$p > 1$ $\quad \int_{\mathbb{D}} |f|^p (1-|z|)^\alpha \, dA$.

$(A^p_\alpha)^* \simeq A^{p'}_\alpha$, $\frac{1}{p} + \frac{1}{p'} = 1$, pairing

pairing $\quad (A^p_w)^* \simeq ?$ $\quad$ WANKER = PAJERO

Hilbert Spaces $\quad$ Genkai Zhang

An inner product (scalar/dot product) on a vector space X is a scalar valued function $\langle \cdot, \cdot \rangle$ on $X \times X$ such that

(1) for each $y \in X$, the function $x \mapsto \langle x, y \rangle$ is linear;

$(\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle)$

(2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\quad x, y \in X$;

$(\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \bar{\alpha} \overline{\langle y, x \rangle} = \bar{\alpha} \langle x, y \rangle)$

(3) $\langle x, x \rangle \geq 0$ for all $x \in X$;

(4) $\langle x, x \rangle = 0 \iff x = 0$. $\quad \langle x, x \rangle \in \mathbb{R}$

Note that by (1), $\langle 0, y \rangle = \langle 0 \cdot 0, y \rangle = 0 \langle 0, y \rangle = 0$ for any $y \in X$, hence also $\langle y, 0 \rangle = 0$ by (2).

Ex $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by

$\langle x, y \rangle = \sum_{j=1}^n x_j \overline{y_j}$, $\quad x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

is an inner product.

More generally, if X is an n-dimensional vector space with basis $\{e_1, \dots, e_n\}$, then $x, y \in X$ can be represented in the form

$$x = \sum \lambda_k e_k, \quad y = \sum \mu_k e_k,$$

and the function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ defined by

$$\langle x, y \rangle = \sum_{k=1}^{n} \lambda_k \overline{\mu_k}$$

is an inner product on X. Note that $\langle \cdot, \cdot \rangle$ depends on the basis chosen.

$\boxed{4}$

<u>Theorem 1.12 (Cauchy-Schwarz inequality)</u>

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space X.

(i) For $x, y \in X$ we have

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle};$$

(ii) The function $x \mapsto \|x\| = \sqrt{\langle x, x \rangle}$ is a norm on X.

<u>Proof</u> (i) If $\langle y, y \rangle = 0$, then $y = 0$ and the inequality is satisfied. Assume that $\langle y, y \rangle > 0$. Then

$$0 \leq \left\langle x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y, x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\rangle$$

$$= \left\langle x, x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \left\langle y, x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y \right\rangle$$

$$= \overline{\left\langle x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y, x \right\rangle} - \frac{\langle x, y \rangle}{\langle y, y \rangle} \overline{\left\langle x - \frac{\langle x, y \rangle}{\langle y, y \rangle} y, y \right\rangle}$$

$$= \overline{\langle x, x \rangle} - \frac{\langle x, y \rangle}{\langle y, y \rangle} \overline{\langle y, x \rangle} - \frac{\langle x, y \rangle}{\langle y, y \rangle} \overline{\langle x, y \rangle} + \frac{\langle x, y \rangle}{\langle y, y \rangle} \frac{\langle x, y \rangle}{\cancel{\langle y, y \rangle}} \cancel{\langle y, y \rangle}$$

$$= \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}\langle y,x\rangle}{\langle y,y\rangle} - \cancel{\frac{|\langle x,y\rangle|}{\langle y,y\rangle}} + \cancel{\frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}}$$

$$= \langle x,x\rangle - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle}$$

$$= \langle x,x\rangle - \frac{|\langle x,y\rangle|^2}{\langle y,y\rangle}$$

and the assertion follows.

(ii) For $x,y \in X$ we have

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,y\rangle + \overline{\langle x,y\rangle} \\
&= \langle x,x\rangle + \langle y,y\rangle + 2\,\mathrm{Re}\langle x,y\rangle \\
&\le \langle x,x\rangle + \langle y,y\rangle + 2|\langle x,y\rangle| \\
&\overset{(i)}{\le} \langle x,x\rangle + \langle y,y\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \\
&= \left(\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}\right)^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

The other properties are trivial

- $\|x\| = \sqrt{\langle x,x\rangle} \ge 0$ by the def of $\langle\cdot,\cdot\rangle$,
- $\|x\| = 0 \iff \langle x,x\rangle = 0 \iff x = 0$ — " —
- $\|\lambda x\| = (\langle \lambda x, \lambda x\rangle)^{\frac12}$
  $= (\lambda\langle x, \lambda x\rangle)^{\frac12}$
  $= (\lambda\bar\lambda\langle x,x\rangle)^{\frac12} = |\lambda|\,\langle x,x\rangle^{\frac12} = |\lambda|\,\|x\|$. □

The norm $\|x\| = \langle x,x\rangle^{\frac12}$ on X is called the induced norm (by the inner product). If not otherwise stated, this is always the norm we use in an inner product space. Thus Theorem 1.12. (i) reads

$$|\langle x,y\rangle| \le \|x\|\,\|y\|, \quad x,y \in X \qquad (*)$$

and is sometimes called the Cauchy-Schwarz inequality.

This inequality implies that the function $\langle\cdot,\cdot\rangle : (X,\|\cdot\|)\times(X,\|\cdot\|) \longrightarrow \mathbb{F}$ is continuous. In particular, for $y \in X$ fixed,

$x \mapsto \langle x, y\rangle$ is a continuous linear functional.

Each inner product induces a norm as we have already seen, but each norm is not induced by an inner product.

<u>Lemma 1.13</u> Let $(X, \|\cdot\|)$ be an inner product space and $x, y, u, v \in X$. Then the following assertions hold:

(a) $\langle u+v, x+y\rangle - \langle u-v, x-y\rangle = 2\langle u, y\rangle + 2\langle v,$

(b) $\langle u+v, x+y\rangle - \langle u-v, x-y\rangle$
$+ i\langle u+iv, x+iy\rangle - i\langle u-iv, x-iy\rangle = 4\langle u, y\rangle,$
$\mathbb{K} = \mathbb{C}$

<u>Proof</u> (a) $\langle u+v, x+y\rangle - \langle u-v, x-y\rangle$
$= \cancel{\langle u, x\rangle} + \langle u, y\rangle + \langle v, x\rangle + \cancel{\langle v, y\rangle}$
$- \cancel{\langle u, x\rangle} + \langle u, y\rangle + \langle v, x\rangle - \cancel{\langle v, y\rangle}$
$= 2\langle u, y\rangle + 2\langle v, x\rangle$

(b) $i\langle u+iv, x+iy\rangle - i\langle u-iv, x-iy\rangle$
$= i[\cancel{\langle u, x\rangle} - i\langle u, y\rangle + i\langle v, x\rangle - ii\cancel{\langle v, y\rangle}]$

$- i[\cancel{\langle u, x\rangle} + i\langle u, y\rangle - i\langle v, x\rangle - ii\cancel{\langle v, y\rangle}]$

$= 2\langle u, y\rangle - 2\langle v, x\rangle,$

and assertion follows by this and (a). $\square$

<u>Theorem 1.14</u> Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$

Then the following assertions hold:

(a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$;

parallelogram rule

(b) If $\mathbb{K} = \mathbb{R}$, then

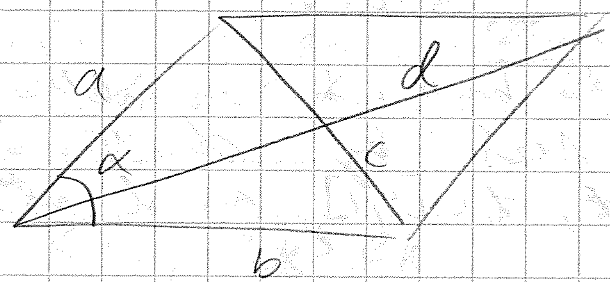
$4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$

(c) If $\mathbb{K} = \mathbb{C}$, then

$4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$

The identities (b) and (c) are called the <u>polarization identity</u>.

$X = \mathbb{R}^2$



$c^2 + d^2 = 2(a^2+b^2)$

<u>Lause</u> Jos $c = d$, niin $\alpha = 90°$.

<u>Tod</u>

$c = d \Rightarrow \cancel{2}c^2 = \cancel{2}(a^2+b^2)$

$\Rightarrow c^2 = a^2+b^2$

Käänteinen Pythagoras

$\Rightarrow$

$\alpha = 90°$

Jos $\alpha = 90°$, niin $c = d$.

$c^2 + d^2 = 2(a^2+b^2)$

$\alpha = 90° \Rightarrow - \quad c^2 = a^2+b^2$

$d^2 = a^2+b^2 = c^2 \Rightarrow c = d$

If $(X, \|\cdot\|)$ is a normed space and the norm satisfies the parallelogram rule, then the function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ defined by the polarization identity is an inner product.

<u>Ex 1.15</u> $C[0,1]$ endowed with the standard norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| = \max_{t \in [0,1]} |f(t)|$$

is not defined by an inner product. This can be seen by choosing $f(t) = 1$, $g(t) = t$, $t \in [0,1]$, and noting that

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 2 + 1 = 3 \neq 4 = 2(1+1) = 2(\|f\|_\infty^2 + \|g\|_\infty^2)$$

Thus the parallelogram rule does not hold and hence the norm cannot be induced by an inner product.

<u>Orthogonality</u>

If $\mathbb{K} = \mathbb{R}$, then the Cauchy-Schwartz inequality (*) implies

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \leq 1, \quad x, y \in X \setminus \{0\}.$$

Therefore the "angle" between x and y can be defined to be

$$\theta = \cos^{-1}\left(\frac{\langle x, y \rangle}{\|x\| \, \|y\|} \right).$$

If $\mathbb{K} = \mathbb{C}$, the situation is more complex. However, an important special case can be considered: if $\langle x, y \rangle = 0$, the vectors x and y are regarded perpendicular or orthogonal, denoted by $x \perp y$. Moreover, the set $\{e_1, \dots, e_n\} \subset X$ is orthonormal if $\|e_k\| = 1$ for all $k = 1, \dots, n$, and $\langle e_m, e_n \rangle = 0$ for all $m \neq n$.

Lemma 1.16 (a) An orthonormal set $\{e_1, \dots, e_n\}$ in an inner product space X is linearly independent. In particular, if $\dim X = n$, the set $\{e_1, \dots, e_n\}$ is an orthonormal basis of X and each $x \in X$ can be written in the form

$$x = \sum_{k=1}^{n} \langle x_n, e_k \rangle e_k$$

where the scalars $\langle x, e_k \rangle$ are the components of x with respect to the basis $\{e_1, \dots, e_n\}$.

(b) Let $\{v_1, \dots, v_n\}$ be a linearly independent subset of an inner product space X, and let $S = \operatorname{span}\{v_1, \dots, v_n\}$. Then there is an orthonormal basis $\{e_1, \dots, e_n\}$ of S.

Proof If $\sum_{k=1}^{n} \alpha_k e_k = 0$ for some $\alpha_k \in \mathbb{K}$, then

$$0 = \langle 0, e_m \rangle = \left\langle \sum_{k=1}^{n} \alpha_k e_k, e_m \right\rangle = \alpha_m, \quad m = 1, \dots, n,$$

and thus $\{e_1, \dots, e_n\}$ is linearly independent. Now, if $\{e_1, \dots, e_n\}$ is a basis and $x \in X$, there exists $\lambda_k = \lambda_k(x) \in \mathbb{K}$ s.th.

$x = \sum_{k=1}^{n} \lambda_k e_k$. Then

$$\langle x, e_m \rangle = \left\langle \sum_{k=1}^{n} \lambda_k e_k, e_m \right\rangle = \sum_{k=1}^{n} \lambda_k \langle e_k, e_m \rangle = \lambda_m,$$

$m = 1, \dots, n.$

(b) The proof is by induction on n.

If $n=1$, we can take $e_1 = \frac{v_1}{\|v_1\|}$, since $v_1 \neq 0$. $\{e_1\}$ is the desired basis.

Now suppose the assertion is true for some $k \in \mathbb{N}$. Let $\{v_1, \ldots, v_{k+1}\}$ be a linearly independent set and let $\{e_1, \ldots, e_k\}$ be the orthonormal basis of $\operatorname{span}\{v_1, \ldots, v_k\}$. Since $\{v_1, \ldots, v_{k+1}\}$ is linearly independent, $v_{k+1} \notin \operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{e_1, \ldots, e_k\}$. Set

$$b_{k+1} = v_{k+1} - \sum_{j=1}^{n} \langle v_{k+1}, e_j \rangle e_j .$$

Then $b_{k+1} \in \operatorname{span}\{v_1, \ldots, v_{k+1}\}$ and $b_{k+1} \neq 0$, for otherwise we would have $v_{k+1} \in \operatorname{span}\{e_1, \ldots, e_k\}$. Also, for each $m = \{1, \ldots, k\}$,

$$\langle b_{k+1}, e_m \rangle = \langle v_{k+1}, e_m \rangle - \sum_{j=1}^{k} \langle v_{k+1}, e_j \rangle \langle e_j, e_m \rangle$$

$$= \langle v_{k+1}, e_m \rangle - \langle v_{k+1}, e_m \rangle = 0 .$$

Thus $b_{k+1} \perp e_j$ for all $j = 1, \ldots, k$. Set $e_{k+1} = \frac{b_{k+1}}{\|b_{k+1}\|}$. Then $\{e_1, \ldots, e_{k+1}\}$ is an orthonormal set with $\operatorname{span}\{e_1, \ldots, e_{k+1}\} \subset \operatorname{span}\{v_1, \ldots, v_{k+1}\}$. But the dimensions of both of these sets is $k+1$, thus they are equal. $\square$

The formula

$$b_{k+1} = v_{k+1} - \sum_{j=1}^{k} \langle v_{k+1}, e_j \rangle e_j , \quad e_{k+1} = \frac{b_{k+1}}{\|b_{k+1}\|} ,$$

is called the Gram-Schmidt algorithm.

The following result is a generalization of Pythagoras' theorem.

$\boxed{5}$

<u>Theorem 1.17</u> Let X be an n-dimensional inner product space and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of X. Then for $\alpha_j \in \mathbb{K}$, $j = 1, \dots, n$, we have

$$\Big\| \sum_{j=1}^{n} \alpha_j e_j \Big\|_X^2 = \sum_{j=1}^{n} |\alpha_j|^2.$$

<u>Proof</u> Clearly

$$\Big\| \sum_{j=1}^{n} \alpha_j e_j \Big\|_X^2 = \Big\langle \sum_{j=1}^{n} \alpha_j e_j, \sum_{k=1}^{n} \alpha_k e_k \Big\rangle$$

$$= \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{n} \overline{\alpha_k} \langle e_j, e_k \rangle = \sum_{j=1}^{n} \alpha_j \overline{\alpha_k} = \sum_{j=1}^{n} |\alpha_j|^2.$$

$\square$

<u>Remark</u> One can use induction to see that if X is an inner product space, $\{x_1, \dots, x_n\} \subset X$ and $x_j \perp x_k$, $j \neq k$, then

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2 \quad \text{HT}$$

Complete inner product spaces are called <u>Hilbert spaces</u>. For example $L^2_\mu(X)$ and l^2 are Hilbert. Also each finite dimensional inner product space is Hilbert. But l^p and L^p are not Hilbert unless $p = 2$.

<u>Lemma 1.18</u> A linear subspace Y of a Hilbert space X is Hilbert $\Leftrightarrow$ Y is closed in X.

<u>Orthogonal complements.</u>

Let X be an inner product space and $A \subset X$. The orthogonal complement of A is the set

$A^{\perp} = \{x \in X : \langle x, a \rangle = 0$ for all $a \in A\}$.

The set of those $x \in X$ that are orthogonal to every vector in A.

$A = \emptyset$: $A^{\perp} = X$

<u>Ex</u> $X = \mathbb{R}^3$, $A = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$,
$A^{\perp} = \{(0, 0, a_3) : a_3 \in \mathbb{R}$.

If X is an inner product space, $\dim X = n \in \mathbb{N} \setminus \{1\}$, $\{e_1, ..., e_n\}$ is an orthonormal basis of X and $A = \operatorname{span}\{e_1, ..., e_p\}$, $1 \leq p \leq n$, then $A^{\perp} = \operatorname{span}\{e_{p+1}, ..., e_n\}$.

<u>Lemma 1.19</u> Let X be an inner product space and $A \subset X$. Then

(a) $0 \in A^{\perp}$,
(b) If $0 \in A$, then $A \cap A^{\perp} = \{0\}$, for otherwise $A \cap A^{\perp} = \emptyset$,
(c) $\{0\}^{\perp} = X$, $X^{\perp} = \{0\}$,
(d) If $A \supset B(a, r)$ (open ball centered at a and of radius r) then $A^{\perp} = \{0\}$; in particular, if $A \neq \emptyset$ is open, then $A^{\perp} = \{0\}$,
(e) If $B \subset A$, then $A^{\perp} \subset B^{\perp}$,
(f) $A^{\perp}$ is a closed linear subspace of X,
(g) $A \subset (A^{\perp})^{\perp}$.

<u>Proof</u> (a) $\langle 0, a \rangle = 0$ for all $a \in A \Rightarrow 0 \in A$

(b) If $x \in A \cap A^{\perp}$, then $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

(c) If $A = \{0\}$, then $\langle x, a \rangle = \langle x, 0 \rangle = 0 \quad \forall x \in X$ and hence $A^{\perp} = X$.
If $A = X$ and $x \in A^{\perp}$ then $\langle x, a \rangle = 0$ for all $a \in X$. In particular $\langle x, x \rangle = 0$ and hence $x = 0$. Thus $A^{\perp} = \{0\}$

(d) Suppose that $x \in A^\perp \setminus \{0\}$ and let $y = \frac{x}{\|x\|}$. If $b \in A$, then

$$\langle y, b\rangle = \left\langle \frac{x}{\|x\|}, b \right\rangle = \frac{1}{\|x\|}\langle x, b\rangle = 0.$$

But since $a + \frac{1}{2} r y \in A$, we must have

$$0 = \langle y, a + \tfrac{1}{2} r y\rangle = \langle y, a\rangle + \tfrac{1}{2} r\langle y, y\rangle = \tfrac{r}{2}\langle y, y\rangle$$

which implies $\langle y, y\rangle = 0$ and hence $y = 0$. This is a contradiction.

(e) Let $x \in A^\perp$ and $b \in B$. Then $b \in A$ so $\langle x, b\rangle = 0$. This holds for each $b \in B$, and so $x \in B^\perp$. Hence $A^\perp \subset B^\perp$.

(f) Let $x, y \in A^\perp$, $\alpha, \beta \in \mathbb{K}$ and $a \in A$. Then

$$\langle \alpha x + \beta y, a\rangle = \alpha\langle x, a\rangle + \beta\langle y, a\rangle = 0 + 0 = 0,$$

so $\alpha x + \beta y \in A^\perp$ and hence $A^\perp$ is a linear subspace of X. Next, let $\{x_n\}$ be a sequence in $A^\perp$ converging to $x \in X$. Then by the continuity of $\langle \cdot, \cdot \rangle$, we have for each $a \in A$

$$0 = \lim_{n\to\infty}\langle x - x_n, a\rangle = \langle x, a\rangle - \lim_{n\to\infty}\langle x_n, a\rangle = \langle x, a\rangle$$

Hence $x \in A^\perp$ and so $A^\perp$ is closed.

(g) Let $a \in A$, then for all $x \in A^\perp$ $\langle a, x\rangle = \overline{\langle x, a\rangle} = 0$, so $a \in (A^\perp)^\perp$. Thus $A \subset (A^\perp)^\perp$.

Note that if $y_1, y_2 \in X$ and $\langle x, y_1\rangle = \langle x, y_2\rangle$ for all $x \in X$, then $y_1 = y_2$. Namely

$0 = \langle x, y_1 - y_2\rangle$ for all $x \in X$ and hence

$y_1 - y_2 \in X^\perp = \{0\}$, thus $y_1 = y_2$. □

Lemma 1.20 Let Y be a linear subspace of an inner product space X. Then $x \in Y^{\perp}$ if and only if $\|x-y\| \geq \|x\|$ for all $y \in Y$.

Proof Let first $x \in Y^{\perp}$ and $y \in Y$. Then

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 - 0 - 0 + \|y\|^2 \geq \|x\|^2,$$

from which we deduce what we want.

Assume now that $x \in X$ and $\|x-y\|^2 \geq \|x\|^2$ for all $y \in Y$. Then, as Y is a linear space,

$$0 \leq \|x - \alpha y\|^2 - \|x\|^2 = -\langle x, \alpha y\rangle - \langle \alpha y, x\rangle + \|y\|^2 |\alpha|^2, \alpha \in$$

Let
$$\beta = \begin{cases} \dfrac{|\langle x,y\rangle|}{\langle y,x\rangle}, & \langle x,y\rangle \neq 0 \\ 1, & \langle x,y\rangle = 0 \end{cases}$$

so that $\beta\langle y,x\rangle = |\langle x,y\rangle|$, and let $\alpha = t\beta$, where $t>0$. Then

$$0 \leq -\overline{\alpha\langle y,x\rangle} - \alpha\langle y,x\rangle + |\alpha|^2\|y\|^2$$
$$= -t|\langle x,y\rangle| - t|\langle x,y\rangle| + t^2\|y\|^2$$

$|\langle x,y\rangle| \leq \frac{1}{2}t\|y\|^2$ for all $t>0$. By letting $t \to 0+$, we deduce $\langle x,y\rangle = 0$ and that $x \in Y^{\perp}$. $\square$

Let X be Hilbert and $A \subset X$. When do we find $x_0 \in A$ such that $\|x_0\| = \inf\{\|x\| : x \in A\}$?

Ex Let $e_n = (0, \ldots, 0, \overset{n}{1}, 0, \ldots) \in \ell^2$ and

$$A = \left\{ \frac{n+2}{n+1} e_n : n \in \mathbb{N} \right\}.$$ Now

$$\left\| \frac{n+2}{n+1} e_n \right\|_{\ell^2}^2 = \left(\frac{n+2}{n+1}\right)^2 \leq \frac{9}{4}.$$

$$\boxed{\ell^2 : \langle x, y\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}}$$

and hence A is bounded. Also

$$\left\| \frac{n+2}{n+1} e_n - \frac{m+2}{m+1} e_m \right\|_{\ell^2}^2 = \begin{cases} \left(\frac{n+2}{n+1}\right)^2 + \left(\frac{m+2}{m+1}\right)^2 > 2 & m \neq n \\ 0 & m = n \end{cases}$$

and so A is closed. BUT there is no $x \in A$ such that $\|x\| = \inf\{\|y\| : y \in A\} = 1$ $= \lim_{n\to\infty} \left\| \frac{n+2}{n+1} e_n \right\|_{\ell^2}$.

If A is compact, then the minimizing vector x_0 exists, of course, but requiring the compactness would be too restrictive for many purposes.

OK. Note that the closed unit ball

$\overline{B_{\ell^2}} = \{x \in \ell^2 : \|x\|_{\ell^2} \le 1\}$ is not compact,

since $\|e_n - e_m\|_{\ell^2} = \sqrt{2}$, $n \neq m$, when $e_n = (0, \dots, 0, 1, 0, \dots)$.

In Hilbert spaces the minimizing vector x_0 can be found if A is convex. Recall that a subset A of a vector space X is <u>convex</u>, if

$$\lambda x + (1-\lambda) y \in A, \quad x, y \in A, \quad \lambda \in [0,1].$$

Note that open and closed balls are convex.

[$x, y \in B(a, r) \Rightarrow \begin{cases} \|x - a\| < r \\ \|y - a\| < r \end{cases}$

$$\begin{aligned} \|\lambda x + (1-\lambda) y - a\| &= \|\lambda (x-a) + (1-\lambda)(y-a)\| \\ &\le \lambda \|x-a\| + (1-\lambda)\|y-a\| \\ &< \lambda r + (1-\lambda) r = r \quad \text{ok} \end{aligned}$$
]

$\boxed{6}$ Theorem 1.21 Let X be Hilbert, and $A \subset X$, $A \neq \emptyset$, closed and convex. If $p \in X$, then there exists a unique $q \in A$ s.th.

$$\|p - q\| = \inf\{\|p - a\| : a \in A\} = \mathrm{dist}(p, A)$$

If $\dim X < \infty$, then such a q exists whenever $A \neq \emptyset$ is $\underline{\underline{\text{closed}}}$, but it is not necessarily unique. In infinite dimensions closed + bdd $\not\Rightarrow$ compact.

Proof. Set $\delta = \inf\{\|p - a\| : a \in A\}$. Then, for each $n \in \mathbb{N}$, there exists $q_n \in A$ s.th.

$$\delta^2 \le \|p - q_n\|^2 < \delta^2 + \frac{1}{n} \qquad (*)$$

We will show that q_n is Cauchy. By the parallellogram rule,

$$\|(p - q_n) + (p - q_m)\|^2 + \|(p - q_n) - (p - q_m)\|^2$$

$$= 2\underbrace{\|p - q_n\|^2}_{< \delta^2 + \frac{1}{n}} + 2\underbrace{\|p - q_m\|^2}_{< \delta^2 + \frac{1}{m}}$$

$$\Leftrightarrow \quad \|2p - (q_n + q_m)\|^2 + \|q_n - q_m\|^2 < 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right)$$

Since A is convex and $q_n, q_m \in A$, $\frac{1}{2}(q_n + q_m) \in A$, and hence

$$\|2p - (q_n + q_m)\|^2 = 4\left\|p - \frac{q_n + q_m}{2}\right\|^2 \ge 4\delta^2$$

Therefore

$$\|q_n - q_m\|^2 < 4\delta^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - \|2p - (q_n + q_m)\|$$

$$\le 2\left(\frac{1}{n} + \frac{1}{m}\right),$$

and thus $\{q_n\}$ is Cauchy. Since X is complete, there exists $q \in X$ s.th. $q_n \to q$, $n \to \infty$

Further, as A is closed, $q \in A$, and further, by (*)

$$\gamma^2 \le \lim_{n\to\infty} \|p - q_n\|^2 = \|p-q\|^2 \le \lim_{n\to\infty}\left(\gamma^2 + \frac{1}{n}\right) = \gamma^2,$$

and hence $\|p-q\| = \gamma$.

It remains to show that q is unique. Let $w \in A$ s.th. $\|p-w\| = \gamma$. Then $\frac{1}{2}(q+w) \in A$ since A is convex, and so

$$\gamma \le \left\|p - \frac{1}{2}(q+w)\right\| = \left\|\frac{1}{2}(p-q) + \frac{1}{2}(p-w)\right\|$$

$$\le \frac{1}{2}\|p-q\| + \frac{1}{2}\|p-w\| = \frac{1}{2}\gamma + \frac{1}{2}\gamma = \gamma.$$

Further, by the parallelogram rule

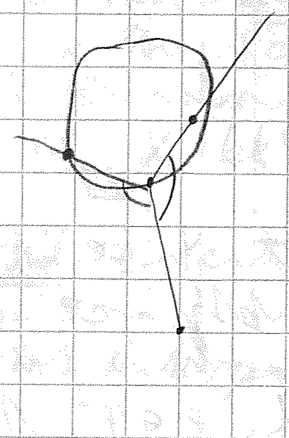
$$\|(p-w)+(p-q)\|^2 + \|(p-w)-(p-q)\|^2$$

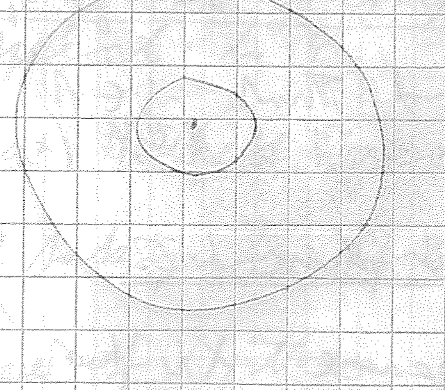
$$= 2\|p-w\|^2 + 2\|p-q\|^2$$

$$\iff \|q-w\|^2 = 2\gamma^2 + 2\gamma^2 - 4\left\|p - \frac{1}{2}(q+w)\right\|^2$$
$$\le 2\gamma^2 + 2\gamma^2 - 4\gamma^2 = 0.$$

Thus $w = q$, and hence q is unique. □

We have shown that the "minimizing" vector q exists and is unique, but we can actually recognize the point quite easily.





Theorem 1.22 Let X be Hilbert and $\emptyset \neq A \subseteq X$, closed and convex. Further, let $x \in X$ and $y \in A$. Then

$$\|x-y\| = \mathrm{dist}(x,A) \Leftrightarrow \mathrm{Re}\langle x-y, a-y\rangle \leq 0$$

for all $a \in A$ $\boxed{y = P(x) \in A}$

Proof Let first $\|x-y\| = \mathrm{dist}(x,A)$ and $0<t<1$. Then

$$y + t(a-y) = (1-t)y + ta \in A$$

by the convexity. Hence

$$\mathrm{dist}(x,A) = \|x-y\| \leq \|x-y-t(a-y)\|$$

$$\Leftrightarrow \cancel{\|x-y\|^2} \leq \cancel{\|x-y\|^2} - 2t\,\mathrm{Re}\langle x-y, a-y\rangle + t^2\|a-y\|^2$$

It follows that

$$\mathrm{Re}\langle x-y, a-y\rangle \leq \frac{t}{2}\|a-y\|^2, \quad 0<t<1.$$

By letting $t \to 0^+$, we deduce $\mathrm{Re}\langle x-y, a-y\rangle \leq 0$ for all $a \in A$.

Conversely assume that $\mathrm{Re}\langle x-y, a-y\rangle \leq 0$. Then

$$\begin{aligned}\|x-a\|^2 &= \|x-y-(a-y)\|^2 \\ &= \|x-y\|^2 \underbrace{-2\mathrm{Re}\langle x-y, a-y\rangle + \|a-y\|^2}_{\geq 0} \\ &\geq \|x-y\|^2, \quad a \in A,\end{aligned}$$

and thus $y \in A$ minimizes the norm. Hence $\|x-y\| = \mathrm{dist}(x,A)$. $\square$

Theorem 1.23 Let Y be a closed linear subspace of a Hilbert space X. For each $x \in X$ there exists unique $y \in Y$ and $z \in Y^\perp$ such that $x = y+z$. Also $\|x\|^2 = \|y\|^2 + \|z\|^2$.

Proof Since Y is convex, nonempty, closed set it follows from Theorem 1.21 that there exists a unique $y \in Y$ such that $\|x - y\| \leq \|x - u\|$ for all $u \in Y$. Set $z = x - y$. Then

$$\|z - u\| = \|x - \overbrace{(y + u)}^{\in Y}\| \geq \|x - y\| = \|z\|$$

Thus, by Lemma 1.20, $z \in Y^{\perp}$. Therefore the desired y and z exist. To prove the uniqueness, assume that $x = y_1 + z_1 = y_2 + z_2$ for $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^{\perp}$. Now

$$y_1 - y_2 = z_2 - z_1 \in Y \cap Y^{\perp} = \{0\}.$$

Hence $y_1 = y_2$ and $z_1 = z_2$.

Finally,

$$\|x\|^2 = \|y + z\|^2 = \langle y + z, y + z\rangle = \|y\|^2 + \cancel{\langle y, z\rangle} + \cancel{\langle z, y\rangle} + \|z\|^2$$
$$= \|y\|^2 + \|z\|^2 .$$

□

Ex $X = \mathbb{R}^2$, $Y = \{(x, 0) : x \in \mathbb{R}\}$, $Y^{\perp} = \{(0, y) : y \in \mathbb{R}\}$. Then Theorem 1.23 coincides with Pythagoras' theorem in $\mathbb{R}^2$. Thus Theorem 1.23 can be considered as a generalization of Pythagoras' theorem.

Let Y be a closed linear subspace of a Hilbert space X and let $x \in X$. The decomposition $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$ is called the orthogonal decomposition of x with respect to Y.

Theorem 1.24 Let X be a Hilbert space.

(a) If Y is a closed linear subspace of X, then $(Y^{\perp})^{\perp} = Y^{\perp\perp} = Y$

(b) If Y is a linear subspace of X, then $Y^{\perp\perp} = \overline{Y}$.