepsilon}$$

Again, trivial !

$$= A|z|^{\varepsilon} \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} |z_j|^{-\varepsilon}.$$

- Now, if $\lambda = 0$, $\sum |z_j|^{-\varepsilon}$ converges and

$$S_1 = O(r^{\varepsilon}) = O(r^{\lambda+\varepsilon}).$$

- If $\lambda = 1$ and $\sum |z_j|^{-1}$ converges, for $\varepsilon < 1$,

$$S_1 \leq A \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z| \sum \left|\frac{z_j}{z}\right|^{1-\varepsilon} |z_j|^{-1}$$

$$\leq 2A|z| \sum |z_j|^{-1} = O(r^{\lambda})$$

- If $0 < \lambda < 1$. (Note that, since $\nu = 0$, $\lambda \leq 1$!).

Take $\varepsilon < \lambda$.

$$S_1 \leq A \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|^{\lambda+\varepsilon} \sum \left|\frac{z}{z_j}\right|^{-\lambda} |z_j|^{-(\lambda+\varepsilon)}$$

$$= A|z|^{\lambda+\varepsilon} \sum \left|\frac{z_j}{z}\right|^{\lambda} |z_j|^{-(\lambda+\varepsilon)}$$

$$\leq 2A|z|^{\lambda+\varepsilon} \sum |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}).$$

Hence, for $\nu = 0$, $S_1 = O(r^{\lambda+\varepsilon}) = S_2$.

$$\Rightarrow \log |Q(z)| = O(r^{\lambda+\varepsilon})$$

$$\Rightarrow \log M(r,Q) = O(r^{\lambda+\varepsilon})$$

$$\Rightarrow \rho(Q) \leq \lambda+\varepsilon \ldots \quad \forall\, \varepsilon<1\,!$$

Now, for $\nu>0$: $\left(\left|\frac{z}{z_j}\right| \geq \frac{1}{2}\right)$.

$$\log \left|E_\nu\left(\frac{z}{z_j}\right)\right| = \log\left|1-\frac{z}{z_j}\right| + \left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^\nu.$$

$\log(1+x) \leq x$
$$\leq 2\left(\left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^\nu\right)$$

$\nu \geq 1!!!!$

& multiply!
$$\leq 2\left|\frac{z}{z_j}\right|^\nu \left(1 + \left|\frac{z_j}{z}\right| + \cdots + \left|\frac{z_j}{z}\right|^{\nu-1}\right)$$

$\left|\frac{z_j}{z}\right| \leq 2$.
$$\leq 2\left|\frac{z}{z_j}\right|^\nu \left(1+2+\cdots+2^{\nu-1}\right)$$

$\sum_{j=0}^{\nu-1} 2^j = \frac{2^\nu - 1}{2}$
$$\leq 2^{\nu+1}\left|\frac{z}{z_j}\right|^\nu$$

$$\leq \frac{2^{\nu+1}}{2}.$$

Now, if $\nu = \lambda - 1$

$$\log\left|E_\nu\left(\frac{z}{z_j}\right)\right| \leq 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda-1} = 2^{\nu+1}\left|\frac{z}{z_j}\right|^\lambda \cdot \left|\frac{z_j}{z}\right|$$

$$\leq 2^{\nu+2}\left|\frac{z}{z_j}\right|^\lambda. \qquad (A).$$

If $\nu \neq \lambda - 1$ & ε is small (so that $\nu < \lambda + \varepsilon \leq \nu + 1$, $(\lambda + \varepsilon + 1 \leq \nu + 2)$),

$$\log \left| E_\nu \left(\frac{z}{z_j}\right) \right| \leq 2^{\nu+1} \left|\frac{z}{z_j}\right|^{\nu} = 2^{\nu+1} \left|\frac{z}{z_j}\right|^{\lambda+\varepsilon} \left|\frac{z_j}{z}\right|^{\lambda+\varepsilon-\nu}$$

$$\leq 2^{\nu+1+\lambda+\varepsilon-\nu} \left|\frac{z}{z_j}\right|^{\lambda+\varepsilon} \leq 2^{\nu+2} \left|\frac{z}{z_j}\right|^{\lambda+\varepsilon} \quad (B)$$

And hence, from (A) and (B) we get

$$\sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log \left| E_\nu \left(\frac{z}{z_j}\right) \right| \leq 2^{\nu+2} |z|^{\lambda+\varepsilon} \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} |z_j|^{-\lambda+\varepsilon}$$

$$\leq 2^{\nu+2} r^{\lambda+\varepsilon} \sum_j |z_j|^{-\lambda+\varepsilon} = O(r^{\lambda+\varepsilon}).$$

$\blacksquare$

<u>Theorem</u> (Hadamard) Let f be a non-constant entire function of finite order ρ. Then

$$f(z) = z^m Q(z) e^{P(z)},$$

where

(1) $m \geq 0$ is the multiplicity of the zero of f at the origin.

(2) Q is the canonical product formed by the non-zero zeros of f.

(3) P is a polynomial of degree $\leq \rho$.

We need the following lemma.

<u>Lemma</u>. Let Q be a canonical product of order λ. Given $\varepsilon > 0$, there exists a sequence $\{R_n\}$, $R_n \xrightarrow[n\to\infty]{} \infty$ such that for each R_n,

$$\mu(r_n) = \min_{|z| = R_n} |Q(z)| > e^{-R_n^{\lambda+\varepsilon}}.$$

<u>Pf.-</u> Denote by z_j the zeros of Q,

$0 < |z_1| \leq |z_2| \leq \cdots$, and $r_j = |z_j|$.

then, $\sum_j r_j^{-(\lambda+\varepsilon)}$ converges, which implies that the length of the set

$$E = \bigcup_{j=1}^{\infty} \left[r_j - \frac{1}{r_j^{\lambda+\varepsilon}}, r_j + \frac{1}{r_j^{\lambda+\varepsilon}} \right]$$

is finite.

As before,

$$\log|Q(z)| = S_1 + S_2 = \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right|$$

$$+ \sum_{\left|\frac{z}{z_j}\right| < \frac{1}{2}} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right| .$$

Now, $$S_1 = \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log\left|1-\frac{z}{z_j}\right| + \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log\left|e^{Q_\nu(z)}\right|$$

$$= S_{11} + S_{12} .$$

Assume $r \notin E$ is sufficiently large.

then, if $r_j \leq 2r$,

$$\left|1-\frac{z}{z_j}\right| = \frac{|z_j - z|}{|z_j|} \geq \frac{r - r_j}{r_j} \geq r_j^{-\lambda-\varepsilon-1}$$

S_1 is a finite sum!!!! $r >>>$.

$$\geq (2r)^{-1-\lambda-\varepsilon}.$$

Hence, $$S_{11} = \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log\left|1-\frac{z}{z_j}\right| \geq \sum_{\left|\frac{z}{z_j}\right| \geq \frac{1}{2}} \log(2r)^{-1-\lambda-\varepsilon}$$

$$= (-1-\lambda-\varepsilon)\log(2r)\, n(2r) = (-1-\lambda-\varepsilon)\log(2r)\, O(r^{\lambda+\varepsilon})$$

$$\lim_{r\to\infty} \frac{r^{\varepsilon}}{K\log(2r)} = \infty$$

$$\geq -r^{\lambda+2\varepsilon}$$

$$\therefore \quad S_{11} \geq -r^{\lambda+2\varepsilon}.$$

Now, $\quad S_{12} \leq S_1 = O(r^{\lambda+\varepsilon}) = S_2.$

$$\equiv \quad S_{12} \leq K_1 r^{\lambda+\varepsilon}, \quad S_2 \leq K_2 r^{\lambda+\varepsilon}, \quad r>>.$$

therefore,

$$\log|Q(z)| \geq |S_{11}| - |S_{12}| - |S_2| \geq -r^{\lambda+2\varepsilon} - Kr^{\lambda+\varepsilon}$$

$$= -r^{\lambda+2\varepsilon}\left[1+Kr^{-\varepsilon}\right] \geq -2r^{\lambda+2\varepsilon} \geq -r^{\lambda+3\varepsilon}.$$

$$\Longrightarrow \quad |Q(z)| \geq e^{-r^{\lambda+3\varepsilon}} \qquad \blacksquare$$

Proof of Hadamard's thm.

(1) and (2) are clear. We also know that $f(z) = z^m Q(z) e^{g(z)}$, for some $g \in \mathcal{H}(\mathbb{C})$. We just need to prove that g is a polynomial of degree $\leq \rho$.

Since $\rho(f) = \rho$, $\quad M(r,f) \leq e^{r^{\rho+\varepsilon}}, \quad r>>.$

$\rho(Q) = \lambda(Q) = \lambda(f) \leq \rho$. Hence, using the sequence in the previous lemma,

$$\max_{|z|=R_n} |e^{g(z)}| = \max_{|z|=R} e^{\mathrm{Re}\, g(z)} \le \frac{\max_{|z|=R} |f(z)|}{R^m \min_{|z|=R} |Q(z)|}$$

$$\le \frac{e^{r^{\rho+\varepsilon}}}{e^{-r^{\lambda+\varepsilon}}} = e^{r^{\rho+\varepsilon}} \cdot e^{r^{\lambda+\varepsilon}} \le e^{2r^{\rho+\varepsilon}}.$$

$$\Rightarrow \liminf_{r\to\infty} \frac{\max_{|z|=r} \mathrm{Re}\, g(z)}{2r^{\rho+\varepsilon}} < \infty.$$

$\Rightarrow$ $g(z)$ is a poly. of degree $\le \rho+\varepsilon$ $\square$.

<u>Corollary</u> Let f be a non-constant entire function of finite non-integer order ρ. then $\lambda(f) = \rho$.

Pf.- We know $\lambda(f) \le \rho$. So that let us assume $\lambda(f) < \rho$.

$$f(z) = z^m Q(z) e^{P(z)},$$

$\deg P = n \le \rho \notin \mathbb{N}$. , so that $n < \rho$.

Hence, $\rho(z^m) = 0$, $\rho(Q) = \lambda(f) < \rho$

and $\rho(e^P) = n < \rho$.

And we obtain

$$\rho(f) \leq \max\left(\rho(z^m), \rho(Q), \rho(e^P)\right)$$

$$= \max\left(\lambda(f), n\right) < \rho = \rho(f) \quad \rightarrow\leftarrow .$$

$\square$

<u>Corollary</u>. If f is trascendental entire and $\rho(f) \notin \mathbb{N}$, the f has infinitely many zeros.

<u>Pf</u>.- If $\rho > 0$, then $\lambda(f) > 0$. ✓.

If $\rho = 0$, $f(z) = c z^m Q(z)$, $c \in \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$ and Q is not a polynomial[*] $\square$

<u>Remark</u>.- $f_a(z) := f(z) - a$ is trauscendental, entire, and $\rho(f_a) = \rho(f)$.

Hence, $\#\{z : f_a(z) = 0\} = \#\{z : f(z) = a\} = \infty$!

What if $\rho(f) \in \mathbb{N}$?

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* Notice that $\rho(Q) = 0$, Hence Q is the product of terms of type $E_0\left(\frac{z}{z_j}\right)$!

Thm. (Borel). If $\rho(f) = \rho \in \mathbb{N}$, the exponent of convergence $\lambda(a,f)$ of a-points of f (i.e., $z: f(z)=a$) equals ρ, with one possible exceptional value a.

Pf.- Suppose there are two exceptional values a, b such that $\lambda(a,f) < \rho$ and $\lambda(b,f) < \rho$.

By the Hadamard theorem,

$$f(z) - a = z^{m_1} e^{P_1(z)} Q_1(z) \quad , \quad f(z) - b = z^{m_2} e^{P_2(z)} Q_2(z),$$

where $\deg(P_1) = \deg(P_2) = \rho$.

and Q_1, Q_2 are canonical products determined by the non-zero a-poits (b-points) of f, both of order $\rho_j < \rho$.

Then, subtracting,

$$b - a = z^{m_1} e^{P_1(z)} Q_1(z) - z^{m_2} e^{P_2(z)} Q_2(z)$$

$$\equiv z^{m_1} Q_1(z) e^{P_1(z) - P_2(z)} = z^{m_2} Q_2(z) + (b-a) e^{-P_2(z)}$$

$\deg P_2 = \rho \implies \deg(P_1 - P_2) = \rho$.

Differentiating:

$$\left(m_1 z^{m_1-1} Q_1 + z^{m_1} P_1' Q_1 + z^{m_1} Q_1'\right) e^{P_1}$$

$$= \left(m_2 z^{m_2-2} Q_2 + z^{m_2} P_2' Q_2 + z^{m_2} Q_2'\right) e^{P_2}$$

If we prove that differentiation leaves order unchanged, the order of $Q_j' = \rho_j$ so that the functions in the parenthesis are of order $< \rho$, they are entire functions also so another application of Hadamard's theorem gives:

$$z^{m_3} Q_3(z) e^{P_1(z)+P_3(z)} = z^{m_4} Q_4(z) e^{P_2(z)+P_4(z)}$$

Hence, $m_3 = m_4$, $Q_3 = Q_4$ and

$$\underbrace{P_1 - P_2}_{\deg = \rho} = \underbrace{P_4 - P_3}_{\deg < \rho} + 2\pi i n, \quad n \in \mathbb{Z}.$$

!!! $\rightarrow\!\leftarrow$. $\square$

<u>Corollary</u>: (Picard's theorem). Every transcendental entire function f takes all finite complex values a infinitely often, with one possible exceptional value a.

<u>Proposition.</u> Let $g \in \mathcal{H}(\mathbb{C})$. $\rho(g) = \rho(g')$.

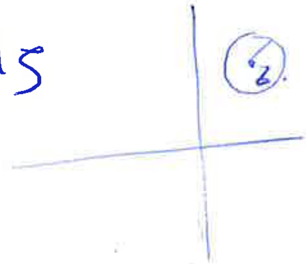
<u>Pf.</u>-
$$g(z) = \int_0^z g'(\zeta)\,d\zeta + g(0)$$

$$\Rightarrow \quad |g(z)| \le r\,M(r, g') + |g(0)|.$$

$$\Rightarrow \quad M(r, g) \le r\,M(r, g') + |g(0)|.$$

$$\Rightarrow \quad \rho(g) \le \rho(g').$$

Also,
$$g'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = R - r} \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta$$



(Here, $|z| = r < R$).

Hence,
$$|g'(z)| \le \frac{2\pi(R-r)}{2\pi(R-r)^2} M(R, g) = \frac{M(R,g)}{R-r}.$$

Hence,
$$M(r, g') \le \frac{M(R,g)}{R-r} \underset{R=2r}{\Longrightarrow} M(r, g') \le \frac{M(2r, g)}{r}$$

& if $r > 1$, $M(r, g') \le M(2r, g)$.

$$\Rightarrow \quad \frac{\log\log M(r, g')}{\log r} \le \frac{\log\log M(2r, g)}{\log r}$$

$$= \frac{\log\log M(2r, g)}{\log 2r - \log 2} = \frac{\log 2r}{\log 2r - \log 2} \, \frac{\log\log M(2r, g)}{\log 2r}$$

$\square$.