Sobolev spaces

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Set $f \in C^1([0,2\pi])$ such that $f(0) = f(2\pi)$. Then an integration by parts yields

$$\widehat{f'}(k) = \frac{1}{2\pi}\int_0^{2\pi} f'(t)e^{-ikt}\,dt = \frac{1}{2}$$

$$= \frac{1}{2\pi}\Big/_0^{2\pi} f(t)e^{-ikt} - \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}(-ik)\,dt$$

$$= 0 + ik\,\hat{f}(k), \quad k \in \mathbb{Z}.$$

Further, by Parseval's identity,

$$\frac{1}{2\pi}\int_0^{2\pi}\left(|f(t)|^2 + \underline{|f'(t)|^2}\right)dt = \frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt + \frac{1}{2\pi}\int_0^{2\pi}|f'(t)|^2\,dt$$

$$= \sum_{k\in\mathbb{Z}}|\hat{f}(k)|^2 + \overset{?}{\sum_{k\in\mathbb{Z}}} k^2|\hat{f}(k)|^2 = \sum_{k\in\mathbb{Z}}(1+k^2)|\hat{f}(k)|^2.$$

Riesz-Fischer theorem now implies that a kind of analogue of this identity should be valid also for more general functions. This yields to the following problem:

$\boxed{\text{What is the "derivative" of } f \in L^2}$

Set $\Omega = (a,b) \subseteq \mathbb{R}$ (Ω can be $\mathbb{R}$). Recall that the support (kantaja) of $\psi : \mathbb{R} \to \mathbb{R}$ is

$$\operatorname{supp}(\psi) = \overline{\{x \in \mathbb{R} : \psi(x) \neq 0\}},$$ RAJAPISTEET

Set $D(\Omega) = \{\psi \in C^\infty(\mathbb{R}) : \operatorname{supp}(\psi) \subseteq \Omega$ is compact$\}$

family of test functions, and recall that $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded in $\mathbb{R}^n$ by the Heine-Borel theorem. Note also that $\psi^{(k)} \in D(\Omega)$ for all $k \in \mathbb{N}$, if $\psi \in D(\Omega)$.

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$\operatorname{supp}(\psi') \subset \operatorname{supp}(\psi)$

If now $f \in C^1(\Omega)$, then integration by parts gives

$$(\#) \quad \int_\Omega f'(x)\,\psi(x)\,dx = -\int_\Omega f(x)\,\psi'(x)\,dx, \quad \psi \in D(\Omega),$$

because ψ vanishes near and on the boundary of Ω. By using this observation we may try to identify the derivative of $f \in C^1(\Omega)$ with the linear map

$$\psi \longmapsto -\int_\Omega f(x)\,\psi'(x)\,dx, \quad \psi \in D(\Omega).$$

This leads us to the definition of the weak derivative

Definition 2.8 Set $\Omega = (a,b)$ and $f \in L^2(\Omega)$. The function $g \in L^2(\Omega)$ is the weak derivative (generalized derivative, distributional derivative) of f if

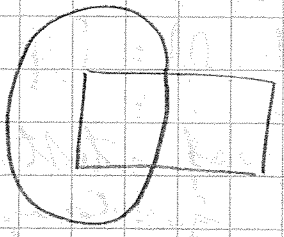
$$\int_\Omega g(x)\,\psi(x)\,dx = -\int_\Omega f(x)\,\psi'(x)\,dx, \quad \psi \in D(\Omega).$$

If f has a weak derivative g, we write $g = f'$.

We prove next that the weak derivative of $f \in L^2(\Omega)$ is unique when it exists, and that the weak derivative coincides with the usual derivative for $f \in C^1(\Omega)$ (provided $f' \in L^2(\Omega)$).

$f(x) = (1-x)^{-\frac{1}{2}+\varepsilon} \in L^2(0,1)$

$f'(x) \sim (1-x)^{-\frac{3}{2}+\varepsilon} \notin L^2(0,1)$



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Lemma 2.4 Let $\Omega \subseteq \mathbb{R}$ be a bounded interval 7 and $f \in L^2(\Omega)$. If $g_1, g_2 \in L^2(\Omega)$ satisfy

$$(*) \quad \int_\Omega g_1(x)\varphi(x)\,dx = -\int_\Omega f(x)\varphi'(x)\,dx = \int_\Omega g_2(x)\varphi(x)\,dx$$

for all $\varphi \in D(\Omega)$, then $g_1 \equiv g_2$ in $L^2(\Omega)$. In particular, if $f \in C^1(\Omega)$ and $f' \in L^2(\Omega)$, then f' is the weak derivative of f.

Proof By the assumption $(*)$ and the linearity of integrals:

$$\int_\Omega (g_2(x) - g_1(x))\varphi(x)\,dx = 0, \quad \varphi \in D(\Omega).$$

This means that $(g_1 - g_2) \perp D(\Omega)$ in $L^2(\Omega)$, where $D(\Omega)$ is a subspace of $L^2(\Omega)$. Therefore it suffices to show that $\overline{D(\Omega)} = L^2(\Omega)$ with respect to $\|\cdot\|_{L^2}$, because then

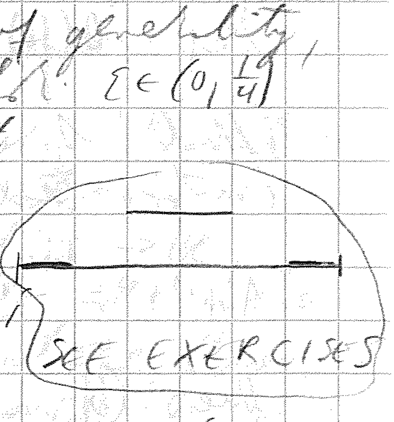
$$D(\Omega)^\perp = \overline{D(\Omega)}^\perp = \{0\}$$ and hence $g_1 \equiv g_2$ in $L^2(\Omega)$.

Since Ω is bounded, Theorem 2.5 yields

$$\overline{C^\infty(\Omega) \cap L^2(\Omega)} = L^2(\Omega).$$

We may assume, without loss of generality, that $\Omega = (0,1)$. Note first that for $\varepsilon \in (0, \frac{1}{4})$ there exists $\eta_\varepsilon \in C^\infty(\Omega)$ such that

① $0 \le \eta_\varepsilon(x) \le 1, \quad x \in \Omega;$
② $\eta_\varepsilon(x) = 0 \quad$ for $x \in (0,\varepsilon] \cup (1-\varepsilon, 1)$;
③ $\eta_\varepsilon(x) = 1 \quad$ for $x \in (2\varepsilon, 1-2\varepsilon)$.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If $f_0 \in C^\infty(\Omega) \cap L^2(\Omega)$, then $\eta_\varepsilon f_0 \in D(\Omega)$. The Lebesgue dominated convergence theorem yields

$$\|f_0 - \eta_\varepsilon f_0\|_{L^2}^2 = \int_0^1 |1 - \eta_\varepsilon(x)|^2 |f_0(x)|^2\,dx \to 0, \quad \varepsilon \to 0^+.$$

Therefore $C^\infty(\Omega) \cap L^2(\Omega) \subset \overline{D(\Omega)}$ in $L^2(\Omega)$.

If $f \in C^1(\Omega)$ and $f' \in L^2(\Omega)$, then (#) and the uniqueness show that f' is the weak derivative of f in L^2. □

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(α)

Remark The consideration above on weak derivatives leads to distributions (generalized functions). This happens, for example, if one requires

(a) each continuous function is a distribution
(b) each distribution has derivatives of all orders and these derivatives are distributions as well. If $f \in C^1$, then its derivative as distribution is its standard derivative
(c) standard rules for e.g. sums and differences should be valid (product turns out to be a problem)
(d) distributions should have good convergence properties so that one can handle limiting processes.

In fact, a distribution is defined as a linear map $\Lambda : D(\Omega) \to \mathbb{K}$. Each continuous $f \in C(\Omega)$ defines the linear map $\Lambda_f : D(\Omega) \to \mathbb{K}$ by $\Lambda_f(\varphi) = \int_\Omega f\varphi$, $\varphi \in D(\Omega)$.

(that is (a) is valid). If we now define (motivated by (#))

$$\Lambda'(\varphi) = -\Lambda(\varphi'), \quad \varphi \in D(\Omega),$$

then (b) and (c) are fulfilled. To deal with (d), the space $D(\Omega)$ of test functions must be endowed with a suitable topology τ and then distributions are precisely the continuous linear maps from

$(D(\Omega), \tau)$ to $\mathbb{K}$. This requires a bit of extra work and is therefore omitted in this course.

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Example 2.10 Set $\Omega = (-1,1)$ and consider the Heaviside function

$$H(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}.$$

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Then $H \in L^2(\Omega)$, but for all $\varphi \in D(\Omega)$ we have

$$-\int_{-1}^{1} \varphi'(x)\, H(x)\, dx = -\int_{0}^{1} \varphi'(x)\, dx = \varphi(0) - \varphi(1) = \varphi(0)$$

On the other hand, since the standard derivative satisfies $H'(x) = 0$ for all $x \neq 0$, there does not exist a function $g \in L^2(\Omega)$ such that

$$\int_\Omega g(x)\, \varphi(x)\, dx = \varphi(0).$$

for all $\varphi \in D(\Omega)$ (see also Lemma 2.9).

The derivative of the Heaviside function as distribution is the so-called Dirac delta functional δ for which

$$\begin{cases} \delta(x) = 0, \quad x \neq 0, \\ \int_\Omega \delta(x)\, dx = 1. \end{cases}$$

Hence δ cannot be an ordinary function, but it is a distribution: In fact δ is a linear map $\varphi \mapsto \varphi(0) : D(\Omega) \to \mathbb{K}$. Note also that in this example the weak derivative is a different creature than the derivative as distribution.

(*) The proof of Lemma 2.9 is based on Theorem 2.5 and hence Ω must be bounded. If Ω is unbounded, we apply Theorem 2.5 on bounded intervals $\Omega_1 \subset \Omega_2 \subset \dots$ for which $\Omega = \bigcup \Omega_j$.

Let $\Omega = (a,b)$ be an open interval. The *Sobolev space* $H^1 = H^1(\Omega)$ consists of $f \in L^2(\Omega)$ such that its weak derivative $f' \in L^2$, that is,

$$H^1 = \{ f \in L^2(\Omega) : f \text{ has the weak derivative } f' \in L^2(\Omega) \}$$

The number 1 refers to the order of the derivative and H to the Hilbert space. Another commonly used notation is $H^1 = W_2^1 = W^{1,2}$.

Theorem 2.11 H^1 is a Hilbert space with respect to the inner product

$$(\alpha) \quad \langle f, h \rangle = \int_\Omega \left(f(x)\overline{h(x)} + f'(x)\overline{h'(x)} \right) dx, \quad f, g \in H^1$$

Proof Since $f, h, f', h' \in L^2(\Omega)$, the Cauchy-Schwarz implies that (α) is well defined. Moreover, by Lemma 2

$$\|f-h\|_{H^1}^2 = \int_\Omega (f-h)\overline{(f-h)} + (f-h)'\overline{(f-h)'} \, dx$$

$$= \|f-h\|_{L^2}^2 + \|f'-h'\|_{L^2}^2, \quad f, h \in H^1,$$

because

$$\int_\Omega (f'(x) - h'(x))\varphi(x)\,dx = \int_\Omega f'(x)\varphi(x)\,dx - \int_\Omega h'(x)\varphi(x)\,dx$$

$$= -\int_\Omega f(x)\varphi'(x)\,dx + \int_\Omega h(x)\varphi'(x)\,dx$$

$$= -\int_\Omega (f(x) - h(x))\varphi'(x)\,dx \quad \text{for all } \varphi \in D(\Omega).$$

16 Let now $\{f_n\} \in H^1$ be a Cauchy sequence. Then $\{f_n\}$ and $\{f_n'\}$ are Cauchy in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there exists f and g in $L^2(\Omega)$ such that $f_n \to f$ and $f_n' \to g$ in $L^2(\Omega)$. Therefore it suffices to show that g is the weak derivative of f, i.e. $g = f'$.

Set $\varphi \in D(\Omega)$. Now

$$\int_\Omega f_n(x)\varphi'(x)\,dx = -\int_\Omega f_n'(x)\varphi(x)\,dx, \quad n \in \mathbb{N},$$

and the Cauchy-Schwarz inequality yields

$$\left| \int_\Omega f(x)\varphi'(x)\,dx + \int_\Omega g(x)\varphi(x)\,dx \right|$$

$$= \left| \int_\Omega (f(x) - f_n(x))\varphi'(x)\,dx + \int_\Omega (g(x) - f_n'(x))\varphi(x)\,dx \right|$$

$$\le \|f - f_n\|_{L^2}\|\varphi'\|_{L^2} + \|g - f_n'\|_{L^2}\|\varphi\|_{L^2} \to 0, \quad n \to \infty.$$

Thus $\int_\Omega g(x)\varphi(x)\,dx = -\int_\Omega f(x)\varphi'(x)\,dx, \quad \varphi \in D(\Omega)$,

that is, $f' = g \in L^2(\Omega)$.

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⇒ KANDITAATILLA HALUTUT OMINAISUUDET]

4, We will need some basic properties of Sobolev functions when dealing with the Sturm-Liouville problem. First we prove a version of the fundamental theorem of analysis for weak derivatives.

Lemma 2.12 If $f \in L^1(\Omega)$ and its weak derivative $f' \equiv 0$, that is, $\int_\Omega f(x)\varphi'(x)\,dx = 0$ for all $\varphi \in D(\Omega)$, then $f \equiv$ const. almost everywhere on Ω. In other words there exists a constant C such that the set $\{x \in \Omega : f(x) \neq C\}$ is of measure zero.

1 Proof Set $\Omega = (a,b)$. Fix a function $\psi \in D(\Omega)$ such that $\int_\Omega \psi(x)\,dx = 1$.

For $w \in D(\Omega)$, set

$$\varphi(x) = \int_a^x \left(w(t) - \left(\int_\Omega w(y)\,dy \right) \psi(t) \right) dt.$$

We claim that $\varphi \in D(\Omega)$.

To see this, note first that

$$\varphi'(x) = w(x) - \int_\Omega w(y)\,dy\ \psi(x).$$

Set now $[c,d] \subset (a,b) = \Omega$ such that $\operatorname{supp}(\psi) \cup \operatorname{supp}(w) \subset [c,d]$. Then

$\varphi(c') = 0$ for all $c' \in (a,c]$, and, for $d' \in [d,b)$ we have

$$\varphi(d') = \varphi(d') - \varphi(c') = \int_{c'}^{d'} \varphi'(t)\,dt$$

$$= \int_{c'}^{d'} \Big(w(t) - \Big(\int_\Omega w(x)\,dx\Big)\psi(t)\Big)\,dt$$

$$= \int_\Omega w(t)\,dt - \Big(\int_\Omega w(x)\,dx\Big)\int \psi(t)\,dt = 0,$$

because $\int_\Omega \psi(x)\,dx = 1$. Hence $\operatorname{supp}(\varphi) \subset \Omega$ is compa

By the hypothesis and Fubini's theorem

$$0 = \int_\Omega f(x)\varphi'(x)\,dx = \int_\Omega f(x)\Big(w(x) - \Big(\int_\Omega w(y)\,dy\Big)\psi(x)\Big)\,dx$$

$$= \int_\Omega f(x)w(x)\,dx - \int_\Omega f(x)\Big(\int_\Omega w(y)\,dy\Big)\psi(x)\,dx$$

$$= \int_\Omega f(y)w(y)\,dy - \int_\Omega w(y)\int_\Omega f(x)\psi(x)\,dx\,dy$$

$$= \int_\Omega w(y)\Big(f(y) - \int_\Omega f(x)\psi(x)\,dx\Big)\,dy,$$

Since $w \in D(\Omega)$ was arbitrary and $\overline{D(\Omega)} = L^2$ by the proof of Lemma 2.9, we deduce

$$f(y) - \int_\Omega f(x)\psi(x)\,dx = f(y) - \text{const} = 0$$

for almost every $y \in \Omega$.

□

If $f \in L^2(\Omega)$, then its weak derivative $f' \in L^2(\Omega)$ iff $f \in H^1(\Omega)$ by the definition. We will show that Sobolev functions are smooth in a certain sense. More precisely, we will prove the following special case of the Sobolev embedding theorem.

Theorem 2.13 Let $\Omega = (a,b)$ be a bounded interval, $f \in H^1(\Omega)$ and let $f' \in L^2(\Omega)$ be its weak derivative. Then the following assertions hold:

(a) there exists a (uniformly) continuous function $\tilde{f} : [a,b] \to \mathbb{K}$ such that $\tilde{f} = f$ almost everywhere on Ω, that is, the equivalence class determined by f contains a continuous member;

(b) $$f(x) = \int_{x_0}^{x} f'(t)\,dt + f(x_0) \quad \text{a.e. } x, x_0 \in (a,b)$$

Proof We may assume that $(a,b) = (0,1)$. Let $x_0 \in (0,1)$ and define

$$h(x) = \int_{x_0}^{x} f'(t)\,dt;$$

this function is well-defined because $f' \in L^2(\Omega) \subset L^1(\Omega)$ by the Cauchy-Schwarz inequality as Ω is bounded. If now $0 < x < y < 1$, then C-S yields

$$|h(x) - h(y)| = \left| \int_{x_0}^{x} f'(t)\,dt - \int_{x_0}^{y} f'(t)\,dt \right| = \left| \int_{x}^{y} f'(t)\,dt \right|$$

$$\leq \int_{x}^{y} |f'(t)|\,dt \leq \left(\int_{x}^{y} |f'(t)|^2\,dt \right)^{\frac{1}{2}} \left(\int_{x}^{y} 1\,dt \right)^{\frac{1}{2}}$$

$\leq \|f\|_{H^1} |x-y|^{\frac{1}{2}}$. Therefore h is uniformly continuous scalar valued function on $(0,1)$, and thus also continuous on $[0,1]$. We claim that $f - h = \text{const.}$ almost everywhere on $(0,1)$. The assertions (a) and (b) will follow from this. To prove the claim, assume that $x_0 = 0$ and $\varphi \in \mathcal{D}(\Omega)$. Fubini's theorem, the definition of the weak derivative and the fact that $\operatorname{supp}(\varphi)$ is compact in $(0,1)$ give

$$\int_0^1 \varphi'(x) h(x)\,dx = \int_0^1 \varphi'(x) \left(\int_0^x f'(t)\,dt \right) dx$$

$$= \int_0^1 \varphi'(x) \int_0^1 \chi(t \le x)\, f'(t)\,dt\,dx$$

$$= \int_0^1 f'(t) \int_t^1 \varphi'(x)\,dx\,dt$$

$$= -\int_0^1 f'(t)\,\varphi(t)\,dt = \int_0^1 f(t)\,\varphi'(t)\,dt,$$

and hence

$$\int_0^1 \varphi'(t)\big(h(t) - f(t)\big)\,dt = 0, \quad \varphi \in D(\Omega).$$

Therefore $f - h = \text{const.}$ almost everywhere on $\Omega = (0,1)$ by Lemma 2.12. $\square$

Remarks ① The existence of a continuous member in the equivalence class determined by a given $f \in H^1(\Omega)$ is a stronger property than the existence of an a.e. continuous member. For example $\chi_{[0,1]}$ is almost everywhere continuous on $\mathbb{R}$, but $\nexists\, h \in C(\mathbb{R})$ such that $h = \chi_{[0,1]}$ a.e. on $\mathbb{R}$.

② If f is continuous such that $f'(x)$ exists almost everywhere on Ω and $f' \in L^2(\Omega)$, then (b) is not necessarily valid: Cantor function $f : [0,1] \to [0,1]$ satisfies

$$f' = 0 \text{ a.e. } [0,1], \quad f(0) = 0 \text{ and } f(1) = 1.$$

We discuss two consequences of Theorem 2.13. We assume that the Sobolev function $f \in H^1(\Omega)$ is continuous (in the sense of Theorem 2.13 (a)) and in particular, $f(a) = \lim_{x \to a+} f(x)$ and $f(b) = \lim_{x \to b-} f(x)$, $\Omega = (a,b)$, exist. In this sense we have $H^1(\Omega) \subset C(\Omega)$, $\Omega = (a,b)$.

Corollary 2.14 If $f \in H^1(\Omega)$ and its weak derivative satisfies $f' \in C(\Omega)$, then $f \in C^1(\Omega)$, where $\Omega = (a,b)$.

Proof By Theorem 2.13 and the continuity of f',

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt, \quad x, x_0 \in [a,b],$$

and thus $f \in C^1(\Omega)$. □

Corollary 2.15 $H_0^1(a,b) = \{ f \in H^1(a,b) : f(a) = 0 = f(b) \}$ is a closed subspace of $H^1(a,b)$ and hence Hilbert.

Proof Ex. □

The (continuous) 2π-periodic functions in $H^1(0,2\pi)$ can be characterized in terms of Fourier coefficients: $f \in H^1(0,2\pi)$ and $f(0) = 2\pi$ is equivalent to

$$f \in L^2(0,2\pi) \text{ and } \sum_{k \in \mathbb{Z}} (1+k^2) |\hat{f}(k)|^2 < \infty.$$

See Ex (ass. first f is a trigonometric polynomial and approximate to obtain the general case = hint).

Applications to differential equations

Hilbert space-methods can be used to solve differential equations. We consider the classical example of Sturm-Liouville equations: Set $p \in C^1[0,1]$, $q \in C(0,1)$ and we try to find $u \in C^2[0,1]$ such that

$$\text{(SL)} \quad \begin{cases} -(p(x)u'(x))' + q(x)u(x) = 0, & x \in (0,1) \\ u(0) = \alpha, \quad u(1) = b \end{cases}$$

We make the additional assumption that there exists $\delta > 0$ such that

$$(\#) \quad p(x), q(x) \geq \delta, \quad x \in [0,1].$$

A function $u \in H^1(0,1)$ is a weak solution of (SL) if $u(0)=\alpha$, $u(1)=\beta$ and

$$\int_0^1 \underbrace{p(x)u'(x)}_{}\varphi'(x)\,dx + \int_0^1 \underbrace{q(x)u(x)}_{}\varphi(x)\,dx = 0$$

for all functions $\varphi \in \mathcal{D}(0,1)$. In other words, u is a weak solution of (SL) if qu is the weak derivative of pu'

Theorem 2.16 If $p \in C^1([0,1])$ and $q \in C([0,1])$ s.th. (#) is satisfied, then (SL) has a unique solution $u \in C^2([0,1])$.

Proof The proof is divided into several steps. The first one consists of proving the following claim.

Claim 1 If $u \in C^2$ is the classical solution of (SL) i.e. u'' is continuous on $[0,1]$ and satisfies the boundary conditions, then u is a weak solution.

Proof of Claim 1 Set $\varphi \in \mathcal{D}(0,1)$. Integration by parts gives

$$\int_0^1 (pu'\varphi' + qu\varphi)(x)\,dx = \Big|_0^1 p(x)u'(x)\varphi(x)$$

$$= \Big|_0^1 p(x)u'(x)\varphi(x) - \int_0^1 \varphi(x)\underbrace{\Big[(pu')'(x) - q(x)u(x)\Big]}_{=0}dx$$

$$= 0 - 0 - 0.$$

Thus u is also a weak solution. □

As the second step we endow $H^1(0,1)$ with the inner product

$$\langle u,v\rangle = \int_0^1 p(x)u'(x)\overline{v'(x)}\,dx + \int_0^1 q(x)u(x)\overline{v(x)}\,dx.$$

Clearly, $\langle u,v\rangle$ is linear with respect to u, $\langle u,v\rangle = \overline{\langle v,u\rangle}$ and $\langle u,u\rangle$ is strictly positive because $p(x), q(x) \geq \delta$

(unless $u \equiv 0$). Thus $\langle \cdot, \cdot \rangle$ is an inner product in $H^1(0,1)$.

Claim 2 $H^1(0,1)$ endowed with $\langle \cdot,\cdot \rangle$ is Hilbert.

Proof of Claim 2 It remains to show that $H^1(0,1)$ is complete with respect to the norm

$$\|u\| = \sqrt{\langle u, u\rangle}, \quad u \in H^1(0,1). \qquad \frac{1}{1-x}$$

Since p and q are continuous and (#) is satisfied, there exists $M>0$ s.th.

$$0 < \delta \le p(x), q(x) \le M < \infty, \quad x \in [0,1].$$

Therefore

$$\|u-v\|^2 = \int_0^1 p(x)|u'(x)-v'(x)|^2 + q(x)|u(x)-v(x)|^2 \, dx$$

$$\le M \int_0^1 |u'(x)-v'(x)|^2 + |u(x)-v(x)|^2 \, dx = M\|u-v\|^2_{H^1},$$

and similarly, $\delta \|u-v\|^2_{H^1} \le \|u-v\|^2$. Therefore these norms are equivalent and thus $(H^1, \|\cdot\|)$ is complete by Theorem 2.11. $\square$

As the third step we will show that (SL) has a weak solution.

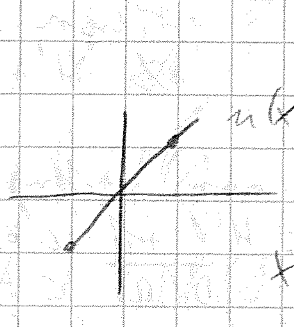
Claim 3 Equation (SL) has a weak solution.

$u(x) = \frac{1}{x}$ $\quad u(-1) = -1$

$u(1) = 1$

$u'(x) = -\frac{1}{x^2}$

$x u'(x) + u(x) = 0$



$u(x) = x$

$u'(x) = 1$

$x u'(x) = x$

Proof of Claim 3 Denote $X = (H^1, \langle \cdot,\cdot \rangle)$, where $\langle \cdot,\cdot \rangle$ is the inner product from step 2, and thus X is Hilbert. Fix $f \in X$ such that $f(0) = \alpha$ and $f(1) = \beta$, for example, $f(x) = \alpha + (\beta - \alpha)x$ does the job. By corollary 2.15, H^1_0 is a closed subspace of H^1 and hence, by the proof of Claim 2, H^1_0 is also a closed subspace of X. Therefore we may apply Theorem 1.21 to find a *unique* $g \in H^1_0$ such that

$$\|f - g\| = \inf \{ \|f - h\| : h \in H^1_0 \},$$

and further, by Theorem 1.25, $f - g \perp H^1_0$ with respect to $\langle \cdot,\cdot \rangle$. Denote $u = f - g$. Then

$$u(0) = f(0) - g(0) = \alpha - 0 = \alpha$$
$$u(1) = f(1) - g(1) = \beta - 0 = \beta$$

Since clearly $D(0,1) \subset H^1_0(0,1)$, for each $\varphi \in D(0,1)$ we have

$$0 = \langle f - g, \varphi \rangle = \int_0^1 \left(p(x) u'(x) \varphi'(x) + q(x) u(x) \varphi(x) \right) dx,$$

that is, u is a weak solution of (SL). □

The 4th step consists of showing that (SL) has at most one solution. Then, by Claim 3, (SL) has exactly one weak solution. We will need the following lemma.

Lemma 2.17 Let $\Omega = (a,b)$ be *bounded*. Then $\overline{D(\Omega)} = H^1_0$ with respect to H^1-norm.

Proof We may assume that $\Omega = (0,1)$. Since $H^1_0 \subset H^1$ is closed and $D(\Omega) \subset H^1_0(\Omega)$, we deduce $\overline{D(\Omega)} \subset H^1_0(\Omega)$.

Conversely, if $f \in H^1_0(\Omega)$, then

$f(x) = \int_0^x f'(t)\,dt$

by Theorem 2.13. Further, by the proof of Lemma 2.9, $\overline{D(\Omega)} = L^2(\Omega)$ with respect to $\|\cdot\|_{L^2}$, so there exists $g \in D(\Omega)$ such that $\|f' - g\|_{L^2} < \varepsilon$. Now, since Ω is bounded, $\|f' - g\|_{L^1} \le \|f' - g\|_{L^2} < \varepsilon$, and hence

$$(*) \qquad \left| \int_0^1 g(x)\,dx \right| = \left| \int_0^1 g(x)\,dx - \int_0^1 f'(x)\,dx \right| \le \|g - f'\|_{L^1} < \varepsilon.$$

To prove the assertion we now want to construct $h \in D(0,1)$ such that both $\|f - h\|_{L^2}$ and $\|f' - h'\|_{L^2}$ are small. A good candidate seems to be

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$$(**) \qquad h(x) = \int_0^x g(t)\,dt,$$

but the question is if it belongs to $D(0,1)$? This occures if $h(1) = 0$, which means that

$$c = \int_0^1 g(x)\,dx = 0,$$

but we only know $(*)$. To deal with this matter, we argue as in the proof of Lemma 2.12. Fix $\varphi \in D(0,1)$ such that $\varphi(x) \ge 0$ for all $x \in [0,1]$ and $\|\varphi\|_{L^1} = \int_0^1 \varphi(t)\,dt = 1$, and let $\tilde{g} = g - c\varphi$. Then $\tilde{g} \in D(0,1)$, $\int_0^1 \tilde{g}(t)\,dt = 0$, and

$$\|g - \tilde{g}\|_{L^2} = |c|\,\|\varphi\|_{L^2} = \left| \int_0^1 g(x)\,dx \right| \|\varphi\|_{L^2} < \varepsilon \|\varphi\|_{L^2}$$

by $(*)$. By replacing g by $\tilde{g}$ in $(**)$ we have $h \in D(0,1)$. Finally,

$$\|f - h\|_{H^1}^2 = \int_0^1 |f(x) - h(x)|^2\,dx + \int_0^1 |f'(x) - h'(x)|^2\,dx$$

$$= \int_0^1 \left| \int_0^x f'(t)\,dt - \int_0^x \tilde{g}(t)\,dt \right|^2 dx + \int_0^1 |f'(x) - \tilde{g}(x)|^2\,dx$$

$$\leq \int_0^1 \left(\int_0^x |f'(t) - \tilde{g}(t)| \, dt \right)^2 dx + \| f' - g \|_{L^2}^2$$

$$\leq \int_0^1 |f'(t) - \tilde{g}(t)|^2 \, dt + \| f' - \tilde{g} \|_{L^2}^2$$

$$= 2 \| f' - \tilde{g} \|_{L^2}^2$$

$$\leq 2 \left(\| f' - g \|_{L^2} + \| g - \tilde{g} \|_{L^2} \right)^2 \leq 2 \left(\varepsilon + \varepsilon \| \varphi \|_{L^2} \right)^2$$

$$\leq 2 \varepsilon^2 \left(1 + \| \varphi \|_{L^2} \right)^2 .$$

Thus $H_0^1(\Omega) \subset \overline{D(\Omega)}$. □

<u>Claim 4</u> If (SL) has a solution (this is true by Claim 3), then it is unique

<u>Proof of Claim 4</u> In the third step we proved that if $g \in H_0^1$ is the unique element for which $f - g \perp H_0^1$, then $u = f - g$ satisfies (SL).

Conversely, if u_1 satisfies (SL), then the boundary conditions yield $u_1 = f - g_1$ (recall that $f(0) = \alpha$, $f(1) = \beta$), where $g_1 \in H_0^1(0,1)$. By the definition of a weak solution $\langle u_1, \varphi' \rangle = 0$ for all $\varphi \in D(0,1)$, that is, $f - g_1 \perp D(0,1)$. But $\overline{D(0,1)} = H_0^1$ by Lemma 2.17, so $f - g_1 \perp H_0^1$. Theorem 1.25 now implies $\| f - g_1 \| = \mathrm{dist}(f, H_0^1)$, so Theorem 1.21 gives $u_1 = f - g_1 = f - g = u$, that is, the weak solution is unique. □

As the last step we show that the weak solution is also the classical solution.

<u>Claim 5</u> A weak solution of (SL) satisfies $u \in C^2([0,1])$.

<u>Proof of Claim 5</u> We know that $u \in H^1$ and so $u' \in L^2(\Omega)$. Since $p \in C^1([0,1])$, we have $pu' \in L^2(\Omega)$. Further, the weak derivative of pu' is $qu \in L^2(0,1)$ by Claim 3,

and hence $pu' \in H^1(0,1)$. Theorem 2.13 (a) now shows that pu' is continuous. Moreover, since $p(x) \geq \delta > 0$, we have $u' \in C(0,1)$ and that $u \in C^1([0,1])$ by Theorem 2.13 (b). Now $(pu')' = qu \in C([0,1])$ and hence $pu' \in C^1$, and thus $u \in C^2([0,1])$.

□

Comments ① The method employed above can be applied also in higher dimensions to elliptic equations, the prototype of which is the Laplace equation $\boxed{18}$

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a domain and $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. Strategy is as above:

(i) classical solution is a weak solution;
(ii) ∃ a weak solution;
(iii) the weak solution is unique (thus the classical solution is unique as well);
(iv) the weak solution is sufficiently regular (above $u \in C^2$).

② Few words about the history. Bernhard Riemann (1826 - 1866) solved $\Delta u = f$ by using the Dirichlet principle: If

$$J(u) = \int_0^1 \left(p(x)|u'(x)|^2 + q(x)|u(x)|^2\right) dx = \|u\|_1^2$$

then a function $u_0 \in H^1(0,1)$ such that $u_0(0) = \alpha$, $u_0(1) = \beta$ and

$$J(u_0) = \inf\{ J(u) : u(0) = \alpha \text{ and } u(1) = \beta \}$$

is a weak solution of (SL), see the proof of Claim 3.

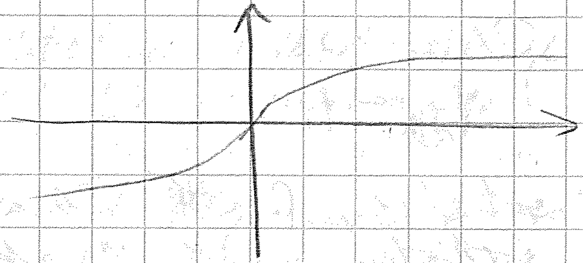
Karl Weierstraß (1815 - 1897) ("Über das sogenannte Dirichletische Princip") gave the following counter example to the Dirichlet principle. If

$$J(u) = \int_{-1}^{1} x^2 |u'(x)|^2 \, dx$$

there does not exist a function u_0 for which $u_0(1) = 1$, $u_0(-1) = -1$ and

$$J(u_0) = \inf \{ J(u) : u(-1) = -1 \text{ and } u(1) = 1 \}.$$

Indeed, if



$$\varphi(x) = \frac{\arctan \frac{x}{\varepsilon}}{\arctan \frac{1}{\varepsilon}}, \quad \varepsilon > 0,$$

then $$\varphi'(x) = \frac{1}{1 + \left(\frac{x}{\varepsilon}\right)^2} \frac{1}{\varepsilon} \frac{1}{\arctan \frac{1}{\varepsilon}}$$

$$= \frac{\varepsilon}{\left(\arctan \frac{1}{\varepsilon}\right)\left(x^2 + \varepsilon^2\right)},$$ and hence

$$J(\varphi) = \int_{-1}^{1} \underline{\underline{x^2}} \frac{\varepsilon^2 \, dx}{\left(\arctan \frac{1}{\varepsilon}\right)^2 \left(x^2 + \varepsilon^2\right)^2}$$

$$= \underbrace{\frac{\varepsilon^2}{\left(\arctan \frac{1}{\varepsilon}\right)^2}}_{= A} \int_{-1}^{1} \frac{x^2}{\left(x^2 + \varepsilon^2\right)^2} \, dx$$

$$= 2A \int_0^1 \frac{x^2 + \varepsilon^2 - \varepsilon^2}{\left(x^2 + \varepsilon^2\right)^2} \, dx = 2A \left[\int_0^1 \frac{dx}{x^2 + \varepsilon^2} - \int_0^1 \frac{\varepsilon^2}{\left(x^2 + \varepsilon^2\right)^2} \, dx \right.$$

$$= 2A \int_0^1 \frac{1}{\varepsilon} \frac{\frac{1}{\varepsilon}}{1 + \left(\frac{x}{\varepsilon}\right)^2} \, dx - \ldots$$

$$= \frac{2A}{\varepsilon} \Big|_0^1 \arctan\left(\frac{x}{\varepsilon}\right) - \ldots$$

$$< \frac{2\varepsilon}{\arctan\left(\frac{1}{\varepsilon}\right)} \to 0, \quad \varepsilon \to 0^+.$$

The reason why the Dirichlet principle does not work here is that H^1 (or H_0^1) is not complete with respect to

$$\|u\| = \int_{-1}^{1} x^2 |u'(x)|^2 \, dx.$$

That is why we assumed $p(x), q(x) \geq \delta$ in (SL). The assumption $q(x) \geq \delta$ can be removed but the situation changes radically if p has a zero. $(p(x) = x^2)$.

3, Linear operators

In section 2 we showed that the partial sums of the Fourier series of $f \in L^2$ converge to f in L^2. We also saw that if f is continuous 2π-periodic, then the arithmetic means of the partial sums of the FS of f converge uniformly to f (Fejér).

Question 1 Does the Fourier series converge in L^1-norm if $f \in L^1$?

Question 2 Does the FS of a continuous function converge pointwise? That is, if $f \in C([0,2\pi])$ and $f(0) = f(2\pi)$, is it true that

$$f(x) = \lim_{n \to \infty} \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} = \lim_{n \to \infty} s_n(f,x), \quad x \in [0,2\pi] ?$$

To answer these questions we will consider $s_n(f,\cdot)$ as a linear operator $f \longmapsto s_n(f,\cdot)$. We begin with recalling the basic concepts.

Let X and Y be vector spaces with the scalar field $\mathbb{K}$. A map $T: X \to Y$ is linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, $x, y \in X$, $\alpha, \beta \in \mathbb{K}$.

We already know that a linear operator

$T: X \to Y$ is continuous if and only if its operator norm $\|T\|_{X\to Y} = \sup_{\|x\|_X \le 1} \|T(x)\|_Y$ is bounded.

In this section we consider spaces of operators. For normed spaces X and Y, denote

$$L(X,Y) = \{T : X \to Y : T \text{ linear}\}$$
$$B(X,Y) = \{T \in L(X,Y) : T \text{ is continuous/bounded}$$

and if $Y = X$, we write $B(X) = B(X,X)$. Recall that by Proposition 1.11, $B(X,Y)$ endowed with the operator norm $\|\cdot\|_{X\to Y}$ is a normed space (with the scalar field $\mathbb{K}$ if X and Y are normed spaces $(\leftrightarrow \mathbb{K})$), and further $B(X,Y)$ is Banach if Y is Banach. Moreover, $B(X,Y)$ is a vector subspace of $L(X,Y)$.

If X and Y are normed spaces and $A \subset X$, we say that a sequence of maps $f_n : A \to Y$ converge pointwise to $f : A \to Y$ in A when

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in A.$$

In this case we write $f_n \to f$, $n \to \infty$, in A. In the proof of Proposition 1.11 we showed that:

Proposition 3.1 Let X, Y be normed spaces $\{T_n\} \subset L(X,Y)$ such that $T_n \to T$, $n \to \infty$, in X. Then $T \in L(X,Y)$.

Next, one would ask if the continuity is conserved in the pointwise convergence. The answer is in general negative as the following example shows.

Example 3.2 Let $P = P[0,1] = \{x : [0,1] \to \mathbb{R} :$

x is a polynomial with real coefficients}

and $\|x\|_\infty = \sup_{0\le t\le 1} |x(t)|$, $x \in P$. Define

$$T_n(x) = n\left(x(1) - x\left(1-\frac{1}{n}\right)\right),\ x\in P.$$

Then clearly, $T_n \in L(P,\mathbb{R})$ and further, $T_n \in B(P,\mathbb{R})$, because

$$\|T_n(x)\|_{\mathbb{R}} = |T_n(x)| = n\left|x(1) - x\left(1-\frac{1}{n}\right)\right|$$

$$\le n\left(|x(1)| + \left|x\left(1-\frac{1}{n}\right)\right|\right) \le 2n\|x\|_\infty,$$

and thus $\|T_n\|_{P\to\mathbb{R}} \le 2n$. On the other hand,

$$\lim_{n\to\infty} T_n(x) = \lim_{x\to\infty} \frac{x(1)-x\left(1-\frac{1}{n}\right)}{\frac{1}{n}} = x'(1),$$

and hence $T_n \to T$ in P, $T: P\to\mathbb{R}$, $T(x) = x'(1)$. T is of course linear, but it is not bounded: if $x_n(t) = t^n$, then $\|x_n\|_\infty = 1$ and

$$|T(x_n)| = |x_n'(1)| = n \Rightarrow \|T\|_{P\to\mathbb{R}} \ge \sup_n |T(x_n)| = \infty.$$

We will see later that the completeness plays a role here.

If $\{T_n\} \subset L(X,Y)$ and $T\in L(X,Y)$, we say that T_n converges to T in *operator norm* if

$$\lim_{n\to\infty}\|T_n - T\|_{X\to Y} = \lim_{n\to\infty}\left(\sup_{\|x\|_X\le 1}\|T_n(x) - T(x)\|_Y\right) = 0.$$

In this case we write $T_n\to T$, $n\to\infty$, or $\lim_{n\to\infty} T_n = T$ $\left(\not\Leftrightarrow \lim_{n\to\infty} T_n(x) = T(x)\ \forall x\in X\right)$

Obviously, $\lim_{n\to\infty} T_n = T$ implies

$\lim_{n\to\infty} T_n(x) = T(x)$ for all $x\in X$ because

$$\|T_n(x) - T(x)\| = \|(T_n - T)(x)\|_Y \le \|T_n - T\|_{X\to Y}\,\|x\|_X \to 0,\ n\to\infty.$$

The converse implication is false in general.

Example 3.3 Set $T_n : l^1 \to l^1$ by

$T_n(x) = (0, \dots, 0, x_n, x_{n+1}, \dots)$, $x = \{x_n\}_{n=1}^{\infty} \in l^1$, $n \in \mathbb{N}$.

Then $\|T_n(x)\|_{l^1} = \sum_{k=n}^{\infty} |x_k| \le \sum_{k=1}^{\infty} |x_k| = \|x\|_{l^1}$, $x \in l^1$,

and hence T_n is bounded and $\|T_n\|_{l^1 \to l^1} \le 1$ for all $n \in \mathbb{N}$. Further, $T_n(x) \to 0 \in l^1$ for all $x \in l^1$, that is, $T_n \to 0 \in l^1$, $n \to \infty$, pointwise. Indeed,

$\|T_n(x) - 0\|_{l^1} = \sum_{k=n}^{\infty} |x_k| \to 0$, $n \to \infty$, for all $x \in l^1$.

However, T_n does not converge to zero operator in the operator norm because each $e_n = (0, \dots, 0, 1, 0, \dots)$ is mapped to itself by T_n, $T_n(e_n) = e_n$, and hence $\|T(e_n)\|_{l^1} = \|e_n\|_{l^1} = 1$ yielding $\|T_n\|_{l^1 \to l^1} = 1$ for all $n \in \mathbb{N}$.

We state Proposition 1.11 here for further reference.

Proposition 3.4 If X is a normed space and Y is Banach, then $B(X, Y)$ endowed with $\|\cdot\|_{X \to Y}$ is Banach.

ALGEBRAIC

Spaces of operators have also algebraic structures. This is because we may define the product of operators (operator product) as the composition of operators.

Theorem 3.5 Let X, Y, Z be normed spaces and $T \in B(X, Y)$ and $S \in B(Y, Z)$. Then $ST = S \circ T \in B(X, Z)$ and

$\|ST\|_{X \to Z} \le \|S\|_{Y \to Z} \|T\|_{X \to Y}$

that is, the operator norm is submultiplicative.

Proof Clearly, $ST \in L(X, Z)$, further, if $x \in X$ and $\|x\|_X \le 1$, then

$$\|(ST)x\|_Z = \|S(T(x))\|_Z \le \|S\|\|T(x)\|_Y \le \|S\|\|T\|\|x\|_X \le \|S\|\|T\|.$$

Hence $ST \in B(X, Z)$ and $\|ST\| \le \|S\|\|T\|$. □

Recall also that by the elementary topology, $S.T$ is continuous as the composition of two continuous maps.

Remarks A Banach space X is called Banach algebra, if there exists a product $X \times X \to X$ such that

$\|xy\| \le \|x\|\|y\|$, $x, y \in X$, and ALGEBRA

$$\left.\begin{array}{ll} x(yz) = (xy)z & , x, y, z \in X; \\ x(y+z) = xy + xz & \quad " \quad ; \\ (x+y)z = xz + yz & \quad " \quad ; \\ (\lambda x)y = \lambda(xy) = x(\lambda y) & \quad " \quad , \lambda \in \mathbb{K}. \end{array}\right\}$$

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Example 3.6 (1) By Proposition 3.4 and Theorem 3.5, $B(X)$ is a Banach algebra, when X is Banach, the norm is the operator norm and the operator product is the composition

(2) If X is compact, then $C(X)$ endowed with the sup-norm and with the product $(fg)(x) = f(x)g(x)$, $x \in X$, is a Banach algebra.

Next we consider the boundedness of the inverse operator T^{-1} when $T: X \to Y$ is bijective. By the definition, $T^{-1}(y) = x$ if and only if $T(x) = y$, where $x \in X$ and $y \in Y$.

Theorem 3.7 Let X, Y be normed spaces and $T: X \to Y$ a linear bijection. Then T^{-1} is linear and

(*) T^{-1} is bounded $\iff \exists\, \alpha > 0$ s.th.

$$\|T(x)\|_Y \geq \alpha \|x\|_X \quad , \quad x \in X$$

that is, T is bounded below.

Proof If $x, y \in Y$ and $\lambda, \mu \in \mathbb{K}$, then

$$T(\overbrace{\lambda T^{-1}(x) + \mu T^{-1}(x)}) = \lambda T(T^{-1}(x)) + \mu T(T^{-1}(y))$$
$$= \lambda x + \mu y = T(\underbrace{T^{-1}(\lambda x + \mu y)})$$

Since T is a bijection, this implies that

$$\lambda T^{-1}(x) + \mu T^{-1}(y) = T^{-1}(\lambda x) + T^{-1}(\mu y).$$

Therefore T^{-1} is linear. It remains to prove (*).

Assume first that T^{-1} is bounded. Since $T^{-1}(x) = 0 \iff x = 0$, we may assume that $X \neq \{0\}$ and $Y \neq \{0\}$. Then there exists $y_0 \in Y \setminus \{0\}$ and so

$$0 < \frac{\|T^{-1}(y_0)\|_X}{\|y_0\|_Y} \leq \|T^{-1}\|_{Y \to X} < \infty$$

Thus $1/\|T^{-1}\|$ is finite. If $x \in X$ is arbitrary, then

$$\|x\|_X = \|T^{-1}(T(x))\|_X \leq \|T^{-1}\| \, \|T(x)\|_Y$$

and hence $\dfrac{1}{\|T^{-1}\|} \|x\|_X \leq \|T(x)\|_Y \quad , \quad x \in X.$

The assertion with $\alpha = \|T(x)\|^{-1}$ follows.

Assume now that there exists $\alpha > 0$ such that $\|T(x)\|_Y \geq \alpha \|x\|_X$ for all $x \in X$. For $y \in Y$ and $x = T^{-1}(y)$ we have $T(x) = T(T^{-1}(y)) = y$ and hence

$$\|T^{-1}(y\|_X = \|x\| \leq \frac{1}{\alpha} \|T(x)\|_Y = \frac{1}{\alpha} \|y\| \quad , \quad y \in Y.$$

Hence $\|T^{-1}\| \le \alpha^{-1} < \infty$ and thus T^{-1} is bounded. □

Corollary 3.8 Let X, Y be normed spaces and $T \in L(X, Y)$ bijection. Then T is a homeomorphism if and only if there exists $\alpha, \beta > 0$ such that

$$(\#) \quad \alpha \|x\|_X \le \|T(x)\|_Y < \beta \|x\|_X, \quad x \in X.$$

Proof This follows by Proposition 1.11 and Theorem 3.7. □

If a bijective $T \in B(X,Y)$ satisfies (#), we say that T is a linear *isomorphism* and that the spaces X and Y are (linearly) *isomorphic*.

Theorem 3.9 Let X, Y be normed spaces and $T \in B(X, Y)$ an isomorphism. Then X is complete if and only if Y is complete.

Proof Assume X is complete and let $\{y_n\}$ be a Cauchy sequence in Y. Then

$$\|T^{-1}(y_n) - T^{-1}(y_m)\|_X = \|T^{-1}(y_n - y_m)\|_X$$

$$\le \|T^{-1}\| \|y_n - y_m\|$$

and hence $\{T^{-1}(y_n)\}$ is Cauchy in X. Since X is complete, $T^{-1}(y_n) \to x \in X$, as $n \to \infty$, in $\|\cdot\|$, and hence

$$\|y_n - T(x)\|_Y = \|T(T^{-1}(y_n)) - T(x)\| \le \|T\| \|T^{-1}(y_n) - x\| \to 0, \quad n \to \infty.$$

Thus Y is complete. The converse implication can be proved in the same way. It also follows by symmetry. □

In particular, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in X, then $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete, the identity map I_X is an isomorphism from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$ and vice versa.

Example 3.10 (a) If $\{a_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ and χ_I is the characteristic function of the interval I, then the map

$$T : \{a_n\}_{n\in\mathbb{Z}} \longmapsto \sum_{n\in\mathbb{Z}} a_n \chi_{[n,n+1)}$$

defines an isomorphism $T : \ell^p(\mathbb{Z}) \to X$, where

$$X = \overline{\operatorname{span}}\{\chi_{[n,n+1)} : n \in \mathbb{Z}\}$$

is a closed subspace of $L^p(\mathbb{R})$. In fact, T is an isometry, that is,

$$\|T(x)\|_{L^p(\mathbb{R})} = \|x\|_{\ell^p} \quad \text{for all } x = \{a_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$$

because

$$\int_{\mathbb{R}} \Big| \sum_{n\in\mathbb{Z}} a_n \chi_{[n,n+1)}(x) \Big|^p dx = \sum_{n\in\mathbb{Z}} |a_n|^p, \quad \{a_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$$

(b) We show that the sequence space $c = \{\{x_n\} : \lim_{n\to\infty} x_n \text{ exists}\}$ is isomorphic to c_0 when both are endowed with the sup-norm

$$\|x\|_\infty = \sup_n |x_n|.$$

If $x = \{x_n\} \in c$, denote $l(x) = \lim_{n\to\infty} x_n$. Define $T : c \to c_0$ by setting

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BMO - [illegible] Calderón - Zygmund decomposition

$T(\{x_n\}) = \{y_n\}$, where

$y_1 = l(x)$, $y_n = x_{n-1} - l(x)$, $n \geq 2$,

Then $T(\{x_n\}) \in c_0$ because
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_{n-1} - l(x) = l(x) - l(x) = 0,$$
and $T : c \to c_0$ is linear (check). Moreover,

$$\|T(x)\|_\infty \leq \sup_n |x_n| + |l(x)| \leq 2\|x\|_\infty,$$

and consequently T is bounded with $\|T\| \leq 2$. Define $S : c_0 \to c$ by setting $S(\{x_n\}) = \{x_{n+1} + x_1\}$, $\{x_n\} \in c_0$. Then $S : c_0 \to c$ is linear and

$$\|S(x)\|_\infty = \sup_n |x_{n+1} + x_1| \leq 2\|x\|_\infty$$

so $\|S\| \leq 2$. Moreover, $T \circ S = I_{c_0}$ and $S \circ T = I_c$, and hence T is a bijection and $S = T^{-1}$. Namely

$$T(S(\{x_n\})) = T(\{\overset{\nearrow 0}{x_{n+1}} + x_1\})$$

$$= \{x_1, x_2 + x_1 - x_1, x_3 + x_1 - x_1, \dots\} = \{x_n\}$$

for all $\{x_n\} \in c_0$. The identity $S \circ T = I_c$ can be proven similarly.

(c) If $p \neq q$, then l^p is not isomorphic to l^q (proof omitted) and analogously L^p is isomorphic to L^q if and only if $p = q$. Further, l^2 is isomorphic to some closed subspace of L^q if $q \geq 2$, but this fails for $q < 2$. Furthermore, each separable Banach space is isomorphic to a closed subspace of $C([0,1])$. The proofs of these facts are not trivial.

(d) With regard to (SL) we considered

$$\|f\|_{H^1} = \left(\int_0^1 |f'(x)|^2 + |f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

and

$$\|f\| = \left(\int_0^1 \left(|f'(x)|^2 p(x) + |f(x)|^2 q(x) \right) dx \right)^{\frac{1}{2}},$$

where $0 < \delta \le p(x), q(x) \le M < \infty$, and proved that

$$\sqrt{\delta} \, \|f\|_{H^1} \le \|f\| \le \sqrt{M} \, \|f\|_{H^1}.$$

Therefore $\|\cdot\|_{H^1}$ and $\|\cdot\|$ induce the same topology to H^1 and H^1 is complete with respect to both norms.

Next, we prove a useful extension property of continuous operators.

Theorem 3.11 Let X be a normed space, Y Banach space, $M \subset X$ a vector space (not necessarily closed) and $T \in B(M, Y)$. Then there exists a unique $\hat{T} \in B(\bar{M}, Y)$ such that $\hat{T}(z) = T(z)$ for all $z \in M$ and $\|\hat{T}\|_{\bar{M} \to Y} = \|T\|_{M \to Y}$.