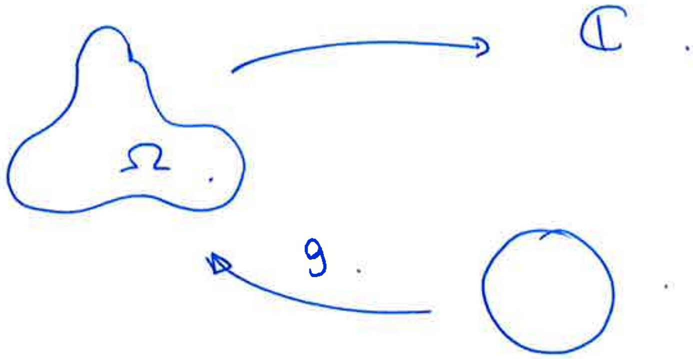
Ch 13. Univalent functions

Let $f \in \mathcal{H}(\Omega)$, Ω simply-connected and f one-to-one.



By the Riemann mapping thm. $\exists\, g \in \mathcal{H}(\mathbb{D})$ $g(\mathbb{D}) = \Omega$, g one-to-one.

Here $F = f \circ g$ is analytic and one-to-one (univalent) in $\mathbb{D}$. $\equiv F \in \mathcal{U}(\mathbb{D})$

Of course $F \in \mathcal{U}(\mathbb{D}) \iff \dfrac{F - F(0)}{F'(0)} \in \mathcal{U}(\mathbb{D})$.

Hence, for different analysis it is convenient to consider functions in the class

$$S = \{ f \in \mathcal{U}(\mathbb{D}) : f(0) = 1 - f'(0) = 0 \}.$$

$S \equiv$ Schlicht.

Note that $f \in S \Rightarrow f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$.

It is easy to check that the following transformations preserve the class S:

→ Conjugation: $f(z) \longmapsto \overline{f(\bar{z})}$

→ Rotation: $f(z) \longmapsto e^{-i\theta} f(e^{i\theta} z)$, $\theta \in \mathbb{R}$.

→ Dilation: $f(z) \longmapsto \dfrac{f(rz)}{r}$, $0 < r < 1$.

→ Disk automorphism:

$$f(z) \longmapsto \frac{f\left(\frac{\alpha+z}{1+\bar{\alpha}z}\right) - f(\alpha)}{(1-|\alpha|^2)\, f'(\alpha)}, \quad \alpha \in \mathbb{R}.$$

→ Omitted value: $f \in S$ and $f(z) \neq \omega \;\; \forall z \in \mathbb{D}$,

$$g(z) = \frac{\omega f}{\omega - f} \in S.$$

$\ulcorner$ $g \in \mathcal{H}(\mathbb{D})$, $g(0) = 0$ & $g'(0) = 1$.

$$\frac{\omega f(z_1)}{\omega - f(z_1)} = \frac{\omega f(z_2)}{\omega - f(z_2)} \Leftrightarrow f(z_1) = f(z_2) \;\lrcorner$$

→ Square-root: $f(z) \in S \Rightarrow g(z) = \sqrt{f(z^2)} \in S$.

$\ulcorner$ $f \in S \Rightarrow f(z^2) = z^2 + \sum_{n=2}^{\infty} a_n z^{2n}$

$$= z^2 \left(1 + \sum_{n=2}^{\infty} a_n z^{2n-2}\right)$$

$$= z^2 \underbrace{\left(1 + a_2 z^2 + a_3 z^4 + \dots\right)}_{\neq 0!}.$$

$=0 \Leftrightarrow \dfrac{f(z^2)}{z^2} = 0$.

Only possibility: $z = 0$.

$$\lim_{z \to 0} \frac{f(z^2)}{z^2} = f'(0) \neq 0 \,!$$

Hence $\sqrt{f(z^2)} = z\sqrt{1 + a_2 z^2 + a_3 z^4 + \dots} = g(z)$.

Is $g \in \mathcal{U}(\mathbb{D})$?

Note that $g(-z) = -z\sqrt{1+a_2z^2+\cdots} = -g(z)$

Hence $g(z) = z + c_3z^3 + c_5z^5 + \ldots$

Suppose that $g(z_1) = g(z_2)$.

then $f(z_1^2) = f(z_2^2) \Rightarrow$ $z_1 = z_2$ ✓ or $z_1 = -z_2$.

But if $z_1 = -z_2 \to g(z_1) = g(z_2) = -g(z_1) \Rightarrow g(z_1) = 0$

$\Rightarrow z_1 = 0 \Rightarrow z_1 = z_2 = 0$.

Some examples

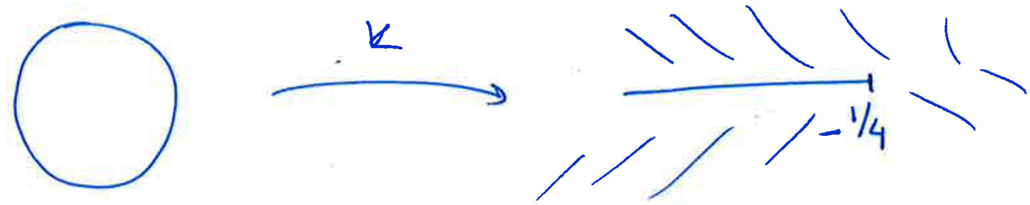
(i) $f(z) = I(z) = z$.

(ii) $f(z) = \dfrac{z}{1-z}$ $\quad \left(f(\mathbb{D}) = \{\operatorname{Re}\{w\} > -\frac{1}{2}\} \right)$

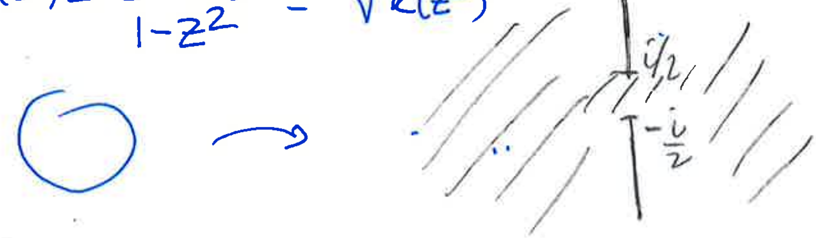
(iii) $f(z) = \frac{1}{2}\log\dfrac{1+z}{1-z}$ $\quad \left(f(\mathbb{D}) = \{-\frac{\pi}{4} < \operatorname{Im} w < \frac{\pi}{4}\} \right)$.

(iv) $f(z) = K(z) = \dfrac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n$

$$K(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right].$$



(v) $f(z) = \dfrac{z}{1-z^2} = \sqrt{K(z^2)}$



Remark. — $f_1, f_2 \in S \not\Rightarrow f_1 + f_2 \in S$!

$$\frac{z}{1-z} + \frac{z}{1+iz} = F(z) \text{ satisfies}$$

$$F'\left(\frac{1+i}{2}\right) = 0 \,!$$

Another related class is

$$\Sigma = \{ f \in \mathcal{U}(\Delta) \text{ except for a simple pole at infinity with residue} = 1 \}.$$

$\Delta = \{ |z| > 1 \}$.

Any $g \in \Sigma$ maps Δ onto the complement of a compact connected set E.

the class $\Sigma' = \{ f \in \Sigma : 0 \in E_f \}$

[Note that just a translation transforms

$$g \in \Sigma \longrightarrow g - a \in \Sigma' \,].$$

$$g \in \Sigma' \Rightarrow g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

$$\left[g\left(\frac{1}{z}\right) = \frac{1}{z} + a_0 + a_1 z + \cdots \right].$$

Note that if $f \in S$, the function

$$g(z) = \frac{1}{f\left(\frac{1}{z}\right)} \in \Sigma'$$

$\to g \in \mathcal{U}(\Delta)$.

$\to g\left(\frac{1}{z}\right) = \frac{1}{f(z)} = \frac{1}{z(1 + a_2 z + \cdots)}$ ✓]

$\to 0 \notin g(\Delta)$.

$0 \notin g(\Delta)$.

$0 \in g(\Delta) \iff f\left(\frac{1}{z}\right) = \infty$ for some $|z| > 1$.

$\iff f(s) = \infty \quad \forall |s| < 1$. No!

$\lim_{z \to 0} g\left(\frac{1}{z}\right) = \lim_{z \to 0} \frac{1}{f(z)} = \infty$.

Hence g has a pole at $z = \infty$.

Moreover, $\lim_{z \to 0} z\, g\left(\frac{1}{z}\right) = \lim_{z \to 0} \frac{z}{f(z)} = 1$.

$\Rightarrow$ the pole is simple and has residue $= 1$!

Note: $\dfrac{1}{f\left(\frac{1}{z}\right)} = \dfrac{1}{\frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots}$

$$1 \;\Big|\; \frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$$

$$z - a_2 - (a_3 - a_2^2) z + \cdots$$

$$-\frac{a_2}{z} - \frac{a_3}{z^2} - \cdots$$

$$\frac{a_2}{z} + \frac{a_2^2}{z^2} + \frac{a_2 a_3}{z^3}$$

$$-\frac{(a_3 - a_2^2)}{z^2} + \cdots$$

In fact, notice that if $g \in \Sigma'$

$$g\left(\frac{1}{z}\right) = \frac{1}{z} + b_0 + b_1 z + \cdots$$

$$\Rightarrow \frac{1}{g\left(\frac{1}{z}\right)} = \frac{z}{\underbrace{1 + b_0 z + b_1 z + \cdots}_{\neq 0}}.$$ and g one-to-one.

Hence this transformation $T: S \longrightarrow \Sigma'$, $f \longmapsto \frac{1}{f\left(\frac{1}{z}\right)}$ stablishes a one-to-one correspondence between these two classes.

The Green theorem.

Let $G \subset \mathbb{R}^2$ be a smooth domain with positively oriented boundary. Let $F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ be a differentiable vector field in G.

$$\int_{\partial G} F_1(x,y)dx + F_2(x,y)dy = \iint_G \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

Area Theorem.

Let $g(z) = z + b_0 + \sum_{j=1}^{\infty} b_j z^{-j} \in \Sigma$.

then $\sum_{j=1}^{\infty} j|b_j|^2 \leq 1$.

Consequence.

(1) $g \in \Sigma \Rightarrow |b_1| \leq 1$. with equality

$\Longleftrightarrow \quad g(z) = z + b_0 + \frac{b_1}{z}, \quad |b_1| = 1$.

Note that $g \in \Sigma \Leftrightarrow \bar{\lambda} g(\lambda z) \in \Sigma$

Hence, let's try to understand how the transformation

$$g(z) = z + \frac{1}{z} \quad \text{behaves.}$$

$|z| = 1 \Rightarrow \quad g(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

$|z| > 1, \quad g(z) = re^{i\theta} + \frac{e^{-i\theta}}{r} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$.

$= x + iy$.

$$\frac{x^2}{\left(r+\frac{1}{r}\right)^2} + \frac{y^2}{\left(r-\frac{1}{r}\right)^2} = 1. \rightarrow \text{elipses.}$$

(2) $|b_n| \leq \frac{1}{\sqrt{n}} \quad \forall n \geq 1.$

But this inequality is not sharp for any $n \geq 2$:

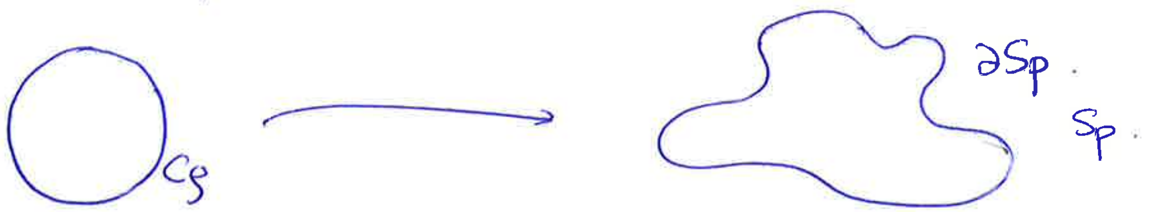
$$g(z) = z + \frac{1}{\sqrt{n}} \bar{z}^{n}$$

$$g'(z) = 1 - \sqrt{n}\, z^{-n-1} = 0 \Leftrightarrow \frac{1}{\sqrt{n}} = z^{-n-1}$$

$$\Longleftrightarrow \quad z = n^{\frac{1}{2(n+1)}} \; !$$

Proof of the area theorem.

Consider $\partial S_\rho = g(\partial C_\rho)$, $C_\rho = \{z : |z| = \rho > 1\}$.



$g(z) = u(x,y) + i v(x,y)$.

Define $\vec{F} = -\frac{1}{2} v \vec{i} + \frac{1}{2} u \vec{j} = (-\frac{1}{2} v, \frac{1}{2} u) = (F_1, F_2)$

Note that $\frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial v} = 1$.

Hence, $A(S_p) = \frac{1}{2} \int_{\partial S_p} u\,dv - v\,du$.

Now, $u\,dv - v\,du = \mathrm{Im}\left((u - iv)(du + i dv)\right)$

$= \mathrm{Im}(\bar{g}\,dg)$

Notice that

$$\int_{\partial S_\rho} \operatorname{Re} \bar{g}\, dg = \int_{\partial S_\rho} u\,du + v\,dv = \frac{1}{2}\int_{\partial S_\rho} d(u^2+v^2)$$

$$= \frac{1}{2}(u^2+v^2)\Big|_{\partial S_\rho} = 0 \quad (\partial S_\rho \text{ is simple \& closed!}).$$

therefore $\boxed{\int \bar{g}\,dg = \int \operatorname{Re} \bar{g}\,dg + i\int \operatorname{Im} \bar{g}\,dg}$

$$A(S_\rho) = \frac{1}{2}\int_{\partial S_\rho} \operatorname{Im}\{\bar{g}\,dg\} \overset{\flat}{=} \frac{1}{2i}\int_{\partial S_\rho} \bar{g}\,dg = \frac{1}{2i}\int_{\partial S_\rho} \bar{g}\,\frac{dg}{d\theta}\,d\theta$$

Now, $g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots$

therefore, $\overline{g(\rho e^{i\theta})} = \rho e^{-i\theta} + \bar{b}_0 + \bar{b}_1 \rho^{-1} e^{i\theta} + \bar{b}_2 \rho^{-2} e^{2i\theta} \dots$

and

$$\frac{\partial g(\rho e^{i\theta})}{d\theta} = i\rho e^{i\theta} - i b_1 \rho^{-1} e^{-i\theta} - 2b_2 i \rho^{-2} e^{-2i\theta} - \dots$$

$$= i(\rho e^{i\theta} - b_1 \rho^{-1} e^{-i\theta} - 2b_2 \rho^{-2} e^{-2i\theta} - \dots)$$

$$\Rightarrow A(S_\rho) = \frac{1}{2}\int_0^{2\pi} (\rho e^{-i\theta} + \bar{b}_0 + \bar{b}_1 \rho^{-1} e^{i\theta} + \dots)(\rho e^{i\theta} - b_1 \rho^{-1} e^{-i\theta} \dots)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\rho e^{-i\theta} + \sum_{n=0}^{\infty} \bar{b}_n \rho^{-n} e^{in\theta}\right)\left(\rho e^{i\theta} - \sum_{n=1}^{\infty} n b_n \rho^{-n} e^{-in\theta}\right) d\theta$$

$$\overset{\circledast}{=} \frac{1}{2}\int_0^{2\pi} \rho^2\,d\theta - \frac{1}{2}\int_0^{2\pi} \sum_{n=1}^{\infty} n |b_n|^2 \rho^{-2n}\,d\theta$$

$\lceil * \int_0^{2\pi} e^{ij\theta}\, d\theta = 0 \quad \forall j \neq 0! \rfloor$.

$$= \pi \left(\rho^2 - \sum_{n=1}^{\infty} n |b_n|^2 \rho^{-2n} \right) \quad , \rho > 1!$$

Let $\rho \to 1^+$ to get

$$1 - \sum_{n=1}^{\infty} n |b_n|^2 \geq 0 \ . \quad \square.$$

Here we have an important consequence.

<u>the Bieberbach theorem.</u>

Let $f(z) = z + a_2 z^2 + \cdots \in S$.

then, $|a_2| \leq 2$.

$\lceil$ Moreover, equality holds iff. f is the Koebe function or some of its rotations. $\rfloor$

<u>Pf.</u>- $f \in S \Rightarrow \begin{cases} \dfrac{1}{f(\frac{1}{z})} \in \Sigma. \\ \sqrt{f(z^2)} \in S. \end{cases}$

$$\Rightarrow g(z) = \frac{1}{\sqrt{f\left(\frac{1}{z^2}\right)}} \in \Sigma.$$

$$g(z) = z - \frac{a_2}{2} z^{-1} + \cdots \Rightarrow \left| \frac{a_2}{2} \right| \leq 1 . \square$$

$$[g(z)]^2 = \frac{1}{f\left(\frac{1}{z^2}\right)}$$

$$= \frac{1}{\frac{1}{z^2} + \frac{a_2}{z^4} + \frac{a_3}{z^6} + \cdots}$$

$$= z^2 - a_2 - \frac{(a_3 - a_2^2)}{z^2} - \cdots$$

$$\begin{array}{l|l} 1 & \frac{1}{z^2} + \frac{a_2}{z^4} + \frac{a_3}{z^6} \\ \hline -1 - \frac{a_2}{z^2} - \frac{a_3}{z^4} - \cdots & z^2 - a_2 \\ -\frac{a_2}{z^2} - \frac{a_3}{z^4} - \cdots & \\ +\frac{a_2}{z^2} + \frac{a_2^2}{z^4} - \cdots & \\ \hline -\frac{(a_3 - a_2^2)}{z^4} & \end{array}$$

$$= z^2 \left[1 - \frac{a_2}{z^2} - \frac{(a_3 - a_2^2)}{z^4} - \cdots\right].$$

$$g(z) = z \underbrace{\sqrt{1 - \frac{a_2}{z^2} + }}_{\varphi}$$

$$= z\left(\varphi(0) + \frac{\varphi'(0)}{z} + \frac{\varphi''(0)}{z^2}\right.$$

$$= z\left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right].$$

$$g(z) = z - \frac{a_2}{z} - \cdots$$

~~$\lim_{z\to 0} z\, g\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{z}{\sqrt{f(z^2)}} = 1.$~~

$$g(-z) = -z - b_0 + \frac{a_2}{z} + \cdots = -\left[z + b_0 - \frac{a_2}{z}\right] \cdots$$

$$g(z) = z + \frac{e^{i\theta}}{z} = \frac{1}{\sqrt{f\left(\frac{1}{z^2}\right)}}$$

$$\equiv \left(z + \frac{e^{i\theta}}{z}\right)^2 = \frac{1}{f\left(\frac{1}{z^2}\right)}$$

$$\equiv z^2 + e^{i\theta} + \frac{e^{2i\theta}}{z^2} = \frac{1}{f\left(\frac{1}{z^2}\right)}$$

$$\equiv \frac{1}{f(z^2)} = \frac{1}{z^2} + 2e^{i\theta} + e^{2i\theta} z^2$$

$$\equiv \frac{1}{f(z)} = \frac{1 + 2e^{i\theta}z + e^{2i\theta}z^2}{z}$$

$$\equiv f(z) = \frac{z}{(1 + e^{i\theta}z)^2}. \qquad \square.$$

$$\sqrt{1 + a_2 z^2 + a_3 z^4 + \dots} = 1 +$$

thm (Köebe)

Let $f \in S$. then $\{w : |w| \leq \frac{1}{4}\} \subset f(\mathbb{D})$.

Pf.- suppose $f(z) \neq w$, $z \in \mathbb{D}$.

then $g(z) = \dfrac{w f(z)}{w - f(z)} = z + (a_2 + \frac{1}{w}) z^2 + \dots \in S$.

$\Rightarrow |a_2 + \frac{1}{w}| \leq 2$.

then,

$$\left|\frac{1}{w}\right| - |a_2| \leq \left|a_2 + \frac{1}{w}\right| \leq 2$$

$\Rightarrow \left|\frac{1}{w}\right| \leq 4 \Rightarrow |w| \geq \frac{1}{4}$. $\square$

Theorem.- $\forall f \in S$, $\forall$ $r = |z| < 1$,

$$\left| z \frac{f''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}.$$

Pf.- Fix $\zeta \in \mathbb{D}$, $|\zeta| = r$.

Consider $F_\zeta(z) = \dfrac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2) f'(\zeta)}$

$= z + A_2(\zeta) z^2 + \dots$

$$A_2(\zeta) = \frac{1}{2}\left((1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta} \right).$$

Hence,
$$\left| (1-|\zeta|^2) \frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta} \right| \le 4.$$

$$\equiv \quad \left| \zeta \frac{f''(\zeta)}{f'(\zeta)} - \frac{2r}{1-r^2} \right| \le \frac{4r}{1-r^2} \quad \square.$$

The distortion thm.

$\Rightarrow S$ is normal... and compact!!!

$f \in S$, $|z| = r < 1$.

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

Pf. $f \in S \Rightarrow f \in \mathcal{U}(\mathbb{D}) \Rightarrow f' \neq 0$.

$\Rightarrow \exists \log f'(z)$, analytic, that vanishes at $z=0$. $\equiv \log 1 = 0!$.

Note that

$$\frac{\partial}{\partial r} \operatorname{Re} \log(f(re^{i\theta})) = \frac{\partial}{\partial r} \frac{1}{2} \log |f'(re^{i\theta})|^2$$

$$= \frac{1}{2} \frac{\partial}{\partial r} \log f'(re^{i\theta}) \overline{f'(re^{-i\theta})}$$

$$= \frac{1}{2} \frac{\partial}{\partial r} \log f'(re^{i\theta}) + \frac{1}{2} \frac{\partial}{\partial r} \log \overline{f'(re^{i\theta})}$$

$$= \frac{1}{2} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} e^{i\theta} + \frac{1}{2} \frac{\partial}{\partial r} \overline{\left(\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right)} e^{-i\theta}$$

$$= \operatorname{Re}\left\{ e^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\}.$$

thus, $r \frac{\partial}{\partial r} \operatorname{Re}\{\log f'(z)\} = \operatorname{Re}\left\{ z \frac{f''(z)}{f'(z)} \right\}.$

Note that $|\alpha| \leq c \Rightarrow |\operatorname{Re}\alpha| \leq c \Rightarrow -c \leq \operatorname{Re}\alpha \leq c$

(Previous thm) $\Rightarrow$ $\frac{2r^2-4r}{1-r^2} \leq \operatorname{Re}\left\{ \frac{z f''(z)}{f'(z)} \right\} \leq \frac{2r^2+4r}{1-r^2}$

$$\equiv \frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r} \log |f'(re^{i\theta})| \leq \frac{2r+4}{1-r^2}$$

$\int_0^r \Rightarrow \log \frac{1-r}{(1+r)^3} \leq \log |f'(re^{i\theta})| \leq \log \frac{1+r}{(1-r)^3}$. □

the growth thm.

$f \in S$, $\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$, $z \in \mathbb{D}, |z| = r$.

Pf. $f(0)=0$.

Hence

$$|f(z)| = \left|\int_0^{|z|} f'(s)\,ds\right| \leq \int_0^r \frac{1+|s|}{(1-|s|)^3}\,|ds| = \frac{r}{(1-r)^2}.$$

Now, note that if $|f(z)| \geq \frac{1}{4}$,

$$\frac{r}{(1+r)^2} < \frac{1}{4} \leq |f(z)| \quad \forall z \in \mathbb{D}!$$

So that we can assume $|f(z)| < \frac{1}{4}$.

But then, $[0, f(z)] \subset f(\mathbb{D})$ so that the curve $\Gamma = f^{-1}([0, f(z)])$ is simple, contained in $\mathbb{D}$ and joints 0 and z.

Being $\mathbb{D}$ simply connected, we have

$$|f(z)| = \left|\int_\Gamma f'(s)\,ds\right| \underset{\uparrow}{=} \int_\Gamma |f'(s)||ds| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3}\,d\rho$$

$\operatorname{Arg}(f'(s)\,ds) = \text{constant!!!}$ ← $u = f(s) = As$. $du = f'(s)\,ds = A\,ds$, $ds \in \mathbb{R}$!

$$= \frac{r}{(1+r)^2}$$

□

Another theorem is:

<u>thm</u>.- $f \in S$; $z \in \mathbb{D}$. then

$$\frac{1-r}{1+r} \leq \left| z \frac{f'(z)}{f(z)} \right| \leq \frac{1+r}{1-r}, \quad |z| = r$$

<u>Pf</u>.- Again, for $\zeta \in \mathbb{D}$,

$$F(\zeta) = \frac{f\left(\frac{\zeta+z}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)\, f'(\zeta)}.$$

then,

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F(-\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}$$

$$\| \qquad \frac{|f(\zeta)|}{(1-|\zeta|^2)|f'(\zeta)|}.$$

$$\equiv \quad \frac{1-|\zeta|}{1+|\zeta|} \leq \left| \frac{f(\zeta)}{\zeta f'(\zeta)} \right| \leq \frac{1+|\zeta|}{1-|\zeta|}. \qquad \square.$$

Something about integral means.

Note that from the growth thm,

we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq \frac{r}{(1-r)^2}.$$

A stronger estimate holds.

Thm.- Let $f \in S$, $0<r<1$.

then $$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \le \frac{r}{1-r}.$$

Pf.- Since $f \in S$, $g(z) = \sqrt{f(z^2)} \in S$.

and $$|g(z)|^2 = |f(z^2)| \le \frac{r^2}{(1-r^2)^2}.$$

$$\Rightarrow \quad |g(z)| \le \frac{r}{1-r^2}.$$

then, $$g(\underbrace{\{|z|<r\}}_{D_r}) \subset \left\{ |z| < \frac{r}{1-r^2} \right\}.$$

Hence, $$A(D_r) \le \pi \frac{r^2}{(1-r^2)^2}.$$

On the other hand, $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$,

$$A(D_r) = \frac{1}{2i}\int_0^{2\pi} \bar{g}\frac{dg}{d\theta} = \pi \sum_{j=1}^{\infty} j|c_j|^2 r^{2j}.$$

therefore,

$$\sum_{j=1}^{\infty} j|c_j|^2 r^{2j-1} \le \frac{r}{(1-r^2)^2}.$$

On the other hand,

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|\, d\theta = \sum_{j=1}^{\infty} |c_j|^2 r^{2j} = r\sum_{j=1}^{\infty} |c_j| r^{j-1}$$

$$\le \frac{r^2}{1-r^2}.$$

We then have,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{2i\theta})|\, d\theta \underset{\uparrow\, t=2\theta}{=} \frac{1}{2\pi}\cdot\frac{1}{2}\int_0^{4\pi} |f(r^2 e^{2it})| dt$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}\cdot 2\int_0^{2\pi} |f(r^2 e^{it})\, dt \le \frac{r^2}{1-r^2}.$$

□

We have proved that $|a_2| \le 2 \ \forall f \in S$. (Bieberbach did, in fact). He suggested in a footnote "Dass $k_n \geq n$ zeigt das Beispiel $\sum n z^n$. Vielleicht ist überhaupt $k_n = n$", where

$$k_n = \max_{f \in S} |a_n(f)|$$

this is the source of his famous conjecture from 1916.

this conjecture was proved by De Branges in 1984.

Thm.- If $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in S$, then

$$|a_j| \le ej \quad \forall j \ge 2.$$

Pf.

$$a_j = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}}\, dz$$

$$\Rightarrow |a_j| \le \frac{1}{2\pi r^j} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta \le \frac{1}{r^{j-1}(1-r)}$$

$z = re^{i\theta}$

$dz = rie^{i\theta} d\theta \to |dz| = r\,d\theta$

$\forall\, 0 < r < 1$!

Hence
$$|a_j| \le \frac{1}{\max\limits_{0<r<1} r^{j-1}(1-r)} = j\left(1 + \frac{1}{j-1}\right)^{j-1} \overset{(*)}{<} ej. \qquad \square$$

(*) Arithmetic-Geometric mean:

$$\sqrt[n+1]{x_1 \cdots x_{n+1}} \le \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}.$$

Choose $x_1 = 1$, $x_2 = x_3 = \cdots = x_{n+1} = 1 + \frac{1}{n}$.

$$\left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}} \le \frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} = \frac{n+2}{n+1} = 1 + \frac{1}{n+1}.$$

$$(1-A)^n \geq 1 - nA.$$

$$\max_{1<r<1} r^{j-1}(1-r) = (1-r)e^{(j-1)\log r} = \varphi(r)$$

$$\varphi'(r) = -e^{(j-1)\log r} + \frac{(1-r)(j-1)}{r} e^{(j-1)} = 0$$

$$\Leftrightarrow (j-1)\frac{(1-r)}{r} = 1 . \equiv (j-1) - (j-1)r = r$$

$$\equiv j-1 = (j-1+1)r \Rightarrow r = \frac{j-1}{j}. \quad \nearrow \searrow.$$

$$\left(\frac{j-1}{j}\right)^{j-1}\left(1-\frac{j-1}{j}\right) = \frac{1}{j}\cdot\left(\frac{j-1}{j}\right)^{j-1}.$$

$$\frac{1}{m_1} \quad j\cdot\left(\frac{j}{j-1}\right)^{j-1} = j\left(1+\frac{1}{j-1}\right)^{j-1}.$$