Proof (a) By Lemma 1.19 (g), $Y \subset Y^{\perp\perp}$. Conversely, if $x \in Y^{\perp\perp}$, then $x = y + z$, where $y \in Y$ and $z \in Y^{\perp}$, by Theorem 1.22. Since $y \in Y$ and $x \in Y^{\perp\perp}$, $\langle y, z\rangle = 0 = \langle x, z\rangle$. Thus

$$0 = \langle x, z\rangle = \langle y + z, z\rangle = \langle y, z\rangle + \langle z, z\rangle = \|z\|^2,$$

implying that $z = 0$ and $x = y \in Y$. Hence $Y^{\perp\perp} \subset Y$ and thus $Y = Y^{\perp\perp}$.

(b) Since $Y \subset \overline{Y}$, Lemma 1.19 (e) yields $\overline{Y}^{\perp} \subset Y^{\perp}$ and hence $Y^{\perp\perp} \subset \overline{Y}^{\perp\perp}$. But $\overline{Y}$ is closed, so Part (a) implies $\overline{Y}^{\perp\perp} = \overline{Y}$ and hence $Y^{\perp\perp} \subset \overline{Y}$.

On the other hand, by Lemma 1.19 (g), $Y \subset Y^{\perp\perp}$, but $Y^{\perp\perp}$ is closed, so $\overline{Y} \subset Y^{\perp\perp}$. Thus $Y^{\perp\perp} = \overline{Y}$. $\square$

Orthogonal projections We take a look to the orthogonal decomposition and the norm minimizing vector from the point of view of operator theory.

Theorem 1.25 Let X be a Hilbert space and $M \subset X$ its closed linear subspace. If $x \in X$ and $y \in M$, then

$$\|x - y\| = \mathrm{dist}(x, M) \iff (x - y) \perp M$$
$$\iff (x - y) \in M^{\perp}.$$

Proof Assume first that $\|x - y\| = \mathrm{dist}(x, M)$. If $z \in M$ and $\lambda \in \mathbb{K}$, then $y + \lambda z \in M$, and Theorem 1.22 yields

$$0 \geq \mathrm{Re}\,\langle x - y, (y + \lambda z) - y\rangle = \mathrm{Re}\,(\overline{\lambda}\langle x - y, z\rangle), \quad \lambda \in \mathbb{K}$$

By choosing $\lambda = \langle x - y, z\rangle$ we deduce $|\langle x - y, z\rangle|^2 \leq 0$ $\Leftrightarrow \langle x - y, z\rangle = 0$ for all $z \in M$. Thus $(x - y) \in M^{\perp}$

Conversely, assume $x - y \in M^\perp$. If $z \in M$, then $z - y \in M$ and so

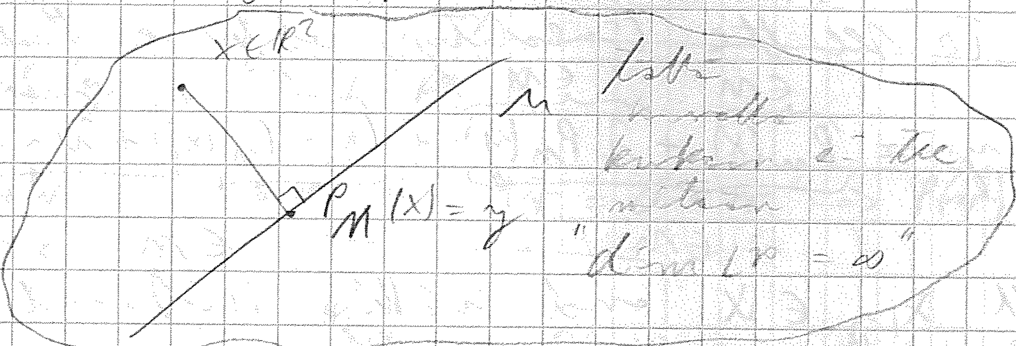
$$0 = \langle x-y, z-y\rangle = \mathrm{Re}\langle x-y, z-y\rangle \text{ for all } z \in M,$$

Theorem 1.22 yields $\|x-y\| = \mathrm{dist}(x, M)$. □

Let $M \subseteq X$ be a closed subspace of a Hilbert space X. Set $P_M : X \to X$ such that

$$P_M(x) = y \iff \begin{cases} y \in M \\ \|x-y\| = \mathrm{dist}(x, M) \end{cases} \quad y \in M$$
$$\iff ((x-y) \perp M).$$

We say that P_M is the orthogonal projection of the vector space X to the vector space M and $y = P_M(x)$ is the orthogonal projection of the vector x to M.



Theorem 1.25 shows that

(*) $y = P_M(x) \iff y \in M$ and $x - y \perp M$

In other words, (*) implies that the orthogonal projection is uniquely determined by

(**) $x = P_M(x) + \underbrace{(x - P_M(x))}_{= P_{M^\perp}(x)}$, where

$P_M(x) \in M$ and $x - P_M(x) \in M^\perp$ for all $x \in X$.
This is the essential content of Theorem 1.23 from which uniqueness follows. Thus, in the orthogonal decomposition,

$x = y + z$, $y \in M$, $z \in M^\perp$,

actually $y = P_M(x)$ and $z = x - P_M(x)$.
From above we deduce, in particular, that

$$P_M(x) = \begin{cases} 0, & x \in M^\perp \\ x, & x \in M \end{cases}$$

VARASIN: DEMOT 2 PE 24.1. klo 14-16 M 106

7 We have defined the orthogonal projection as a metric property but because of the vector space structure of Hilbert spaces we end up with linear operators.

Theorem 1.26 Let M be a closed subspace of a Hilbert space X. Then $P_M : X \to X$ is linear.

Proof To see the linearity, observe that

$$x + \lambda y = \underbrace{\overbrace{P_M(x)}^{\in M} + \overbrace{\lambda P_M(y)}^{\in M}}_{\in M} + \underbrace{\overbrace{(x - P_M(x))}^{M^\perp} + \overbrace{\lambda(y - P_M(y))}^{M^\perp}}_{\in M^\perp}$$

for all $x, y \in X$ and $\lambda \in \mathbb{K}$. This and the uniqueness in $(**)$ yield
$P_M(x + \lambda y) = P_M(x) + \lambda P_M(y)$. Thus P_M is linear. $\square$

Let X be a vector space. A linear map $P : X \to X$ is a projection, if $P^2 = P \circ P = P$. If $U = P(X)$ and $V = \ker(P) = \{x \in X : P(x) = 0\}$, we say that P is a projection of X to U to the direction (of) V

(P on projektio avaruudelle U suuntaan V).

A closely related concept from linear algebra is the so-called direct sum. We say that X is the direct sum of the subspaces U and V, denoted by $X = U \oplus V$, if

$X = U + V = \{u + v : u \in U, v \in V\}$ and $\Rightarrow U \cap V = \{0\}$.

If P is a projection of X to U to the direction (of) V, then $X = U \oplus V$. Namely if $x \in X$, then $x = P(x) + (x - P(x))$, where $P(x) \in U = P(X)$ and $x - P(x) \in \ker(P) = V$, because $P(x - P(x)) = P(x) - P^2(x) = 0$.

Conversely, if $X = U \oplus V$, then the direct sum defines a projection P to U to the direction V. Namely, in this case each $x \in X$ has a unique representation $x = u + v$, where $u \in U$ and $v \in V$. By setting $P(x) = u$, we deduce that P is a linear projection, $U = P(X)$ and $V = \ker(P)$. Linearity follows as in the proof of Theorem 1.26, $P(x) = u \in U$ thus $P(X) \subset U$, $u \in U \subset X$ so $P(u) = u \in P(X)$ thus $U \subset P(X)$ $\Rightarrow U = P(X)$, and clearly $V = \ker P$.

Linearity $\qquad V = \ker P$

$x = u + v \qquad V \ni v \Rightarrow v = 0 + v \Rightarrow P(v) = 0$

$y = a + b$

$x + y = \underbrace{(u + a)}_{U} + \underbrace{(v + b)}_{V} \Rightarrow P(x + y) = u + a = P(x) + P(y)$

$x = u + v$

$\alpha x = \alpha u + \alpha v \Rightarrow P(\alpha x) = \alpha u = \alpha P(x)$

Theorem 1.27 Let M be a closed subspace of a Hilbert space X. Then $X = M \oplus M^\perp$ and P_M is a projection of X to M to the direction $M^\perp$. Moreover,

$$\|P_M(x)\|_X \le \|x\|_X, \quad x \in X,$$

and hence $P_M : X \to M$ is a bounded/continuous linear operator.

Proof. By Lemma 1.19 $M^\perp$ is a (closed) subspace of X. Moreover, by (⋆⋆) each $x \in X$ can be represented uniquely in the form
$x = P_M(x) + (x - P_M(x))$, where $P_M(x) \in M$ and $x - P_M(x) \in M^\perp$. Thus $X = M + M^\perp$.
Moreover, if $x \in M \cap M^\perp$, then
$\langle x, x \rangle = \|x\|^2 = 0$, and hence $x = 0$. Thus $M \cap M^\perp = \{0\}$ and so $X = M \oplus M^\perp$.

Consider the representation
$x = P_M(x) + z$, where $z = x - P_M(x) \in M^\perp$.
Since $P_M(z) = 0$, the linearity yields
$P_M^2(x) = P_M(P_M(x)) = P_M(x - z) = P_M(x) - P_M(z) = P_M(x)$,
and thus P_M is a projection. Moreover, $P(X) = M$ follows by (⋆) by choosing $y = x$.
Clearly, $M^\perp \subset \ker(P_M) = \{x \in X : P_M(x) = 0\}$ and if $x \in X$ such that $P_M(x) = 0$, then the uniqueness in (⋆⋆) implies
$x = 0 + x \in M^\perp$.

$\underline{M = \ker(P_M)}$

ok

(⋆) $\overset{y=x}{\Longrightarrow}$ $0 = P_M(x) \iff 0 \in M$ & $x - 0 = x \perp M \iff x \in M^\perp$

Finally, we estimate the norm by using the Remark after Theorem 1.17 to deduce

$$\|x\|^2 = \|P_M(x) + (x - P_M(x))\|^2 = \|P_M(x)\|^2 + \|x - P_M(x)\|^2 \geq \|P_M(x)\|^2,$$

and the assertion follows. □

The last part of the proof also reveals that $\|P_M\| = 1$. ~~whenever~~ $M \neq X$. ?

Let $A \subset X$ be non-empty, closed and convex. By Theorem 1.21, there exists a metric projection $P_A : X \to X$ such that

$P_A(x) = q$, where $q \in A$ is the unique vector for which $\|x - q\| = \operatorname{dist}(x, A)$.

Metric projections are usually not linear, but they are always contractions

$\|P_A(u) - P_A(v)\| \le \|u - v\|, \quad u, v \in X.$

Compare with Proposition 1.10.

<u>Orthonormal bases</u> Recall that $\{e_n\}$ is called <u>orthonormal</u> if

$$\langle e_j, e_k \rangle = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

that is $e_j \perp e_k$, $j \neq k$, and $\|e_j\| = 1$ for all j.

<u>Ex</u> The vectors $e_n = (0, \dots, 0, \overset{n}{1}, 0, \dots)$ form a basis for ℓ^2. The space

$L^2(0, 2\pi)$ equipped with the inner product

$$\langle x, y \rangle = \int_0^{2\pi} x(t)\overline{y(t)}\, dt$$

is Hilbert, and $\{x_n\}_{n \in \mathbb{Z}}$, where

$$x_n = \frac{1}{\sqrt{2\pi}} e^{int} = \frac{1}{\sqrt{2\pi}}(\cos(nt) + i\sin(nt)), \quad n \in \mathbb{Z},$$

is orthonormal in $L^2(0, 2\pi)$. Namely if $n \neq m$, then

$$\langle x_n, x_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{int} \overline{e^{imt}} = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t}\, dt$$

$$= \frac{1}{2\pi i(n-m)} \Big|_0^{2\pi} e^{i(n-m)t} = 0.$$

Orthonormal sets (or sequences) and vectors norms determined by can be controlled by the Bessel's inequality

<u>Theorem 1.28 (Bessel's inequality)</u> Let $\{e_n\}_{n=1}^{\infty}$ is orthonormal in a Hilbert space X. Now

$$\text{(B)} \quad \sum_{k=1}^{\infty} \langle x, e_k \rangle^2 \le \|x\|^2$$

for all $x \in X$. In particular, $\lim_{k \to \infty} \langle x, e_k \rangle = 0.$

Before proving the theorem, we note that (B) holds also for finite orthonormal sets (with the same proof).

Proof. Consider the vectors

$$x_n' = x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k, \quad n \in \mathbb{N}.$$

Clearly,

$$\langle x_n', e_j\rangle = \langle x, e_j\rangle - \sum_{k=1}^{n} \langle x, e_k\rangle \langle e_k, e_j\rangle$$

$$= \langle x, e_j\rangle - \langle x, e_j\rangle \langle e_j, e_j\rangle = 0$$

and so $x_n' \perp e_j$ for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$. Since orthogonal complements are linear subspaces, we deduce

$$x_n' \perp \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

Therefore Pythagoras' theorem yields

$$\|x\|^2 = \|x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k\|^2 + \|\sum_{k=1}^{n} \langle x, e_k\rangle e_k\|^2$$

$$\geq \|\sum_{k=1}^{n} \langle x, e_k\rangle e_k\|^2$$

$$= \sum_{k=1}^{n} \|\langle x, e_k\rangle e_k\|^2 = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2,$$

and $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$ follows by letting $n \to \infty$. $\square$

$\square$

If $\{e_k\}$ is orthonormal in a Hilbert space X and $x \in X$, then the scalars $\langle x, e_k \rangle$ are called the Fourier coefficients of x with respect to $\{e_k\}$. $\boxed{8}$

Recall that if X is a vector space and $A \subseteq X$, then $\mathrm{span}(A)$, the linear span of A, is the linear subspace of X spanned by A:

$$\mathrm{span}(A) = \left\{ \sum_{j=1}^{n} \lambda_j a_j : n \in \mathbb{N}, \lambda_k \in \mathbb{K}, a_j \in A \right\},$$

and $\overline{\mathrm{span}}(A) = \overline{\mathrm{span}(A)}$ is the closure of $\mathrm{span}(A)$ in X. See Ex 2.

The following result gives a concrete and useful formula for the orthogonal projection to the linear space spanned by a finite orthonormal set in a Hilbert space X.

Theorem 1.29. Let X be Hilbert, $\{e_k\}_{k=1}^{n} \subseteq X$ orthonormal and $M = \mathrm{span}(\{e_1, \dots, e_n\})$. Then the following assertions hold:

(a) M is a closed subspace of X, that is, $M = \overline{\mathrm{span}}(\{e_1, \dots, e_n\})$;

(b) the orthogonal projection P_M of X to M is given by

$$(*) \quad P_M(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \quad x \in X.$$

Proof. The map P defined by

$$P(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \quad x \in X,$$

is linear by the linearity of $x \mapsto \langle x, e_k \rangle$, $k = 1, \dots, n$. Moreover, Pythagoras's theorem and Bessel's inequality yield

$$\| P(x) \|^2 \overset{(A)}{=} \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \overset{(B)}{\leq} \|x\|^2, \quad x \in X,$$

and hence P is continuous and bounded, and $\|P\| \le 1$. MORS THE

If $x \in M$, then $x = \sum_{k=1}^{n} \lambda_k e_k$

for some $\lambda_k \in \mathbb{K}$ (and $n \in \mathbb{N}$). It follows that (see the proof of Lemma 1.16)

$$\langle x, e_j \rangle = \langle \sum_{k=1}^{n} \lambda_k e_k, e_j \rangle = \sum_{k=1}^{n} \lambda_k \langle e_k, e_j \rangle = \lambda_j,$$

$j = 1, \dots, n$, and hence

$$P(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k = \sum_{k=1}^{n} \lambda_k e_k = x.$$

Since $P(x) \in M$ by the definition of M, we deduce $M = \{x \in X : P(x) = x\} = \ker(I - P)$ where $I : X \to X$, $I(x) = x$. As the kernel of the bounded linear operator $(I - P)$, the space $M (= (I-P)^{-1}(\{0\}))$ is closed linear subspace of X.

Finally, as in the beginning of the proof of Bessel's inequality we see that

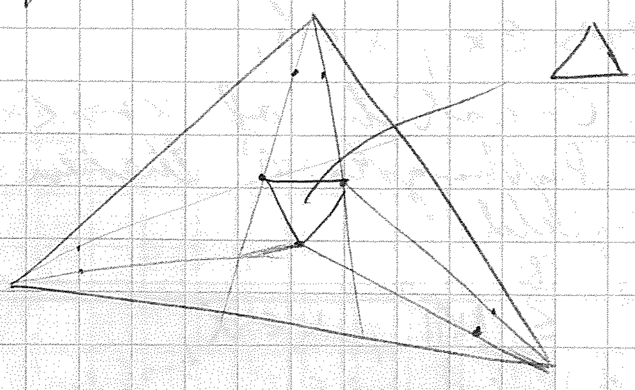
$$e_j \perp (x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k), \quad j = 1, \dots, n,$$

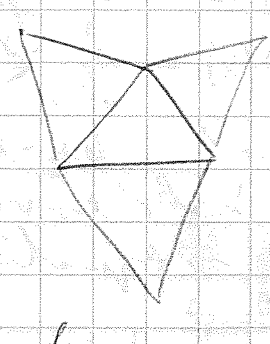
that is, $x - P(x) \in M^{\perp}$ for all $x \in X$. By Theorem 1.25, the conditions $P(x) \in M$ and $x - P(x) \in M^{\perp}$ uniquely determine the orthogonal projection. Thus

$$P_M(x) = P(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k \quad \text{for all } x \in X,$$

and also (b) is proved. $\square$

Morley





Napoleon
Malden

Recall that each finite dimensional subspace of a Banach space is closed (see Ex. 2).

For each finite dimensional subspace M of a Hilbert space X the Gram-Schmidt algorithm gives a basis, and then Theorem 1.25 gives a concrete formula for the orthogonal projection P_M.

We can now combine Theorems 1.25 and 1.24: If $\{e_k\}_{k=1}^n$ is orthonormal in X, then for each $x \in X$, the function

$$(\lambda_1, \dots, \lambda_n) \longmapsto \|x - \sum_{k=1}^n \lambda_k e_k\| , \quad \mathbb{K}^n \to [0, \infty)$$

attains its minimum when $\lambda_k = \langle x, e_k\rangle$ for $k = 1, \dots, n$, and this minimum is

$$\|x - \sum_{k=1}^n \langle x, e_k\rangle e_k\| = \operatorname{dist}(x, M).$$

We next study the summability of series of orthonormal vectors in Hilbert spaces. It will turn out that the summability depends <u>only</u> on the coefficients, something which does not happen in general in Banach spaces.

<u>Theorem 1.30</u> Let $\{e_k\}_{k=1}^\infty$ be orthonormal in a Hilbert space X, and let $\lambda_k \in \mathbb{K}$ for all $k \in \mathbb{N}$. Then

$\sum_{k=1}^\infty \lambda_k e_k$ converges if and only if $\sum_{k=1}^\infty |\lambda_k|^2 < \infty$. Moreover, in this case

$$\|\sum_{k=1}^\infty \lambda_k e_k\|^2 = \sum_{k=1}^\infty |\lambda_k|^2.$$

<u>Proof</u> Set $s_n = \sum_{k=1}^n \lambda_k e_k$, $n \in \mathbb{N}$. Then, as X is complete $\sum \lambda_k e_k$ converges if and only if $\{s_n\}$ is Cauchy.

Let $p, q \in \mathbb{N}$ such that $p < q$. Then, by

Pythagoras' theorem

$$\|s_q - s_p\|^2 = \Big\|\sum_{k=1}^{q} \lambda_k e_k - \sum_{k=1}^{p} \lambda_k e_k\Big\| = \Big\|\sum_{k=p+1}^{q} \lambda_k e_k\Big\|^2 \overset{(P)}{=} \sum_{k=p+1}^{q} |\lambda_k|^2$$

Therefore $\{s_n\}$ is Cauchy iff $\sum |\lambda_k|^2$ converges.

If now $x = \sum_{k=1}^{\infty} \lambda_k e_k$, then

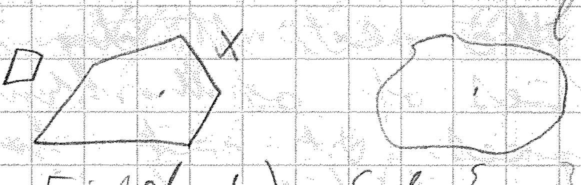
$$\langle x, e_j\rangle = \Big\langle \sum_{k=1}^{\infty} \lambda_k e_k, e_j \Big\rangle = \Big\langle \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k e_k, e_j \Big\rangle$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k \langle e_k, e_j\rangle = \lambda_j$$

for all $j \in \mathbb{N}$, and hence

$$\|x\|^2 = \langle x, x\rangle = \Big\langle \sum_{k=1}^{\infty} \lambda_k e_k, x \Big\rangle = \sum_{k=1}^{\infty} \lambda_k \langle e_k, x\rangle$$

$$= \sum_{k=1}^{\infty} \lambda_k \overline{\lambda_k} = \sum_{k=1}^{\infty} |\lambda_k|^2. \quad \square$$



<u>Corollary 1.31 (Riesz–Fischer)</u> Let $\{e_k\}$ be orthonormal in a Hilbert space X. If $\{\lambda_k\} \in l^2$, then there exists $x \in X$ such that $\langle x, e_k\rangle = \lambda_k$, $k \in \mathbb{N}$. In other words, the map $x \mapsto \{\langle x, e_k\rangle\}$ is surjective from X to l^2.

<u>Proof</u> By Theorem 1.30 $x = \sum \lambda_k e_k$ converges in X, and the Fourier coefficients $\langle x, e_k\rangle$ equals to λ_k for all $k \in \mathbb{N}$ by the proof of Theorem 1.30. $\square$

An orthonormal sequence/set $\{e_k\}$ in a Hilbert space X is called a Hilbert basis (or an orthonormal basis) in X, if

$$\overline{\operatorname{span}(\{e_k : k \in \mathbb{N}\})} = X.$$

A sequence $\{e_k\}$ is a Hilbert basis if and only if $\{e_k\}$ is a maximal orthonormal set in X. Namely, if $M = \overline{\operatorname{span}(\{e_k : k \in \mathbb{N}\})}$, then $M \neq X \iff M^\perp \neq \{0\}$ because $X = M \oplus M^\perp$ by Theorem 1.27. Equivalently, there exists $x \in M^\perp$ such that $\|x\| = 1$, and thus $\{x\} \cup \{e_k : k \in \mathbb{N}\}$ is orthonormal $\iff$ M is not a maximal orthonormal set in X.

If $\dim X < \infty$, then the situation is simple. Namely, if $\dim X = n < \infty$, then an orthonormal set $\{e_1, \dots, e_n\}$ is a Hilbert basis in X, because $\{e_1, \dots, e_n\}$ is linearly independent and the dimension yields $X = \operatorname{span}(\{e_1, \dots, e_n\})$ (we don't even need the closure).

The following characterization of Hilbert basis plays a fundamental role in the theory of Hilbert spaces and its applications. The result is formulated in the case of infinite dimensions.

<u>Theorem 1.32</u> Let $\{e_k\}_{k=1}^\infty$ be orthonormal in a Hilbert space X, $\dim X = \infty$. Then the following conditions are equivalent:

(a) $\{e_k\}_{k=1}^\infty$ is a Hilbert basis;

(b) $\langle x, e_k \rangle = 0$ for all $k \in \mathbb{N}$ if and only if $x = 0$;

(c) $x = \sum_{k=1}^\infty \langle x, e_k \rangle e_k$ for all $x \in X$ (where convergence is in sense of the norm $\|x - \sum_{k=1}^N \langle x, e_k \rangle e_k\| \to 0$, $N \to \infty$);

(d) $\|x\|^2 = \sum_{k=1}^\infty |\langle x, e_k \rangle|^2$ for all $x \in X$; (Parseval)

(e) $\langle x, y \rangle = \sum_{k=1}^\infty \langle x, e_k \rangle \langle e_k, y \rangle$ for all $x, y \in X$. (Plancherel)

9

Remarks (1) Condition (d) is called Parseval's identity or Parseval's formula, and (e) is called Plancherel's formula;

(2) In (c), the coefficients $\langle x, e_k\rangle$ are unique for all $x \in X$;

(3) A similar result is valid when $\dim X = n$ and $\{e_k\}_{k=1}^{n}$ is orthonormal in X: just take the sum from 1 to n in (c)-(e). Proof is similar to the below, but easier.

Proof Note first, that for each $A \subset X$ we have $(\bar{A})^\perp = A^\perp$. If A is a subspace, then this follows by Thm. 1.24 (b): $A^{\perp\perp} = \bar{A} \Rightarrow (\bar{A})^\perp = A^{\perp\perp\perp} = A^\perp$. In general $(\bar{A})^\perp \subset A^\perp$ by Lemma 1.19 c). On the other hand, if $x \in A^\perp = \{x \in X : \langle x, a\rangle = 0 \text{ for all } a \in A\}$, then by the continuity of $\langle \cdot, \cdot \rangle$, $\langle x, a\rangle = 0$ for all $a \in \bar{A}$, and so $x \in (\bar{A})^\perp$. [$\Rightarrow A^\perp \subset (\bar{A})^\perp$] Thus $(\bar{A})^\perp = A^\perp$.

$\boxed{(a) \Leftrightarrow (b)}$ In particular if $M = \overline{\operatorname{span}}\{e_k : k \in \mathbb{N}\}$, then (b) is equivalent to $M^\perp = \{0\}$. This is in turn equivalent to $M = X$, because $X = M \oplus M^\perp$ by Theorem 1.27. Thus (a) and (b) are equivalent.

Clearly (e) $\Rightarrow$ (d) $\Rightarrow$ (b). $\quad \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|^2$

$\boxed{(b) \Rightarrow (c)}$ Set $x \in X$ and let $\quad \sum \lambda_k e_k \text{ conv} \Leftrightarrow \sum |\lambda_k|^2 <$

$$\hat{x} = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$$

where the right hand side converges by Bessel's inequality and Theorem 1.30. Now for all $j \in \mathbb{N}$ we have

$$\langle \hat{x}, e_j\rangle = \lim_{N\to\infty} \left\langle \sum_{k=1}^{N} \langle x, e_k\rangle e_k, e_j \right\rangle = \langle x, e_j\rangle$$

and hence

$$\langle \hat{x} - x, e_j\rangle = 0$$

for all $j \in \mathbb{N}$.

Therefore $\tilde{x} = x$ by (b), and thus (c) holds.

$\boxed{(c) \Rightarrow (e)}$ Finally we prove (c) $\Rightarrow$ (e). By (c)

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \;, \; x \in X,$$

and hence by linearity and continuity,

$$\begin{aligned}\langle x, y \rangle &= \lim_{N\to\infty} \left\langle \sum_{k=1}^{N} \langle x, e_k\rangle e_k , y \right\rangle \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \langle x, e_k \rangle \langle e_k, y \rangle \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} \langle x, e_k\rangle \overline{\langle y, e_k \rangle} \\ &= \sum_{k=1}^{\infty} \langle x, e_k \rangle \overline{\langle y, e_k\rangle},\end{aligned}$$

where the convergence of the right hand side follows by Holder's inequality (with $p=2$, or Cauchy-Schwartz) and Theorem 1.30. One can also prove this implication by arguing as follows

$$\left| \langle x, y\rangle - \sum_{k=1}^{N} \langle x, e_k\rangle \overline{\langle y, e_k\rangle} \right|$$

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$$\begin{aligned}&= \left| \langle x, y \rangle - \left\langle x, \sum_{k=1}^{N} \langle y, e_k \rangle e_k \right\rangle \right| \\ &= \left| \left\langle x, y - \sum_{k=1}^{N} \langle y, e_k\rangle e_k \right\rangle \right| \\ &\overset{C\text{-}S}{\leq} \|x\| \left\| y - \sum_{k=1}^{N} \langle y, e_k \rangle e_k \right\| \to 0, \text{ as } N \to \infty.\end{aligned}$$

$\square$

<u>Theorem 1.33</u> Let X be a Hilbert space. Then X has a Hilbert basis if and only if X is separable.

<u>Proof</u> If X has a Hilbert basis, i.e. there exists an orthonormal set $\{e_k\}$ such that $\overline{\operatorname{span}(\{e_k : k \in \mathbb{N}\})} = X$, then X is clearly separable: a dense numerable set is $\{ \sum \lambda_k e_k : \lambda_k = a + bi \in \mathbb{K},\ a, b \in \mathbb{Q},\ \lambda = \{\lambda_k\} \in \ell^2 \}$.

Conversely, assume that X is separable. We may assume that $\dim X = \infty$, for otherwise the Hilbert basis is found by using the Gram-Schmidt algorithm, see Lemma 1.16 (b). Since X is separable, there exists a dense set $\{x_n\}_{n=1}^{\infty}$. We will use induction to construct a subsequence / set $\{x_{n_j}\}_{j=1}^{\infty}$ such that

(i) $\{x_{n_1}, \dots, x_{n_p}\}$ is linearly independent;
(ii) $x_m \in \operatorname{span}(\{x_{n_1}, \dots, x_{n_p}\})$, provided $1 \le m \le n_p$, that is,
(*) $\operatorname{span}(\{x_{n_1}, \dots, x_{n_p}\}) = \operatorname{span}(\{x_k : 1 \le k \le n_p\})$

<u>Construction</u> Assume we have $n_1 < \dots < n_p$ such that (i) and (ii) are valid. Set $n_{p+1} > n_p$ be the smallest natural number such that $\{x_{n_1}, \dots, x_p, x_m\}$ is linearly independent. Then

$x_k \in \operatorname{span}(\{x_1, \dots, x_p, x_{n_{p+1}}\})$, $n_1 < k < n_{p+1}$.

Do the Gram-Schmidt to the sequence $\{x_{n_k}\}$ to obtain an orthonormal set. The requirement $\overline{\operatorname{span}}(\{e_k : k \in \mathbb{N}\}) = X$ follows by (*), because $\{x_k\}$ is dense in X. □

Set $\{e_j\}_{j \in J}$, where $J = \mathbb{N}$ or $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ be an orthonormal basis of a Hilbert space X. Then each $x \in X$ has a unique representation

$$x = \sum_{k \in J} \langle x, e_k \rangle e_k.$$

Set first $J = \mathbb{N}$, and define the linear operator $T : \ell^2 \to X$ by

$$T(\{\lambda_k\}_{k \in \mathbb{N}}) = \sum_{k \in \mathbb{N}} \lambda_k e_k, \quad \{\lambda_k\} \in \ell^2.$$

Then T is a linear isomorphism from ℓ^2 to X. In other words, T is a linear bijection

$\ell^2 \to X$ such that its inverse T^{-1} is continuous. Namely, by Theorem 1.30,

$$\|T(x)-T(y)\| = \Big\|\sum_{k\in N}(x_k-y_k)\,e_k\Big\| = \Big(\sum_{k\in N}|x_k-y_k|^2\Big)^{\frac{1}{2}}$$
$$= \|x-y\|_{\ell^2}$$

for all $x=\{x_n\}$ and $y=\{y_n\}$ in ℓ^2. In fact, T is an isometry, that is, it preserves the distance. Moreover, T is a linear bijection by Theorems 1.30 and 1.32. To see this, note first that the linearity follows by the properties of convergent series (trivial). If $x \in \{x_k\} \in \ell^2$ and $T(x) = \sum_{k\in N} x_k e_k = 0$, then $x_k = 0$ for all $k \in N$

by the uniqueness of representation (injectivity follows). On the other hand, if $x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k \in X$, then $\{\langle x, e_k\rangle\} \in \ell^2$ by Parseval's formula and

$$x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k = T(\{\langle x, e_k\rangle\}_{k\in N})$$

(surjectivity follows).

If $J = \{1, \dots, n\}$, we obtain a linear map $T : \mathbb{K}^n \to X$ by setting

$$T(\{\lambda_k\}_{k\in J}) = \sum_{k=1}^{n} \lambda_k e_k .$$

Also in this case T is a linear isomorphism from $(\mathbb{K}^n, \|\cdot\|_{\ell^2_n})$ to X. We deduce that:

ALL SEPARABLE HILBERT SPACES ARE ISOMORPHIC TO ℓ^2 OR TO $\ell^2_n = (\mathbb{K}^n, \|\cdot\|_{\ell^2_n})$ FOR SOME $n \in N$.

$\boxed{10}$ Examples

① Set $e_n = (0, \dots, 0, 1, 0, \dots) \in l^2$ for $n \in \mathbb{N}$. Then $\{e_n\}$ is an orthonormal basis of l^2; if $x = \{x_n\} \in l^2$ and $0 = \langle x, e_n \rangle = x_n$ for all $n \in \mathbb{N}$, then $x = 0$ (Theorem 1.32 (b))

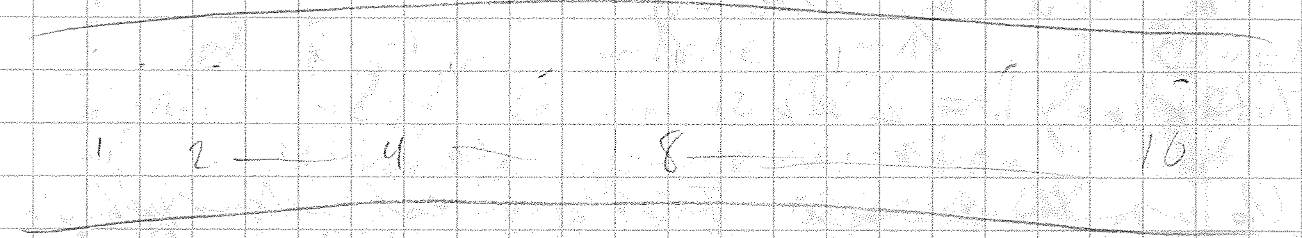
② Set $L^2([0,1])$ be a Hilbert space with $\mathbb{K} = \mathbb{R}$, that consists of the functions $f : [0,1] \to \mathbb{R}$ for which

$\|f\|^p_{L^p}$, $p \in (0,1]$

$$\|f\|^2_{L^2} = \int_0^1 |f(t)|^2 dt < \infty.$$

(To be precise, members in L^2 are equivalence classes.

Haar system $\{h_n(x)\}_{n=0}^{\infty}$ is probably the easiest way to construct a Hilbert basis in $L^2([0,1])$. We begin with the characteristic function of $[0,1]$ and choose $h_0(t) = \chi_{[0,1]}(t) = 1$ for all $t \in [0,1]$. Clearly, $\|h_0\|_{L^2} = 1$. The rest of the functions in the basis are chosen as follows. If $0 \le j < 2^k$, then let $n = 2^k + j$



$$\Delta_n = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right] \subset [0,1]$$

$$\Delta_n^+ = \left(\frac{j}{2^k}, \frac{j+1/2}{2^k}\right), \quad \Delta_n^- = \left(\frac{j+1/2}{2^k}, \frac{j+1}{2^k}\right)$$

Define $h_n(t) = 2^{\frac{k}{2}} \left(\chi_{\Delta_n^+}(t) - \chi_{\Delta_n^-}(t)\right)$

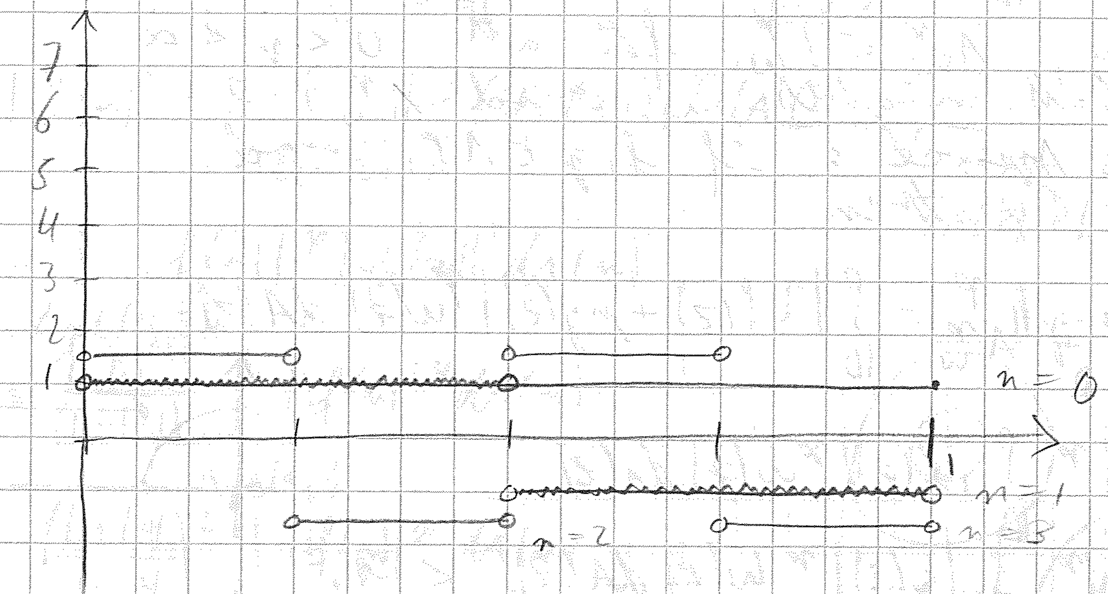
$$= \begin{cases} 2^{\frac{k}{2}}, & t \in \Delta_n^+ \\ -2^{\frac{k}{2}}, & t \in \Delta_n^- \\ 0, & t \notin \Delta_n^+ \cup \Delta_n^- \end{cases}$$

where $n \in \mathbb{N}$ and $n = 2^k + j$ as above.

$n = 1 = 2^0 + 0 \Rightarrow k = 0,\ j = 0,\ \Delta_1 = [0,1],$

$$\begin{cases} n = 2 = 2^1 + 0 \Rightarrow k = 1,\ j = 0,\ \Delta_2 = [0, \tfrac{1}{2}] = \Delta_1^+ \\ n = 3 = 2^1 + 1 \Rightarrow k = 1,\ j = 1,\ \Delta_3 = [\tfrac{1}{2}, 1] = \Delta_1^- \end{cases}$$

$$\begin{cases} n = 4 = 2^2 + 0 \Rightarrow k = 2,\ j = 0,\ \Delta_4 = [0, \tfrac{1}{4}] = \Delta_2^+ \\ n = 5 = 2^2 + 1 \Rightarrow k = 2,\ j = 1,\ \Delta_5 = [\tfrac{1}{4}, \tfrac{1}{2}] = \Delta_2^- \\ n = 6 = 2^2 + 2 \Rightarrow k = 2,\ j = 2,\ \Delta_6 = [\tfrac{1}{2}, \tfrac{3}{4}] = \Delta_3^+ \\ n = 7 = 2^2 + 3 \Rightarrow k = 2,\ j = 3,\ \Delta_7 = [\tfrac{3}{4}, 1] = \Delta_3^- \end{cases}$$



It turns out that this Haar system $\{h_n(x)\}_{n=0}^{\infty} \subset L^2([0,1])$ is a Hilbert basis. To prove this rigorously requires a bit of measure theory.

The Fourier coefficients with respect to the Haar basis of a function $f \in L^2([0,1])$ are obtained from the formulas

$$\langle f, h_0 \rangle = \int_0^1 f(t)\,dt$$

$$\langle f, h_n \rangle = 2^{\frac{k}{2}} \left(\int_{\Delta_n^+} f(t)\,dt - \int_{\Delta_n^-} f(t)\,dt \right),$$

when $n = 2^k + j$ as above. Thus for each $f \in L^2([0,1])$, we have

$$f(t) = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n(t)$$

and the series converges in $L^2([0,1])$ in the norm of $L^2([0,1])$.

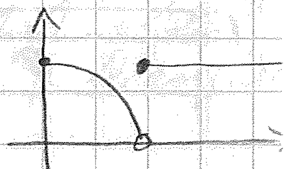
③ Weighted Bergman Spaces

Set $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $w : \mathbb{D} \to (0, \infty)$ be continuous and integrable over $\mathbb{D}$. For such w and $0 < p < \infty$, the weight Bergman space A^p_w consists of those $f \in \mathcal{H}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} \}$ such that

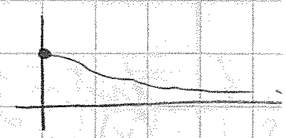
$$\| f \|^p_{A^p_w} = \int_{\mathbb{D}} |f(z)|^p w(z) \, dA(z) < \infty.$$

Clearly $A^p_w \subset L^p_w$ for all $0 < p < \infty$ and all weights w, and A^p_w is a vector space: if $f, g \in A^p_w$ and $\lambda, \mu \in \mathbb{C}$, then

$$\| \lambda f + \mu g \|^p_{A^p_w} = \int_{\mathbb{D}} | \lambda f(z) + \mu g(z) |^p w(z) \, dA(z)$$



$$\leq 2^p |\lambda|^p \int_{\mathbb{D}} |f(z)|^p w(z) \, dA(z)$$
$$+ 2^p |\mu|^p \int_{\mathbb{D}} |g(z)|^p w(z) \, dA(z) < \infty.$$



If $g \in \mathcal{H}(\mathbb{D})$, then $|g|^p$ is subharmonic in $\mathbb{D}$, and hence

$$|g(0)|^p \lesssim \int_{D(0, \frac{1}{2})} |g(z)|^p \, dA(z).$$

$$g(0) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) \, d\theta$$

$$|g(0)| r \lesssim \int_0^{2\pi} |g(re^{i\theta})| \, d\theta \, r$$

$$\int_0^{\rho} |g(0)|\, r\,dr \le \int_0^{\rho}\int_0^{2\pi} |g(re^{i\theta})|\, d\theta\, r\,dr$$

$$\Leftrightarrow \frac{1}{2}\rho^2 |g(0)| \le \int_{D(0,\rho)} |g(z)|\, dA(z) \qquad \rho = \frac{1}{2}$$

$$|g(0)|^p \lesssim \int_{D(0,\frac{1}{2})} |g(z)|^p\, dA(z).$$

By applying this to the function $g = f\circ\varphi_a$, we deduce

$$\begin{cases} \varphi_a(z) = \dfrac{a-z}{1-\bar a z} \\ \varphi_a(0) = 0 \\ \varphi_a'(z) = \dfrac{-1+\bar a z + \bar a(a-z)}{(1-\bar a z)^2} = \dfrac{|a|^2-1}{(1-\bar a z)^2} \end{cases}$$

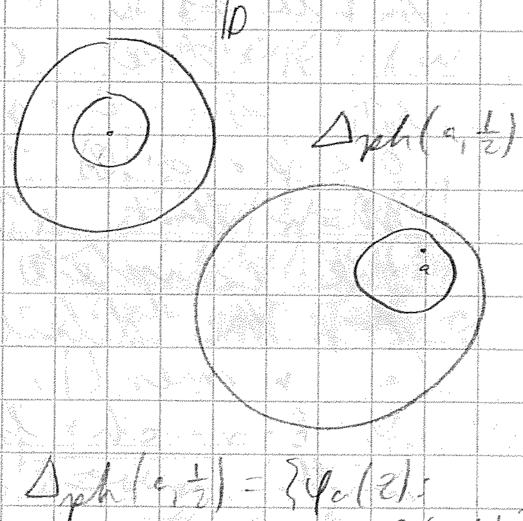
$$|f(a)|^p \lesssim \int_{D(0,\frac{1}{2})} |f(\varphi_a(z))|^p\, dA(z)$$, $u = \varphi_a(z) \Leftrightarrow z = \varphi_a(u)$, $dA(z) = |\varphi_a'(u)|^2\, dA(u)$

$$= \int_{\Delta_{ph}(a,\frac{1}{2})} |f(u)|^p |\varphi_a'(u)|^2\, dA(u)$$

$$\le \frac{(1-|a|^2)^2}{(1-|a|)^4}\int_{\Delta_{ph}(a,\frac{1}{2})} |f(u)|^p\, dA(u)$$

$$= \left(\frac{1+|a|}{1-|a|}\right)^2 \int_{\Delta_{ph}(a,\frac{1}{2})} |f(u)|^p\, dA(u)$$

$$\le \frac{4}{(1-|a|)^2}\,\frac{1}{\min_{u\in\Delta_{ph}(a,\frac{1}{2})} \omega(u)} \int_{\Delta_{ph}(a,\frac{1}{2})} |f(u)|^p \omega(u)\, dA(u)$$



$\Delta_{ph}(a,\frac{1}{2}) = \{\varphi_a(z) : z \in D(0,\frac{1}{2})\}$

$|\varphi_a'(u)| = \dfrac{1-|a|^2}{|1-\bar a u|^2}$

and hence

$$(*) \quad |f(a)| \lesssim \frac{\|f\|_{A^p_\omega}}{(1-|a|)^{\frac{2}{p}}\left(\min_{u\in\Delta_{ph}(a,\frac{1}{2})} \omega(u)\right)^{\frac{1}{p}}}, \quad a\in\mathbb{D},\ f\in A^p_\omega.$$

In fact, we proved a bit more, namely

$$|f(a)| = o\left(\frac{1}{(1-|a|)^{\frac{2}{p}}\left(\min\limits_{u\in\Delta_{ph}(a,\frac{1}{2})} \omega(u)\right)^{\frac{1}{p}}}\right), \quad |a|\to 1^-.$$

$$f(a) = o\left(\frac{1}{1-|a|}\right), \quad |a|\to 1^-$$

$$\Leftrightarrow |f(a)|(1-|a|) \to 0, \quad |a|\to 1^-$$

$$|f(a)| = O\left(\frac{1}{1-|a|}\right)$$

$$\Leftrightarrow |f(a)|(1-|a|) \le C, \quad |a|\to 1^-$$

$$\Leftrightarrow \limsup_{|a|\to 1^-} |f(a)|(1-|a|) < \infty$$

The inequality (α) shows that the norm convergence implies locally uniform convergence, that is, if $f_n, f \in A^p_\omega$, $n\in\mathbb{N}$, and $\|f_n - f\|_{A^p_\omega} \to 0$, $n\to\infty$, then $f_n \to f$ uniformly on each compact subset of $\mathbb{D}$.

Another consequence of (α) is that A^p_ω is a complete metric space for all $0<p<\infty$. (α) To see this it suffices to show that A^p_ω is a closed subspace of the complete metric space L^p_ω. We skip the details of this fact (at least for a moment). We deduce that A^p_ω is Banach for $p\ge 1$, and A^2_ω is Hilbert with inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\,\overline{g(z)}\,\omega(z)\,dA(z), \quad f, g \in A^2_\omega.$$

(α) ~~whenever ω is radial, that is, $\omega(z) = \omega(|z|)$ for all $z\in\mathbb{D}$.~~

$\boxed{\text{TERVEISIN PIERASI}}$

Let now $\{e_n\}$ be an orthonormal basis of A^2_ω. Then

$\boxed{11}$

$$\langle e_n, e_m\rangle = \delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

the Kronecker delta function. Furthermore, each $f \in A^2_\omega$ has a unique representation $f = \sum \lambda_n e_n$, convergent in norm, and therefore uniformly convergent in compact subsets:

$$f(z) = \sum \lambda_n e_n(z), \quad z \in \mathbb{D},$$

where $\lambda_n = \langle f, e_n\rangle$. By Parseval's formula, $\sum |\lambda_n|^2 = \|f\|^2_{A^2_\omega}$.

Assume now that $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$ that is, ω is radial. Then the monomials $1, z, z^2, \dots$ form an orthogonal set in A^2_ω and

$$(+) \int_{\mathbb{D}} z^n \bar{z}^m \omega(z)\, dA(z) = \int_0^1 \omega(r) r \int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta}\, d\theta\, dr$$

$$= \int_0^1 \omega(r) r^{n+m+1} \int_0^{2\pi} e^{i\theta(n-m)}\, d\theta\, dr$$

moments of ω

$$= 2\pi \int_0^1 \omega(r) r^{n+m+1}\, dr\, \delta_{nm}.$$

$\omega(z) = \omega(|z|)$
$\omega(z) = \omega\left(\frac{z}{|z|}\right)$

Denote $\omega_n = \int_0^1 r^{2n+1}\omega(r)\, dr$. Then we deduce that the functions

$$e_n(z) = \frac{z^n}{\sqrt{2\pi\, \omega_n}}, \quad n \in \mathbb{N} \cup \{0\},$$

are orthonormal in A^2_ω. To see that they form a basis, we have to verify that $\overline{\operatorname{span}(\{e_n\})} = A^2_\omega$. Equivalently, by Theorem 1.22, we must show the Parseval's formula

$$(**) \sum_{n=0}^{\infty} |\langle f, e_n\rangle|^2 = \|f\|^2_{A^2_\omega}, \quad f \in A^2_\omega.$$

But if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\langle f, e_k\rangle = \langle \sum_{n=0} a_n z^n, e_k\rangle$$

$$= a_k \frac{1}{\sqrt{2\pi w_k}} \langle z^k, z^k\rangle$$

$$= \frac{a_k}{\sqrt{2\pi w_k}} \int_{\mathbb{D}} |z|^{2k} w(z)\, dA(z)$$

$$= \frac{a_k}{\sqrt{2\pi w_k}} 2\pi w_k = a_k \sqrt{2\pi w_k}$$

and hence (x x) is equivalent to

$$2\pi \sum_{n=0}^{\infty} |a_n|^2 w_n = \|f\|^2_{A^2_w}, \quad f(z) = \sum a_n z^n \in A^2_w$$

To see this, let $\rho \in (0,1)$ and consider

$$T_n(f)(z) = \sum_{k=0}^{n} a_k z^k, \quad n \in \mathbb{N},$$

the partial sum of the Maclaurin series of f. Then (+) yields

$$\int_{D(0,\rho)} |T_n(f)(z)|^2 w(z)\, dA(z) = \int_0^{2\pi}\int_0^{\rho} T_n(f)(re^{i\theta})\overline{T_n(f)(re^{i\theta})}\, w(r)\, r\, dr\, d\theta$$

$$= 2\pi \sum_{k=0}^{n} |a_k|^2 \int_0^{\rho} w(r) r^{2k+1}\, dr$$

But $T_n(f) \to f$ uniformly on compact subsets of $\mathbb{D}$ (CA I b), so it follows that

$$\int_{D(0,\rho)} |f(z)|^2 w(z)\, dA(z) = 2\pi \sum_{k=0}^{\infty} |a_k|^2 \int_0^{\rho} w(r) r^{2k+1}\, dr$$

Now both sides are increasing in ρ, so by letting $\rho \to 1^-$, we deduce

$$\|f\|^2_{A^2_w} = 2\pi \sum_{k=0}^{\infty} |a_k|^2 w_k,$$

and thus $\{e_n\}$ forms a basis of A^2_w.

We also deduce an important consequence from this reasoning. Namely, since $\{e_n\}$ span the space A^2_w in the sense that $\overline{\operatorname{span}}\{e_n\} = A^2_w$, and $\{e_n\}$ is a basis of A^2_ω, the polynomials form a dense subset of A^2_ω. This is not true in general if $\mathbb{D}$ is replaced by a domain Ω, even if Ω is simply connected domain (we do not prove this here).

$\mathcal{F}$, a collection of subsets of X, is called σ-algebra in X if $\mathcal{F}$ has the following properties:

(i) $X \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, (iii) $A_j \in \mathcal{F} \Rightarrow A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$

Members in $\mathcal{F}$ are called measurable sets in the measurable space X. The members in the smallest σ-algebra in a topological space X containing all open sets are called the Borel sets of X.

A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a Borel measure on X. Topological space is Hausdorff if for each pair of distinct points p and q in X there exists neighbourhoods $U \ni p$ and $V \ni q$ s.th. $U \cap V = \phi$.

A positive measure is a function μ, defined on a σ-algebra $\mathcal{F}$ whose range is in $[0, \infty]$ and which is countably additive, that is if $\{A_j\}$ is a disjoint countable collection of members of $\mathcal{F}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

$$\|f\|^p_{L^p_\mu} = \int_X |f|^p d\mu$$

$$\|f\|_{L^\infty_\mu} = \mu\text{-ess} \sup_{x \in X} |f(x)|$$

<u>Theorem 1.34</u> Let μ be a positive measure in a measure space X and $1 \le p \le \infty$. Then $L^p_\mu = L^p_\mu(X)$ is complete.

<u>Proof</u> Let first $1 \le p < \infty$, and let $\{f_n\}$ be Cauchy in L^p_μ. Then

there exists a subsequence $\{f_{n_j}\}$,
$n_1 < n_2 < \dots$, such that

$$\|f_{n_{j+1}} - f_{n_j}\|_{L^p_\mu} < 2^{-j}, \quad j \in \mathbb{N}.$$

Set $g_k = \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}|$, $g = \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}|$

By Minkowski's inequality,

$$\|g_k\|_{L^p_\mu} = \left(\int_X g_k^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \left(\int_X \left(\sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}|\right)^p d\mu\right)^{\frac{1}{p}}$$

$$\leq \sum_{j=1}^{k} \left(\int_X |f_{n_{j+1}} - f_{n_j}|^p \, d\mu\right)^{\frac{1}{p}}$$

$$= \sum_{j=1}^{k} \|f_{n_{j+1}} - f_{n_j}\|_{L^p_\mu} < \sum_{j=1}^{\infty} 2^{-j} < 1, \quad k \in \mathbb{N}.$$

Hence an application of Fatou's lemma gives

$$\|g\|_{L^p_\mu}^p = \int_X g^p \, d\mu = \int_X \liminf_{k\to\infty} g_k^p(x) \, d\mu(x)$$

$$\leq \liminf_{k\to\infty} \int_X g_k^p \, d\mu = \liminf_{k\to\infty} \|g_k\|_{L^p_\mu}^p \leq 1.$$

In particular, $g(x) < \infty$ for almost every $x \in X$, and hence

$$f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x))$$

converges absolutely for almost every $x \in X$.
Denote

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x)), & \text{if the sum converges} \\ 0, & \text{for otherwise.} \end{cases}$$

Since

$$f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) = f_{n_k},$$

we see that $f(x) = \lim_{j\to\infty} f_{n_j}(x)$ for almost all $x \in X$. Thus we have found a function f which is the pointwise limit almost everywhere of $\{f_n\}$. We now show that f is the L^p_μ-limit of $\{f_n\}$.

Set $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$\|f_n - f_m\|_{L^p_\mu} < \varepsilon$ if $n, m \geq N$. For every $m > N$, Fatou's lemma shows that

$$\int_X |f - f_m|^p d\mu = \int_X \liminf_{j\to\infty} |f_{n_j} - f_m|^p d\mu$$
$$\leq \liminf_{j\to\infty} \int_X |f_{n_j} - f_m|^p d\mu \leq \varepsilon^p.$$

$p \in (0,1)$
$d(\ ) = \|\ \|^p$

Hence $f - f_m \in L^p_\mu$, which in turn implies that $f \in L^p_\mu$, and $\|f - f_m\|_{L^p_\mu} \to 0$, $n \to \infty$.

Thus L^p_μ, $1 \leq p < \infty$, is complete.

Suppose $\{f_n\}$ is a Cauchy sequence in L^∞_μ, and let A_k and $B_{m,n}$ be the sets where $|f_k(x)| > \|f_k\|_{L^\infty_\mu}$ and $|f_n(x) - f_m(x)| > \|f_n - f_m\|_{L^\infty_\mu}$, respectively. Then A_k and $B_{m,n}$ are of measure zero and hence

$$\mu(E) = 0, \quad E = \left(\bigcup_k A_k\right) \cup \left(\bigcup_{m,n} B_{m,n}\right),$$

because E is a countable union of sets of ~~because E is a countable union of sets of~~ zero measure. In the complement of E we have that $\{f_n\}$ converges uniformly to a bounded function f:

$$\left|\lim_{n\to\infty} f_n(x) - f_m(x)\right| = \lim_{n\to\infty} |f_n(x) - f_m(x)|$$

$$\leq \lim_{n\to\infty} \|f_n - f_m\|_{L^\infty_\mu} \leq \varepsilon, \quad m > N.$$

Set $f(x) = 0$, $x \in E$. Then $f \in L^\infty_\mu$ and $\|f_n - f\|_{L^\infty_\mu} \to 0$, $n \to \infty$. $\square$

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TO 20.2 KLO 14-18 SALI !

SOPIKIA DEMOT 5 & 6
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$\boxed{12}$ Corollary 1.35 Set $1 \leq p < \infty$ and $\omega : \mathbb{D} \to [0, \infty)$ such that for each compact $K \subseteq \mathbb{D}$ there exists $C = C(p, \omega, K)$ s.th.

$$|f(z)| \leq C \|f\|_{A^p_\omega}, \quad z \in K, \quad f \in A^p_\omega \ (f \in H(\mathbb{D})),$$

that is, point evaluations are bounded linear functionals in A^p_ω. Then A^p_ω is complete.

$$\left(\begin{array}{l} z \mapsto f(z) \\ L_z : A^p_\omega \to \mathbb{C}, \quad L_z(f) = f(z) \end{array} \right)$$

Proof By Theorem 1.34 we only have to show that A^p_ω is closed in L^p_ω. To see this, let $f_n \in A^p_\omega$ such that $f_n \to f \in L^p_\omega$ in norm, that is, $\|f_n - f\|_{L^p_\omega} \to 0$, $n \to \infty$. Then, by the proof of theorem 1.34 there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j}(z) \to f(z)$ almost everywhere. On the other hand, $\{f_n\}$ is Cauchy in norm of A^p_ω hence by $(\ast\ast)$ a locally uniform Cauchy, so it converges locally uniformly to a function g that is analytic in $\mathbb{D}$. Thus $f(z) = g(z)$ for almost every $z \in \mathbb{D}$, and hence the limit function f can be identified with the function $g \in A^p_w$. □

Note that the proof above applies if $\mathbb{D}$ is replaced by any domain $\Omega \subseteq \mathbb{C}$ (open and connected).

(4) In the next section we will see that the functions $\frac{1}{\sqrt{2\pi}} e^{int}$, $n \in \mathbb{Z}$, form an orthonormal basis in the Hilbert space $L^2([0,2\pi])$, with $\mathbb{K} = \mathbb{C}$.

(5) By using the Gram-Schmidt method to polynomials one can find a Hilbert basis to different (weighted) L^2-spaces. For example, let

$$p_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} \left((t^2-1)^n \right).$$

These polynomials are called Legendre's polynomials. One can show that

$$\langle p_n, p_k \rangle = \int_{-1}^{1} p_n(t) p_k(t)\, dt = \frac{2}{2n+1} \delta_{nk}.$$

Denote $e_n = \sqrt{\frac{2n+1}{2}}\, p_n$. One can show that $\{e_n\}$ is a Hilbert basis in $L^2([-1,1])$. This sequence can also be constructed directly by applying Gram-Schmidt to $\{1, t, t^2, \dots\}$.

(6) Set

$$L^2_\rho(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} |f(t)|^2 \rho(t)\, d\mu(t) < \infty \right\}, \quad \rho(t) = e^{-t^2},$$

where μ stands for the one dimensional Lebesgue measure on $\mathbb{R}$. Then the Hermite's polynomials