Ch. 12: The Riemann Mapping thm.

<u>thm</u>.- Let Ω be a simply connected domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$. Then, there exists an analytic bijection

$$f: \Omega \to \mathbb{D}$$

For each $z_0 \in \Omega$, there is a unique such map f with $f(z_0)=0$, $f'(z_0)>0$.

<u>Pf</u>.- <u>UNIQUENESS</u>

(Sup) there are 2 such bijections f_1, f_2 with $f_1(z_0)=f_2(z_0)=0$, $f_1'(z_0)>0$, $f_2'(z_0)>0$.

Then, $\varphi = f_1 \circ f_2^{-1} : \mathbb{D} \xrightarrow[\text{onto}]{} \mathbb{D}$ & $\varphi(0)=0$.

So that $\varphi(z) = \lambda z$, $|\lambda| = 1$.

$\equiv$ $f_1 = \lambda f_2$, $|\lambda|=1$.

But $f_1'(z_0) = \lambda f_2'(z_0) \Rightarrow \lambda = 1$. #.

<u>EXISTENCE</u>.

Consider $\mathcal{F} = \{ \text{univalent analytic } f : \Omega \to \mathbb{D},\ f(z_0)=0 \}$.

Let us first prove $\mathcal{F} \neq \emptyset$.

Since $\Omega \neq \mathbb{C}$, $\exists\ a \in \mathbb{C} : a \notin \Omega$.

Hence $z - a \neq 0\ \ \forall z \in \Omega$. and, since Ω is simply connected, we can choose a holomorphic branch of $\log(z-a)$. that is, there exists $\ell \in \mathcal{H}(\Omega)$:

$$e^{\ell(z)} = z - a, \quad z \in \Omega.$$

- ℓ is injective. ✓
- $z_1 \neq z_2 \in \Omega$, $\ell(z_1) - \ell(z_2) \notin 2\pi i \mathbb{Z}$.

(Otherwise, $z_1 - a = z_2 - a \ldots$).

In particular, $\ell(z) \neq \ell(z_0) + 2\pi i$. $\forall z \in \Omega$.

Claim $\exists\ \varepsilon > 0 : \ |\ell(z) - \ell(z_0) - 2\pi i| > \varepsilon\ \ \forall z \in \Omega$.

Proof of the claim: (Sup) $\exists\ \{z_n\} \subset \Omega$:

$$\ell(z_n) \longrightarrow \ell(z_0) + 2\pi i, \quad n \to \infty.$$

then,
$$\underbrace{e^{\ell(z_n)}}_{= z_n - a} \longrightarrow e^{\ell(z_0) + 2\pi i} = e^{\ell(z_0)} = z_0 - a.$$

that is, $z_n \longrightarrow z_0$. Hence $\ell(z_n) \longrightarrow \ell(z_0)$ $\rightarrow\!\leftarrow$.

Now consider

$$\tilde{f}(z) = \frac{1}{\ell(z) - \ell(z_0) - 2\pi i}.$$

$\tilde{f}$ is:

→ analytic and one-to-one.

→ Bounded: $|\tilde{f}(z)| \leq \frac{1}{\varepsilon}$ $\forall z \in \Omega$.

Hence $\tilde{f}(\Omega) \subset D(0, \frac{1}{\varepsilon})$

the function $f(z) = \dfrac{\tilde{f}(z) - \tilde{f}(z_0)}{\frac{1}{\varepsilon} + |\tilde{f}(z_0)|}$

is then analytic, one to one, $f(z_0) = 0$

and $f(\Omega) \subset \mathbb{D}$.

the next step is to show that if $f \in \mathcal{F}$ has maximal absolute (value) derivative at z_0, the f is surjective.

$\equiv$ If $f \in \mathcal{F}$ is not surjective, $\exists\, g \in \mathcal{F}$:

$$|g'(z_0)| > |f(z_0)|.$$

Suppose f is not surjective. then $\exists\, w \in \mathbb{D}$:

$f(z) - w \neq 0$ $\forall z \in \Omega$.

Consider the automorphism of $\mathbb{D}$:

$$\varphi_p(z) = \frac{p - z}{1 - \bar{p} z}.$$

then $\varphi_w \circ f \neq 0$ on Ω and therefore, we can choose an analytic branch of $\varphi_w \circ f$ in Ω: $F = \sqrt{\varphi_w \circ f}$

$\left[F(z_0) = \sqrt{\varphi_w(f(z_0))} \right.$

$\left. \equiv f(z_0) = \varphi_w \circ F(z_0)^2 = 0! \right]$.

the function

$$g = \varphi_{F(z_0)} \circ F = \varphi_{F(z_0)}\left(\sqrt{\varphi_w \circ f}\right) \in \mathcal{H}(\Omega),$$

$g(\Omega) \subset \mathbb{D}$, $g(z_0) = 0$ Once the branch is determined!

$$\sqrt{\varphi_w \circ f(z_1)} = \sqrt{\varphi_w \circ f(z_2)} \iff f(z_1) = f(z_2)$$

$$\iff z_1 = z_2!$$

$\therefore g \in \mathcal{F}$.

Note that $\varphi_w\left((\varphi_{F(z_0)} \circ g)^2 \right) = f$.

That is, $f = \varphi \circ g$, $\varphi = \varphi_w \circ (\varphi_{F(z_0)})^2$

It is clear that $\varphi \notin \mathrm{Aut}(\mathbb{D})$ and that $\varphi(0) = 0$.

Hence, $|\varphi'(0)| < 1$ by the Schwarz-Lemma

therefore, $|f'(z_0)| = |\varphi'(g(z_0))| \cdot |g'(z_0)| < |g'(z_0)|$.

Now, let $\lambda = \sup_{f \in \mathcal{F}} |f'(z_0)|$.

2 options. • Any $f \in \mathcal{F}$ is surjective $\rightarrow$ done.

(in fact, $\lambda > 0$ always: f one-to-one $\Rightarrow f' \neq 0$!)

• $\lambda > 0$.

In fact, we will see later that $\lambda < \infty$.

Now, since $\mathcal{F}$ is bounded, $\mathcal{F}$ is normal,

hence $\mathcal{F}'$ is normal too.

Let $f_n \in \mathcal{F}'$: $\lim_{n\to\infty} |f_n'(z_0)| = |f'(z_0)|$.

then, $\exists \{n_k\}$: $\{f_{n_k}\} \underset{K \subset\subset \Omega}{\rightrightarrows} f_0$.

and $f_{n_k}'(z_0) \longrightarrow f_0'(z_0)$.

Moreover, $\underset{\substack{\downarrow \\ f_{n_k}(z)}}{f_{n_k}(z_0)} = \int_0^z f_{n_k}'(\xi)\, d\xi \longrightarrow \int_0^z f_0(\xi) d\xi = f_0(z)$;

Hence $f_0(z_0) = 0$.

Also, $|f_0(z)| = \lim_{k\to\infty} |f_{n_k}(z)| \leq 1 \implies f_0(\Omega) \subset \mathbb{D}$.

<u>thm</u> (Hurwitz).

Let $\{f_n\}$ be analytic and one to one

Suppose that $f_n \underset{K\subset\subset\Omega}{\rightrightarrows} f$. then f is either

constant or one-to-one.

that is, f_0 is one to one ($f_0'(z_0) \neq 0$).

And this ends the proof o Riemann's thm. $\square$.

Some details to be checked also:

(1) $$f'(z_0) = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{(\xi - z_0)^2} d\xi.$$

r: $D(z_0, r) \subset\subset \Omega$. $(r>0!)$.

$$\Rightarrow |f'(z_0)| \le \frac{1}{2\pi} \int \frac{1}{r^2} |ds| = \frac{1}{r} < \infty .$$

(2) A domain $\Omega \subset \mathbb{C}$ is simply connected, then for any analytic function f on Ω with no ceros has an analytic logarithm on Ω.

$$\equiv \exists\, g \in \mathcal{H}(\Omega) : \ f(z) = e^{g(z)} .$$

$$\equiv g(z) = \log f(z) .$$

Pf.- Given $z_0 \in \Omega$, choose a constant c_0:

$$e^{c_0} = f(z_0) .$$

Define

$$g(z) := c_0 + \int_{[z_0, z]} \underbrace{\frac{f'(s)}{f(s)}}_{\text{analytic}} ds .$$

$$\Rightarrow g \in \mathcal{H}(\Omega) \quad \& \quad g'(z) = \frac{f'(z)}{f(z)} .$$

Consider

$$G(z) = f(z)\, e^{-g(z)} \in \mathcal{H}(\Omega) .$$

$$G'(z) = \left[f'(z) - f(z) \cdot \frac{f'(z)}{f(z)} \right] e^{-g(z)} = 0 \Rightarrow G \equiv \text{constant},$$

$$G(z_0) = e^{c_0} e^{-c_0} = 1, \qquad f(z) = e^{g(z)} \quad ▨$$

(3) <u>the Hurwitz theorem.</u>

A more general result is true:

Let Ω be a domain, $\{f_n\} \in \mathcal{H}(\Omega)$,

$$f_n \underset{K \subset\subset \Omega}{\rightrightarrows} f .$$

Suppose $f \not\equiv 0$. but f has a zero of order m at $z_0 \in \Omega$.

then $\forall \rho > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, f_n has precisely m zeros (counting multiplicity) in $|z - z_0| < \rho$.

Furthermore, these zeros converge to z_0 as $n \to \infty$.

Remark.- Notice we are considering zeros in Ω. If f has zeros on $\partial\Omega$, the theorem fails: $f_n(z) = z - 1 + \frac{1}{n}$ has exactly one zero in $\mathbb{D}$ but $f(z) = z - 1$ $\left(= \lim_{\substack{n\to\infty \\ K \subset\subset \mathbb{D}}} f(z)\right)$ has no zeros in $\mathbb{D}$. But $\{\text{zeros of } f_n\} = \{1 - \frac{1}{n}\} \xrightarrow[n\to\infty]{} 1$!

Pf.- Let $f(z) = (z - z_0)^m g(z)$, $g(z_0) \neq 0$.

then, $\exists$ ρ : $f(z) \neq 0$ $\forall$ $0 < |z - z_0| < \rho$, $D(z_0, \rho) \subset\subset \Omega$.

Choose δ : $|f(z)| \geq \delta$ $\forall$ $|z - z_0| = \rho$.

then $|f_k(z)| \geq \frac{\delta}{2}$ $\forall$ $|z - z_0| = \rho$, $k \geq N$.

Hence, for these values of k, $\frac{f_k'(z)}{f_k(z)}$ is well-defined and $\frac{f_k'(z)}{f_k(z)} \rightrightarrows \frac{f'(z)}{f(z)}$, $|z - z_0| = \rho$.

$$m = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)}\,dz = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f_k'(z)}{f_k(z)}\,dz = \lim_{k\to\infty} N_k,$$

N_k = # zeros of f_k in $|z-z_0|=\rho$.

$\therefore \lim\limits_{k\to\infty} N_k = m$. But $N_k \in \mathbb{N}$, hence

$N_k \equiv m$ for k big enough.

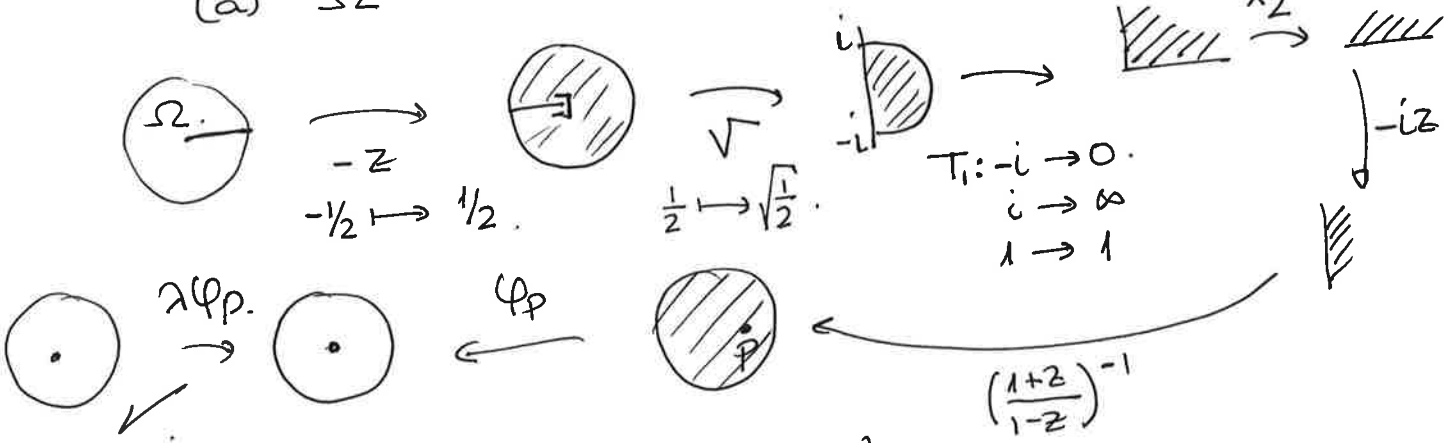
Examples : $\to$ $\varphi_{p_n} = \dfrac{p_n - z}{1-\overline{p_n} z}$, $\{p_n\} \subset \mathbb{D}$.

(1) $p_n \longrightarrow p_0 \in \mathbb{D}$, $\varphi_{p_n} \underset{K\subset\subset\mathbb{D}}{\rightrightarrows} \varphi_{p_0}$.

(2) $p_n \longrightarrow q \in \partial\mathbb{D}$, $\varphi_n \underset{K\subset\subset\mathbb{D}}{\rightrightarrows} \varphi_q = \dfrac{q-z}{1-\bar{q}z} \equiv q$.

$\to$ Find an analytic function φ on Ω onto $\mathbb{D}$ with the given conditions.

(a) $\Omega = \mathbb{D}\setminus[0,1)$, $f(-\tfrac{1}{2})=0$, $f'(-\tfrac{1}{2})>0$.



(b) $\Omega = \{z\in\mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{2}\}$.

the function $\varphi = \frac{1}{2}\log\frac{1+z}{1-z} : \mathbb{D} \to \Omega$.

$f = \varphi^{-1}$.

(c) $\Omega = \{z\in\mathbb{C} : |z|<1, |z-\tfrac{1}{2}| > \tfrac{1}{2}\}$

$T : \begin{cases} -1 \to 0 \\ 0 \to 1 \\ 1 \longmapsto \infty. \end{cases}$

rotate & dilate to obtain Ω in (b)!

Locally univalent analytic functions.

If $f'(z) \neq 0$ for all $z \in \Omega$, then f is locally univalent in Ω. $\equiv$ $\forall z \in \Omega$, $\exists R$: $f|_{D(z,R)}$ is one-to-one.

Assume now that f is analytic and one-to-one in $D(z_0, R)$. then $f'(z_0) \neq 0$.

RK.- f analytic and one to one in $D(z_0, R) \iff g(z) = f(z+z_0)$ is analytic and one to one in $D(0,R)$. So is
$$h(z) = g(z) - g(z_0)$$

Hence, we may assume $z_0 = f(z_0) = 0$.

$\boxed{h(0)=0, \quad h'(0) = f'(z_0)}$.

Let's prove that if $f'(0) = 0$, f cannot be one-to-one in $D(0,R)$.

Pf.- $f(0) = f'(0) = 0$.

Hence, $f(z) = z^k g(z)$, $k \geq 2$, $g(0) \neq 0$, $g \in \mathcal{H}(D(0,R))$.

Now, since $g(0) \neq 0$, $\exists\, r < R$: $g(z) \neq 0$ $\forall |z| \leq r$ $\Rightarrow$ we can define an analytic branch of the logarithm, hence a k-th root of $h(z) \equiv \sqrt[k]{g(z)}$ defines an analytic function in $|z| < r$. and $g(z) = h(z)^k$.

therefore, $f(z) = (z h(z))^k = (\phi(z))^k$

Now, ϕ is analytic, hence open.

therefore, $\phi(D(0,r)) \supset D(0,\delta)$.

In particular, $\exists\ z_1, z_2 \in D(0,r)$:

$$\phi(z_1) = \frac{\delta}{2}, \quad \phi(z_2) = \frac{\delta}{2} e^{\frac{2\pi i}{k}} \neq \delta,$$

since $k \geq 2$.

But this gives $f(z_1) = \frac{\delta^k}{2} = f(z_2)$ $\rightarrow\leftarrow$.