Ch. 11: Normal families.

Remember the chordal metric we defined before (in $\hat{\mathbb{C}}$). via the stereographic projection $\pi : S^2 \to \hat{\mathbb{C}}$,

$$\pi(x_1, x_2, x_3) = \begin{cases} \dfrac{x_1 + i x_2}{1 - x_3}, & x_3 \neq 1 \\ \infty, & x_3 = 1. \end{cases}$$

or rather, its inverse

$$\pi^{-1} : \hat{\mathbb{C}} \to S^2 :$$

$$\pi^{-1}(z) = \begin{cases} \left(\dfrac{2 \operatorname{Re} z}{1+|z|^2}, \dfrac{2 \operatorname{Im} z}{1+|z|^2}, \dfrac{|z|^2 - 1}{1+|z|^2} \right), & z \in \mathbb{C}. \\ (0,0,1), & z = \infty \end{cases}$$

In this way, for $z_1, z_2 \in \hat{\mathbb{C}}$,

$$\chi(z_1, z_2) := d_e(\pi^{-1}(z_1), \pi^{-1}(z_2))$$

$$\overset{*}{=} \begin{cases} \dfrac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}, & z_1, z_2 \neq \infty. \\ \dfrac{2}{\sqrt{1+|z_1|^2}}, & z_2 = \infty, z_1 \neq \infty. \end{cases}$$

$\circledast$ If $z_1 = z_2 = \infty$, $\chi(z_1, z_2) = 0$.

If $z_1 \neq \infty$, $z_2 = \infty$.

$$\chi(z_1, z_2) = \left| \left(\frac{2\operatorname{Re} z_1}{1+|z_1|^2}, \frac{2\operatorname{Im} z_1}{1+|z_1|^2}, \frac{|z_1|^2-1}{1+|z_1|^2} \right) - (0,0,1) \right|$$

$$= \sqrt{\frac{4(\operatorname{Re} z_1)^2 + 4(\operatorname{Im} z_1)^2}{(1+|z_1|^2)^2} + \frac{(4)}{(1+|z_1|^2)^2}}$$

$$= 2\sqrt{\frac{1+|z_1|^2}{(1+|z_1|^2)^2}} = \frac{2}{\sqrt{1+|z_1|^2}}$$

If $z_1, z_2 \neq \infty$,

$$\chi(z_1, z_2) = \left| \left(\frac{2\operatorname{Re} z_1}{1+|z_1|^2} - \frac{2\operatorname{Re} z_2}{1+|z_2|^2}, \frac{2\operatorname{Im} z_1}{1+|z_1|^2} - \frac{2\operatorname{Im} z_2}{1+|z_2|^2}, \frac{|z_1|^2-1}{1+|z_1|^2} - \frac{|z_2|^2-1}{1+|z_2|^2} \right) \right|$$

$$= \sqrt{\boxed{\frac{4(\operatorname{Re} z_1)^2}{(1+|z_1|^2)^2} + \frac{4(\operatorname{Im} z_1)^2}{(1+|z_1|^2)^2} + \frac{(1+|z_1|^4-2|z_1|^2)}{(1+|z_1|^2)^2}} - \frac{8\operatorname{Re} z_1 \operatorname{Re} z_2 + 8\operatorname{Im} z_1 \operatorname{Im} z_2}{(1+|z_1|^2)(1+|z_2|^2)} - \frac{2(|z_1|^2-1)(|z_2|^2-1)}{(1+|z_1|^2)(1+|z_2|^2)}}$$

+ same factor with z_2!

$$= \sqrt{2 - \frac{2(|z_1|^2-1)(|z_2|^2-1)}{(1+|z_1|^2)(1+|z_2|^2)} - \frac{8\operatorname{Re} z_1 \operatorname{Re} z_2 + 8\operatorname{Im} z_1 \operatorname{Im} z_2}{(1+|z_1|^2)(1+|z_2|^2)}}$$

$$= \frac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}.$$

As a consequence, we have:

<u>Corollary</u>: $\forall z_1, z_2 \in \mathbb{C}$,

(i) $\chi(z_1, z_2) \leq 2$.

(ii) $\chi\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \chi(z_1, z_2)$

(iii) $|z_1| \leq |z_2| \leq \infty \Rightarrow \chi(0, z_1) \leq \chi(0, z_2)$.

(iv) $\chi(z_1, z_2) \leq 2|z_1 - z_2|$.

<u>Remark</u>.– It is clear that $\chi(z_1, z_2) \geq 0$ $\forall z_1, z_2 \in \mathbb{C}$, and $=$ iff $z_1 = z_2$.

Also, that $\chi(z_1, z_2) = \chi(z_2, z_1)$.

To obtain that χ satisfies the triangle inequality (and hence deduce that χ is a distance), we just use that the Euclidean distance in $\mathbb{R}^3$ satisfies it, and that the stereographic projection is a homeomorphism:

$$\chi(z_1, z_3) = |\pi^{-1}(z_1) - \pi^{-1}(z_3)| \leq |\pi^{-1}(z_1) - \pi^{-1}(z_2)| + |\pi^{-1}(z_2) - \pi^{-1}(z_3)| = \chi(z_1, z_2) + \chi(z_2, z_3).$$

Define the arc-length element ds on $S^2 \approx \hat{\mathbb{C}}$.

$$ds = \frac{2|dz|}{1+|z|^2} = \frac{2(dx)^2+2(dy)^2}{1+|z|^2}.$$

$$\boxed{\lim_{w\to z} \frac{\chi(w,z)}{|w-z|} = \frac{2}{1+|z|^2} \quad !!!}$$

the corresponding spherical area element is

$$dA = \frac{4\,dx\,dy}{(1+|z|^2)^2}$$

Given a curve $\gamma \subset S^2$, its spherical length is:

$$L(\gamma) = \int_\gamma \frac{2|dz|}{1+|z|^2} = \int_{t_1}^{t_2} \frac{2|\gamma'(t)|\,dt}{1+|\gamma(t)|^2}.$$

$\gamma = \gamma(t),\ t\in[t_1,t_2] \equiv z=\gamma(t)$

Using this definition, we can still define the spherical metric on S^2 by

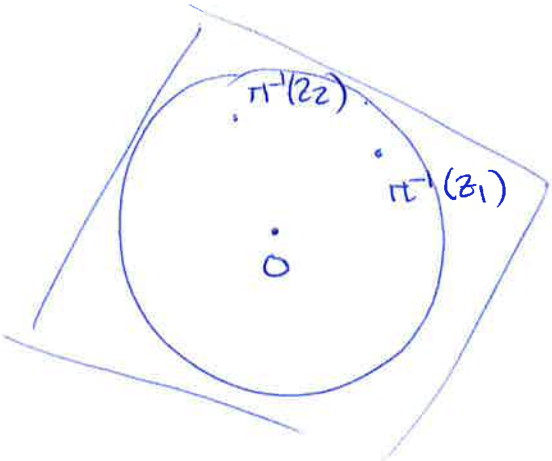
$$\sigma(z_1,z_2) = \inf L(\gamma),$$

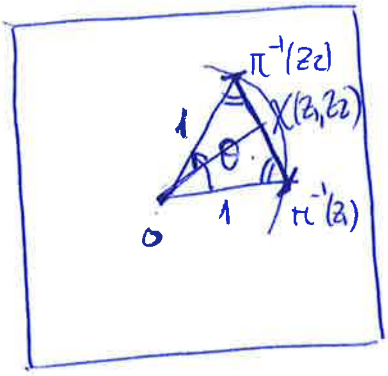
where γ is a curve on S^2 that joints z_1 and z_2.

Notice that $\chi(z_1,z_2) \le \sigma(z_1,z_2)$

But moreover, we also have

$$\sigma(z_1,z_2) \le \frac{\pi}{2}\chi(z_1,z_2)$$





Hence $\chi(z_1,z_2) = |\sin\theta|$ & $\sigma(z_1,z_2) = \theta$.

Therefore, $\frac{\sigma(z_1,z_2)}{\chi(z_1,z_2)} = \frac{|\theta|}{2\left|\sin\frac{\theta}{2}\right|} \underset{⊛}{\leq} \frac{\pi}{2}$ ** $\square$.

Hence, both metrics induce the same topology on S^2.

<u>DEF</u> :- A sequence $\{f_n\}$ of functions $f_n : \mathbb{C} \to \hat{\mathbb{C}}$ converges spherically uniformly to a function $f : \mathbb{C} \to \hat{\mathbb{C}}$ on a set $E \subset \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists\, n = n(\varepsilon)$: if $n \geq n_\varepsilon$

$$\chi(f(z), f_n(z)) < \varepsilon \quad \forall z \in E$$

** Antti's remark! the ratio is bigger then $\theta \to \pi$!!!! and at π, $\frac{\sigma(z_1,z_2)}{\chi(z_1,z_2)} = \frac{\pi}{2}$!

⊛ $\sin x \geq \frac{2x}{\pi}$ $\forall x \in \left[0, \frac{\pi}{2}\right]$; $f(x) = \frac{\pi}{2}\sin x - x$.

$f'(x) = \frac{\pi}{2}\cos x - 1 = 0$, $\cos x = \frac{2}{\pi}$, $x_0 = \arccos\frac{2}{\pi}$ ~~maximum~~ minimum.

& $f\left(\frac{\pi}{2}\right) = 0$, $\lim_{x \to 0} f(x) = 0$. $\therefore$ $f(x) \geq 0$!

Remark – the (Euclidean) uniform convergence on E implies spherical uniform convergence, since $\chi(z_1, z_2) \le 2|z_1, z_2|$.

theorem – If a sequence $\{f_n\}$ of functions $f_n : \mathbb{C} \to \hat{\mathbb{C}}$ converges spherically uniformly to a bounded function $f : \mathbb{C} \to \hat{\mathbb{C}}$ on E, then f_n converges uniformly (in the Euclidean sense) to f on E.

Pf. – Assume $|f(z)| \le M$ on E. then

$$\chi(0, f(z)) \le \chi(0, M) = \frac{2M}{\sqrt{1+M^2}} < 2.$$

Let $\varepsilon < 1 - \frac{M}{\sqrt{1+M^2}}$, and let n_ε be such that

$$\chi(f(z), f_n(z)) < 2\varepsilon \quad , n \ge n_\varepsilon, \ z \in E.$$

then.

$$\frac{2|f_n(z)|}{\sqrt{1+|f_n(z)|^2}} = \chi(0, f_n(z)) \le \chi(0, f(z)) + \chi(f(z), f_n(z))$$

$$< 2\left(\frac{M}{\sqrt{1+M^2}} + \varepsilon\right) := 2m \ (<2).$$

And hence, $|f_n(z)| < \dfrac{m}{\sqrt{1-m^2}} := M_1$, $n \geq n_\varepsilon$

and $z \in E$.

therefore, $\forall\, z \in E$,

$$|f(z) - f_n(z)| = \sqrt{1+|f(z)|^2}\,\sqrt{1+|f_n(z)|^2}\,\frac{1}{2}\,\chi(f(z), f_n(z))$$

$$\leq \sqrt{1+M^2}\,\sqrt{1+M_1^2}\,\frac{1}{2}\,\chi(f(z), f_n(z)) \quad \square.$$

<u>DEF</u> - A function $f: \mathbb{C} \to \hat{\mathbb{C}}$ is spherically continuous at $z_0 \in \mathbb{C}$ if $\forall \varepsilon > 0$, $\exists\, \delta > 0$:

$$\chi(f(z), f(z_0)) < \varepsilon \quad , \quad |z - z_0| < \delta.$$

<u>Proposition</u>.- If $f: \mathbb{C} \to \hat{\mathbb{C}}$ is meromorphic in $\Omega \subset \mathbb{C}$, then f is spherically continuous in Ω.

<u>Pf</u>.- If f is analytic at $z_0 \in \Omega$, then the result holds $(\chi(f(z), f(z_0)) \leq 2\,|f(z) - f(z_0)|)$

If z_0 is a pole of f, then $\frac{1}{f}$ is continuous at z_0, and we again get the result since $\chi(f(z), f(z_0)) = \chi\left(\frac{1}{f(z)}, \frac{1}{f(z_0)}\right) \leq 2\left|\frac{1}{f(z)} - \frac{1}{f(z_0)}\right|$

$\square$

DEF.– Let f be meromorphic in a domain $\Omega \subset \mathbb{C}$.

Suppose $f(z) \in \mathbb{C}$.

$$f^{\#}(z) := \lim_{w \to z} \frac{\chi(f(z), f(w))}{|z-w|}$$

$$= \lim_{w \to z} \frac{2\,|f(z)-f(w)|}{|z-w|} \frac{1}{\sqrt{1+|f(z)|^2}} \frac{1}{\sqrt{1+|f(w)|^2}}$$

$$= 2 \cdot \frac{|f'(z)|}{1+|f(z)|^2} .$$

If z is a pole of f,

$$f^{\#}(z) = \lim_{z' \to z} \frac{|f(z')|}{1+|f(z')|^2}$$

RK.– Obvious that $f^{\#}$ is continuous in $\mathbb{C}$. Also,

$$\left(\frac{1}{f}\right)^{\#} = f^{\#} .$$

DEF.- Let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions defined on a domain $\Omega \subset \mathbb{C}$.
the sequence converges uniformly on compact subsets of Ω to a function $f : \Omega \to \mathbb{C}$ if for any compact set $K \subset \Omega$ and any $\varepsilon > 0$,
$\exists \; N = N(K, \varepsilon)$: if $n \geq N$,

$$|f_n(z) - f(z)| < \varepsilon \quad \forall z \in K.$$

For a sequence $f_n : \Omega \to \widehat{\mathbb{C}}$, we say that the sequence converges spherically uniformly on compact subsets of Ω to a function $f : \Omega \to \widehat{\mathbb{C}}$ if $\forall K \subset\subset \Omega$ and any $\varepsilon > 0$,

$\exists \; N = N(K, \varepsilon)$:

$$\chi\big(f_n(z), f(z)\big) < \varepsilon \quad \forall z \in K.$$

DEF.- A family $\mathcal{F}$ of functions $f : \Omega \to \mathbb{C}$ is locally bounded on a domain Ω if for each $z_0 \in \Omega$, $\exists \; M = M(z_0)$, $0 \leq M < \infty$, and a disk $D(z_0, r) \subset \Omega$: $|f(z)| \leq M \;\; \forall z \in D(z_0, r)$ and all $f \in \mathcal{F}$.

<u>Example.</u> $\mathcal{F} = \left\{ f_\alpha(z) = \dfrac{1}{z - e^{i\alpha}} \, , \alpha \in \mathbb{R} \right\}$.

$\Omega = \mathbb{D}$.

→ Uniformly bounded?.

$\exists M: \ |f_\alpha(z)| \le M \quad \forall \alpha \in \mathbb{R}, \ \forall z \in \mathbb{D}$?

NO: $\lim\limits_{z \to e^{i\alpha}} |f_\alpha(z)| = \infty$. (take $z = re^{i\alpha} \in \mathbb{D}$).

$\therefore$ It is not uniformly bounded.

→ Locally bounded?.

$$|z| \le R < 1 \Rightarrow |f_\alpha(z)| = \frac{1}{|e^{i\alpha} - z|} \le \frac{1}{1 - |z|}$$

$$\le \frac{1}{1-R}.$$

So that for any compact set in $\mathbb{D}$ of the form $|z| \le R$, $|f_\alpha(z)| \le \dfrac{1}{1-R}$.

But any $K \subset\subset \mathbb{D}$ is contained in $|z| \le R$ for some R. Hence, the family is locally bounded.

<u>Theorem.</u>- If $\mathcal{F}$ is a locally bounded family of analytic functions in a domain $\Omega \subset \mathbb{C}$ then $\mathcal{F}' = \{ f' / f \in \mathcal{F} \}$ is locally bounded

<u>Pf</u>.- Let $z_0 \in \Omega$ be arbitrary. then, $\exists\, M$: $|f(z)| \leq M \quad \forall f \in \mathcal{F}$ & $\forall z \in \overline{D}(z_0, r)$.

Choose $z \in D(z_0, \frac{r}{2})$.

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|\zeta - z_0| = r} \frac{|f(\zeta)|\,|d\zeta|}{|\zeta - z|^2}$$

$|\zeta - z|^2 \geq \left(\frac{r}{2}\right)^2$

$$\leq \frac{M}{\cancel{2\pi}} \cancel{2\pi} \cdot \frac{r}{\cancel{2}} \cdot \frac{4}{r^2} = \frac{4M}{r}.$$

$\forall f' \in \mathcal{F}'$.

that is, $\forall z_0$, $\exists\, D(z_0, \frac{r}{2})$: $|f'(z)| \leq \frac{4M}{r}$. $\forall f' \in \mathcal{F}'$, so that $\mathcal{F}'$ is locally bounded $\square$.

<u>Remark</u> - $\mathcal{F} = \{n : n \in \mathbb{N}\}$ is not locally bounded. $\left[\exists\, f_n \subset \mathcal{F} : f_n(z_0) \xrightarrow[n\to\infty]{} \infty\right]$. But $\mathcal{F}' = \{0\}$ is.

So that $\mathcal{F}'$ locally bounded $\not\Rightarrow$ $\mathcal{F}$ is.

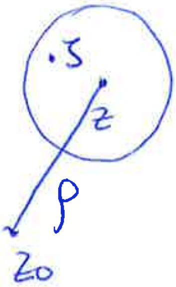
<u>theorem</u>.- Let $\mathcal{F}$ be a family of analytic functions on a domain $\Omega \subset \mathbb{C}$ such that

(i) $\mathcal{F}'$ is locally bounded

(ii) $\exists\, z_0 \in \Omega$: $|f(z_0)| \leq M \quad \forall f \in \mathcal{F}$.

then $\mathcal{F}$ is locally bounded. if Ω is starlike with respect to z_0.

Proof:- Let $z \in \Omega$ be again, arbitrary.
Consider a neighborhood $D(z,r)$



Let $\rho = |z - z_0|$.

If $f \in \mathcal{F}$ and $\zeta \in D(z,r)$, then integrating from z_0 to ζ along the path γ consisting on the line segments $[z_0, z]$ and $[z, \zeta]$ gives

$$|f(\zeta)| \le |f(z_0)| + \int_{z_0}^{z} |f'(w)|\,|dw| + \int_{z}^{\zeta} |f'(w)|\,|dw|.$$

$$\le M + \underbrace{\left[\sup_{w \in \gamma} |f'(w)|\right]}_{} \cdot (\rho + r) \le M + M_1(R+r).$$

$< \infty$, since γ is a compact set and f' is continuous.

DEF.- A family $\mathcal{F}$ of analytic functions on a domain $\Omega \subset \mathbb{C}$ is normal in Ω if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence converging to an analytic limit function f uniformly on each $K \subset\subset \Omega$, or a subsequence converging to ∞ uniformly on each such K.

the following theorem shows that it suffices to consider the property of normality locally. More precisely, we say that $\mathcal{F}$ is normal at $z_0 \in \Omega$ if it is normal in some (open) neighborhood of z_0.

<u>theorem</u>. A family $\mathcal{F}$ of analytic functions is normal in a domain Ω if and only if $\mathcal{F}$ is normal at each point in Ω.

<u>Pf</u>.- Normal $\Rightarrow$ normal at each point. ✓.

<u>$\Leftarrow$)</u>. Suppose $\mathcal{F}$ is normal at each $z \in \Omega$. Choose a countable set dense in Ω. For instance, all $z_n = x_n + i y_n$ with $x_n, y_n \in \mathbb{Q}$ in Ω.

Let $D(z_n, r_n)$ be the largest disk centered at each z_n in which $\mathcal{F}$ is normal.

It is obvious (since $\{z_n\}$ is dense in Ω), that $\bigcup_{n=1}^{\infty} \left(z_n, \frac{r_n}{2}\right) = \Omega$.

Let $\{f_n\} \subset \mathcal{F}$.

By normality at z_1, $\exists \, (f^{(1)}_{n_k}) \subset (f_n)$ which converges uniformly in $D(z_1, \frac{r_1}{2})$ either to an analytic function or to ∞.

the sequence $(f^{(2)}_{n_k})$ has a subsequence that converges uniformly in $D(z_1, \frac{r_1}{2}) \cup D(z_2, \frac{r_2}{2})$

Repeat this process and consider the diagonal sequence

$$\{ f^{(1)}_{n_1}, f^{(2)}_{n_2}, f^{(3)}_{n_3}, \dots \}.$$

which converges in $D(z_n, \frac{r_n}{2})$, $n = 1, 2, \dots$, in each disk separately to an analytic fcn. / ∞.

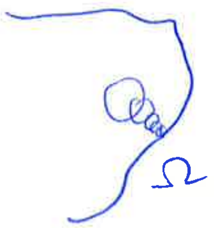
But then,

$$\Omega = \Big[\bigcup_{\to \infty} D(z_n, \tfrac{r_n}{2}) \Big] \mathring{\cup} \Big[\bigcup_{\to \text{analy.}} D(z_n, \tfrac{r_n}{2}) \Big].$$

[Recall $f_n \underset{K \subset\subset \Omega}{\rightrightarrows} f$, f_n analytic, then f analytic!].

then, Ω = one of these sets.

Now, it might happen that $r_n \to 0$.



But, since we just need a finite number of such disks to cover any $K \subset\subset \Omega$, we see that the convergence is uniform in any such K. $\square$

<u>Example</u>.- $\mathcal{F} = \{ f_n(z) = nz,\ n \in \mathbb{N} \}$.

$f_n(0) \equiv 0$ and $f_n(z) \xrightarrow[n \to \infty]{} \infty$.

Hence, if Ω is a domain which does not contain $z=0$, $\mathcal{F}$ is normal in Ω.

But if $\{0\} \in \Omega$, $\mathcal{F}$ is not normal at $z=0$. that is, $\mathcal{F}$ is not normal in any such domain.

<u>DEF</u>.- A family of functions $\mathcal{F}$ on a domain $\Omega \subset \mathbb{C}$ is equicontinuous at $z_0 \in \Omega$ if $\forall \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon, z_0)$:

$$|f(z) - f(z_0)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

& $|z - z_0| < \delta$.

<u>DEF</u>.- Such family of functions is spherically equicontinuous at $z_0 \in \Omega$ if $\forall \varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon, z_0)$:

$$\chi(f(z), f(z_0)) < \varepsilon \quad \forall f \in \mathcal{F}.$$

and $|z - z_0| < \delta$.

the family is equicontinuous (resp. spherically equicontinuous) on a subset $E \subset \Omega$ if it is at each point $z_0 \in E$.

Remark.- Notice that if E is compact, equicontinuity = uniformly continuous.
(E can be covered by a finite number of disks).

Same with respect to spherically equicontinuity.

Also, since $\chi(z,w) \leq 2|z-w|$, we have that equicontinuity $\Rightarrow$ spherical equicontinuity.

Proposition. Let (f_n) be a sequence of (spherically) continuous functions converging (— " —) uniformly to a function f on a compact subset $E \subset \mathbb{C}$. then f is uniformly (spherically) continuous on E and (f_n) form a (spherically) equicontinuous family of functions in E.

Pf.- (for spherical metric only).

$$f_n \underset{\chi}{\rightrightarrows} f \equiv \forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}:$$

$$\chi(f_n(z), f(z)) < \frac{\varepsilon}{3} \quad \forall z \in E,$$

if $n \geq n_0$.

f_{n_0} is spherical continuous on E. Hence,

$\exists\ \delta = \delta(\varepsilon, E) > 0$:

$$\chi(f_{n_0}(z), f_{n_0}(z')) < \frac{\varepsilon}{3}$$

$\forall z, z' \in E$: $|z - z'| < \delta$.

then, $\chi(f(z), f(z')) \leq \chi(f(z), f_{n_0}(z))$
$+ \chi(f_{n_0}(z), f_{n_0}(z')) + \chi(f_{n_0}(z'), f(z')) < \varepsilon$

if $|z - z'| < \delta$.

thus, f is uniformly spherically continuous.

Moreover,

$$\chi(f_n(z), f_n(z')) \leq \chi(f_n(z), f(z)) + \chi(f(z), f(z')) + \chi(f(z'), f_n(z')) < 3\varepsilon,$$

$|z - z'| < \delta$, $n \geq n_0$.

$n < n_0$, f_n is a continuous function on E.

$\Rightarrow \quad \chi(f_n(z), f_n(z')) < 3\varepsilon$, $|z - z'| < \delta_n$.

Choose $\Delta = \min\{\delta_n, \delta\}$. $\square$

Proposition.- A locally bounded family of analytic functions on a domain Ω is equicontinuous on compact subsets of Ω.

Pf.- Since $\mathcal{F}$ is locally bounded, then $\mathcal{F}'$ is locally bounded, hence uniformly bounded on compact subsets of Ω.

Take now a closed disk $K \subset \Omega$ and $M < \infty$: $|f'(z)| \leq M$ $\forall z \in K$ & $\forall f \in \mathcal{F}$.

Given $\varepsilon > 0$ and $z, z' \in K$: $|z - z'| < \frac{\varepsilon}{M}$, we get

$$|f(z) - f(z')| \leq \int_{[z,z']} |f'(s)| |ds| \leq M |z - z'| = \varepsilon.$$

Hence, f is equicontinuous on the compact disk K.

Now, choose any compact $K \subset \Omega$ and cover the compact with open disks. Extract a finite subsequence s.t. $K \subset \bigcup_{1}^{N} D(z_i, r_i)$ and apply the previous arguments to each of the $\overline{D(z_i, r_i)}$

this gives you $|f'(z)| \le M_i$, $i = 1, \dots, N$.

& $\forall$ $z, z' \in D_i$, say, $|z - z'| < \frac{\varepsilon}{M_i}$,

$$|f(z) - f(z')| \le \varepsilon.$$

Let now $M = \max\limits_{1 \le i \le N}$. If $|z - z'| < \frac{\varepsilon}{MN}$, $z, z' \in K$, then $|f(z) - f(z')| \le \varepsilon$ (Use the triangle inequality, if needed) $\square$.

<u>Remark</u>.– $\mathcal{F} = \{z + n : n \in \mathbb{N}\}$ is an example of a family of analytic equicontinuous functions in $\mathbb{D}$ which is not locally bounded.

<u>Montel's theorem</u>.– Let $\mathcal{F}$ be a locally bounded family of analytic functions on a domain $\Omega \subset \mathbb{C}$. Then $\mathcal{F}$ is normal in Ω.

<u>Pf</u>.– Take, as before, a countable dense subset $\{z_n\}$ in Ω.

Take a sequence $(f_n) \subset \mathcal{F}$ and consider the sequence $\{f_n(z_1)\}_{n \in \mathbb{N}}$

$\mathcal{F}$ is locally bounded so that $\exists M$:

$|f_n(z_1)| \le M$ $\forall$ $n \in \mathbb{N}$.

Hence, $\exists$ $\{n_k\} \subset \mathbb{N}$:

$$f_{n_1}^{(1)}(z_1),\ f_{n_2}^{(1)}(z_1), \ldots \qquad \text{converges.}$$

Consider the sequence $\{f_{n_k}^{(1)}(z_2)\}$ and apply the same argument to get

$$f_{n_1}^{(2)}(z_2),\ f_{n_2}^{(2)}(z_2), \ldots$$

that converges for z_1 and z_2.

By repeating this process, we get subsequences $\{f_{n_k}^{(p)}\}$ that converge at $z_1, z_2, \ldots, z_p$.

the diagonal sequence $\{f_{n_k}^{(k)}\}$ converges, then, at every z_n. Let's rename this diagonal sequence as $\{g_k\}$.

Consider $K \subset\subset \Omega$ and $\varepsilon > 0$.

Being $\mathcal{F}$ locally bounded, it is equicontinuous on K. therefor, $\exists\ \delta$:

$$|g_n(z) - g_n(z')| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N},\ |z - z'| < \delta,$$

$z, z' \in K$.

But $K \subset \bigcup_{j=1}^{N} D(z_j, \delta)$ and $g_n(z_j)$ converges for all such j. Hence, $\exists\ n_0 \in \mathbb{N}$:

$$|g_n(z_j) - g_m(z_j)| < \frac{\varepsilon}{3}, \quad n, m \geq n_0.$$

for all j.

Now, given $z \in K$, $\exists j : z \in D(z_j, \delta)$

We then have for $n, m \geq n_0$,

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)|$$
$$+ |g_m(z_j) - g_m(z)| < \varepsilon . \quad \square$$

<u>Theorem</u>.– Let $\mathcal{F}$ be a family of analytic functions in a domain $\Omega \subset \mathbb{C}$.

Suppose that $\mathcal{F}'$ is normal and $\exists z_0 \in \Omega$: $\{f(z_0) \,/\, f \in \mathcal{F}\}$ is bounded.

Then $\mathcal{F}$ is normal.

<u>Pf.</u>–

$\Omega = D(z_0, R)$: Let $(f_n) \subset \mathcal{F}$. Then $\exists f_{n_k}$: f'_{n_k} converges uniformly on $K \subset\subset \Omega$. Also, $\{f_{n_k}(z_0)\}$ is bounded so that, passing to a subsequence if needed, we have

$$f'_{n_k} \underset{K}{\rightrightarrows} g \quad , \quad f_{n_k}(a) \to w \in \mathbb{C} .$$

Define for $z \in \Omega$

$$G(z) = w + \int_{\underbrace{[z_0, z]}_{\Omega \text{ is convex!!!}}} g(\zeta)\, d\zeta .$$

then G is analytic in Ω and $G' = g$.

And we have

$$|f_{n_k}(z) - G(z)| \leq |f_{n_k}(a) - w| + \int_{[a,z]} |f'_{n_k}(s) - g(s)|\,|ds|$$

$$\leq |f_{n_k}(a) - w| + R \sup_{s \in [a,z]} |f'_{n_k}(s) - g(s)|.$$

$$\xrightarrow[k\to\infty]{} 0.$$ uniformly in the compact.

that is $\mathcal{F}$ is normal in every disk, hence is normal $\square$

Some examples

(1) Let $\mathcal{F} = \left\{ f_n(z) = \dfrac{nz^2 - (n+1)z + 1}{2(n+1)} \right\}$.

Is $\mathcal{F}$ normal in $\mathbb{D}$?

Solution. Note that $\lim_{n\to\infty} f_n(z) = \dfrac{z^2 - z}{2}$, $z \in \mathbb{D}$.

We can show:

$\to$ $|f_n(z) - f(z)| < \varepsilon$ $\forall |z| \in K \subset\subset \mathbb{D}$ $(n \geq n_0)$.

$\to$ $\mathcal{F}$ is normal, so that it suffices to show that $\mathcal{F}$ is locally bounded.

But $|f_n(z)| \leq \dfrac{n|z|^2 + (n+1)|z| + 1}{2(n+1)} \leq 1$ $\forall z \in \mathbb{D}$ $(\forall n \in \mathbb{N})$. Hence $\mathcal{F}$ is a bounded family!

(2) $\mathcal{F} = \left\{ f_n(z) = \dfrac{nz^{n+1}}{1 - z^2}, z \in \mathbb{D} \right\}$.

Does $f_n \underset{K \subset\subset \mathbb{D}}{\rightrightarrows}$?

$$|f_n(z)| \leq \frac{n|z|^{n+1}}{1 - |z|^2} \leq \frac{nR^{n+1}}{1 - R^2}$$

$$\left[\ \lim_{n\to\infty} nR^{n+1} = \lim_{n\to\infty} n e^{(n+1)\log R} = \lim_{n\to\infty} \frac{n}{e^{-(n+1)\log R}}\right.$$

$$\left. = \lim_{n\to\infty} \frac{1}{(n+1)\log R\, e^{-(n+1)\log R}} = 0\ \right]$$

But $\lim_{z\to 1} f_n(z) = \infty$!!! Hence $f_n \underset{\mathbb{D}}{\not\rightrightarrows}$.