COMPLEX ANALYSIS II

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Background

This material of Complex Analysis II assumes that the reader is familiar with basic facts of complex analysis as presented in Complex Analysis Ia + Ib. In particular, the reader should be able to understand (and work with) complex numbers including their polar representation, and elementary complex functions such as the exponential function as well as basic trigonometric functions. Of course, the reader should also know the notion of analytic functions as well as Cauchy–Riemann equations, Möbius transformations, power series and complex integration. In particular, we shall apply Cauchy integral theorem, Cauchy integral formula, power series representation of analytic function, Gauss' mean value theorem, Cauchy inequalities, elementary uniqueness theorem of analytic functions, maximum principle and the Schwarz lemma, whenever needed.

1. Singularities for analytic functions

Unless otherwise specified, we are considering analytic functions in domains in question.

Definition. Given f, $z = a$ is an *isolated* singularity of f, if there exists $R > 0$ such that f is analytic in $0 < |z - a| < R$. The point $z = a$ is a *removable* singularity, if there exists an analytic $g\colon B(a, R) \to \mathbb{C}$ such that $g(z) = f(z)$ for all z such that $0 < |z - a| < R$.

Theorem 1.2. *A singularity at $z = a$ is removable if and only if*

$$\lim_{\substack{z\to a\\ z\neq a}} (z-a)f(z) = 0.$$

Proof. (1) As an analytic function, g is continuous, hence bounded around a. Therefore,

$$\lim_{\substack{z\to a\\ z\neq a}} (z-a)f(z) = \lim_{\substack{z\to a\\ z\neq a}} (z-a)g(z) = 0$$

trivially.

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(2) Let us define $h\colon B(a,R) \to \mathbb{C}$ by

$$h(z) := \begin{cases} (z-a)f(z), & z \neq a \\ 0, & z = a. \end{cases}$$

Clearly, h is continuous. We first prove that h is analytic. By the Cauchy integral theorem,

$$\int_\gamma h(\zeta)\, d\zeta = 0,$$

provided γ is a piecewise continuously differentiable closed path in $B(a,R)$. This implies the existence of $H\colon B(a,r) \to \mathbb{C}$ such that $H' = h$. Clearly, H is analytic. Therefore, H is infinitely differentiable, and so $h = H'$ also is differentiable and therefore analytic in $B(a,R)$. This implies that h can be represented as

$$h(z) = \sum_{j=0}^{\infty} a_j (z-a)^j.$$

Since $h(a) = 0$,

$$h(z) = \sum_{j=1}^{\infty} a_j (z-a)^j = (z-a) \sum_{j=0}^{\infty} a_{j+1} (z-a)^j.$$

As a convergent power series, $\sum_{j=0}^{\infty} a_{j+1}(z-a)^j =: g(z)$ determines an analytic function in $B(a,R)$. If $z \neq a$, then

$$(z-a)f(z) = h(z) = (z-a)g(z),$$

and so $f(z) = g(z)$. $\square$

Definition 1.3. An isolated singularity $z = a$ is a *pole*, if $\lim_{z\to a, z\neq a} |f(z)| = \infty$. If an isolated singularity is neither removable nor a pole, then it is called an *essential* singularity.

Theorem 1.4. *For a pole $z = a$ of f, there exists $m \in \mathbb{N}$ and an analytic function $g\colon B(a,R) \to \mathbb{C}$ such that*

$$f(z) = (z-a)^{-m} g(z)$$

for any $0 < |z-a| < R$.

Proof. Since $\lim_{z\to a, z\neq a} \frac{1}{|f(z)|} = 0$, we have

$$\lim_{\substack{z\to a \\ z\neq a}} (z-a)\frac{1}{f(z)} = 0.$$

By Theorem 1.2, $z = a$ is a removable singularity for $\frac{1}{f(z)}$. Therefore, there exists an analytic $h\colon B(a,R) \to \mathbb{C}$ such that

$$h(z) = \frac{1}{f(z)} \qquad \text{for all } 0 < |z-a| < R.$$

By the power series representation, for some $m \in \mathbb{N}$,

$$\begin{aligned} h(z) &= \sum_{j=m}^{\infty} a_j (z-a)^j = (z-a)^m \sum_{j=0}^{\infty} a_{m+j}(z-a)^j \\ &= (z-a)^m h_1(z), \end{aligned}$$

where h_1 is analytic in $B(a, R)$ and $h_1(a) \neq 0$. Since

$$\frac{1}{f(z)} = (z-a)^m h_1(z), \qquad 0 < |z-a| < R,$$

we get

$$(z-a)^m f(z) = \big(h_1(z)\big)^{-1} \tag{1.1}$$

Since $0 < |h_1(a)| < \infty$, it follows that $\frac{1}{h_1(z)}$ is bounded around $z = a$ and so

$$\lim_{z \to a} (z-a) \frac{1}{h_1(z)} = 0.$$

Therefore, $\frac{1}{h_1}$ has a removable singularity at $z = a$ and so there exists an analytic $g\colon B(a, R) \to \mathbb{C}$ so that $g(z) = \frac{1}{h_1(z)}$ for $0 < |z-a| < R$. By (1.1),

$$f(z) = (z-a)^{-m} g(z), \qquad 0 < |z-a| < R. \quad \square$$

Definition 1.5. Assume f has a pole at $z = a$. The smallest integer $m \in \mathbb{N}$ such that $(z-a)^m f(z)$ has a removable singularity at $z = a$, is the *multiplicity* of the pole.

Exercise 1.1. Consider the following functions around $z = 0$:

(1) $f(z) = \frac{1}{z}$

(2) $f(z) = \frac{\sin z}{z}$

(3) $f(z) = \frac{\cos z}{z}$

(4) $f(z) = \frac{1}{1-e^z}$

(5) $f(z) = e^{1/z}$

(6) $f(z) = z \sin \frac{1}{z}$.

Determine whether $z = 0$ is removable, a pole or an essential singularity. In case of a pole, determine also the multiplicity.

Theorem 1.6. (Laurent series). *A function f analytic in an annulus $0 \le R_1 < |z-a| < R_2 \le \infty$ admits a unique representation*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j.$$

The series on the right hand side converges absolutely and uniformly in every annulus $r_1 < |z-a| < r_2$ such that $R_1 < r_1 <_2 < R_2$. The coefficients a_j are determined by

$$a_j := \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - a)^{j+1}}\, d\zeta \qquad (\sim)$$

where $\gamma_r := \{\, |z-a| = r \,\}$, $R_1 < r < R_2$.

Proof. Omitted, see Saff–Snider, Theorem 5.5.14.

Theorem 1.7. *Let $z = a$ be an isolated singularity of f and*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j$$

be its Laurent series expansion in $0 < |z-a| < R$. Then

(1) *$z = a$ is removable if and only if $a_j = 0$ for $j \le -1$,*
(2) *$z = a$ is a pole of multiplicity $m \in \mathbb{N}$ if and only if $a_{-m} \ne 0$ and $a_j = 0$ for $j \le -(m+1)$,*
(3) *$z = a$ is essential if and only if $a_j \ne 0$ for infinitely many negative integers j.*

Exercise 1.2. Prove Theorem 1.7.

Theorem 1.8. (Casorati–Weierstraß). *If f has an essential singularity at $z = a$, then for every $\delta > 0$,*

$$\overline{f\big(B(a,\delta) \setminus \{a\}\big)} = \mathbb{C}.$$

Proof. We have to prove: Given $c \in \mathbb{C}$ and $\varepsilon > 0$, there exists for each $\delta > 0$ a point $z \ne a$ such that $|z-a| < \delta$ and $|f(z) - c| < \varepsilon$. If this is not the case, then there exists $c \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - c| \ge \varepsilon$ for all $z \in B(a,\delta)$, $z \ne a$. But then

$$\lim_{\substack{z \to a \\ z \ne a}} \left| \frac{f(z) - c}{z - a} \right| = \infty.$$

This means that $\frac{f(z)-c}{z-a}$ has a pole at $z = a$. Let m be the multiplicity. Then $m \ge 1$ and

$$g(z) := (z-a)^m \frac{f(z) - c}{z - a}$$

has a removable singularity. Therefore

$$0 = \lim_{z \to a} (z-a) g(z) = \lim_{z \to a} (z-a)^m \big(f(z) - c\big).$$

Then
$$\lim_{z\to a}(z-a)^m f(z) = \lim_{z\to a}\big[(z-a)^m\big(f(z)-c\big) + c(z-a)^m\big] = 0$$
and so
$$\lim_{z\to a}(z-a)\big(f(z)(z-a)^{m-1}\big) = 0.$$
Hence,
$$f(z)(z-a)^{m-1}$$
has a removable singularity at $z = a$. By Definition 1.1, there exists an analytic $g\colon B(a,\delta)\to\mathbb{C}$ such that
$$f(z) = \frac{g(z)}{(z-a)^{m-1}}, \qquad 0 < |z-a| < \delta.$$
If $m > 1$, then $\lim_{z\to a}|f(z)| = \infty$, hence f has a pole at $z = a$, and if $m = 1$, then $f(z)$ has a removable singularity at $z = a$. Both cases contradict the assumption of an essential singularity at $z = a$. $\square$

2. The residue theorem

Let $z = a$ be an isolated singularity of f and let

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-a)^j$$

be its Laurent expansion around $z = a$. Define now the *residue of f at $z = a$* by

$$\operatorname{Res}(f, a) := a_{-1}.$$

Theorem 2.1. (Residue theorem). *Assume that $f\colon G \to \overline{\mathbb{C}}$ is analytic in a convex region G except for finitely many poles $a_1, \dots, a_n$ and let γ be a piecewise continuously differentiable closed path in G such that $a_j \notin \gamma(I)$, $j = 1, \dots, n$. Then*

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta = \sum_{j=1}^{n} n(\gamma, a_j) \operatorname{Res}(f, a_j),$$

where $n(\gamma, a_j)$ denotes the winding number of γ around $z = a_j$ counterclockwise.

Remarks. (1) Intuitively, the winding number tells how many times one goes around $z = a_j$ as one follows the path γ from $\gamma(0)$ to $\gamma(1)$. We omit the exact definition.

(2) The residue theorem holds good even in a number of more general situations. We omit these considerations.

Proof of Theorem 2.1. Let

$$f(z) = \sum_{j=-\mu_k}^{\infty} a_{j,k}(z-a_k)^j = S_k(z) + \sum_{j=0}^{\infty} a_{j,k}(z-a_k)^j$$

be the Laurent expansions of $f(z)$ around $z = a_k$, $k = 1, \dots, n$. Clearly, $g(z) = f(z) - \sum_{k=1}^{n} S_k(z)$ is analytic in G. By the Cauchy theorem,

$$\begin{aligned} 0 = \int_\gamma g(\zeta)\, d\zeta &= \int_\gamma f(\zeta)\, d\zeta - \sum_{k=1}^{n} \int_\gamma S_k(\zeta)\, d\zeta \\ &= \int_\gamma f(\zeta)\, d\zeta - \sum_{k=1}^{n} \sum_{j=-\mu_k}^{-1} a_{j,k} \int_\gamma (\zeta - a_k)^j\, d\zeta. \end{aligned}$$

Therefore, it suffices to compute

$$\int_\gamma (\zeta - a_k)^{-m}\, d\zeta$$

for $1 \leq k \leq n$ and for any $m \in \mathbb{N}$. This integral is independent of the path and so we may assume γ to be a circle centered at a_k. Since $(\zeta - a_k)^{-m}$ has a primitive for $m \geq 2$, then $\int_\gamma (\zeta - a_k)^{-m} = 0$ for $m \geq 2$. If $m = 1$, then

$$\int_\gamma (\zeta - a_k)^{-1} \, d\zeta = 2\pi i n(\gamma, a_k)$$

by the Cauchy integral formula. Therefore,

$$\begin{aligned} 0 &= \int_\gamma f(\zeta) \, d\zeta - \sum_{k=1}^{n} a_{-1,k} \cdot 2\pi i n(\gamma, a_k) \\ &= \int_\gamma f(\zeta) \, d\zeta - 2\pi i \sum_{k=1}^{n} n(\gamma, a_j) \operatorname{Res}(f, a_j). \quad \square \end{aligned}$$

Theorem 2.2. *If $f(z)$ has a pole of multiplicity m at $z = a$ and*

$$g(z) := (z - a)^m f(z),$$

then

$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Proof. Clearly,

$$f(z) = \sum_{j=-m}^{\infty} a_j (z - a)^j$$

and so

$$g(z) = a_{-m} + a_{-m+1}(z - a) + \cdots + a_{-1}(z - a)^{m-1} + \cdots,$$

hence

$$g^{(m-1)}(a) = (m-1)! \, a_{-1}. \quad \square$$

Corollary 2.3. *If $f(z)$ has a simple pole at $z = a$ and $g(z) := (z - a)f(z)$, then*

$$\operatorname{Res}(f, a) = g(a) = \lim_{z \to a} (z - a) f(z).$$

Example 2.4. To compute

$$\int_{-\infty}^{+\infty} \frac{dx}{1 + x^2},$$

consider

$$f(z) = \frac{1}{1 + z^2} = \frac{1}{2i} \left(\frac{1}{z - i} + \frac{1}{z + i} \right).$$

$f(z)$ is analytic in $\mathbb{C} \setminus \{i, -i\}$, with simple poles at $z = \pm i$. By Corollary 2.3,

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \frac{1}{2i}.$$

Assume $R > 1$, and compute $\int_\gamma f(\zeta)\, d\zeta$, where γ is as in the figure. By the residue theorem

$$\int_\gamma \frac{d\zeta}{1 + \zeta^2} = 2\pi i \operatorname{Res}(f, i) = \pi.$$

On the other hand,

$$\int_\gamma \frac{d\zeta}{1 + \zeta^2} = \int_{-R}^{R} \frac{dx}{1 + x^2} + \int_{K_R} \frac{d\zeta}{1 + \zeta^2},$$

where K_R is the half-circle part of γ. But $\zeta = Re^{i\varphi}$ on γ and so $d\zeta = iRe^{i\varphi}\, d\varphi$, hence

$$\left| \int_{K_R} \frac{d\zeta}{1 + \zeta^2} \right| = \left| \int_0^\pi \frac{iRe^{i\varphi}}{1 + \zeta^2}\, d\varphi \right| \leq R \int_0^\pi \frac{d\varphi}{|1 + \zeta^2|} \leq \frac{R\pi}{R^2 - 1} \to 0 \qquad \text{as } R \to \infty,$$

since $|1 + \zeta^2| \geq \big| |\zeta|^2 - 1 \big| = R^2 - 1$ on K_R. Therefore

$$\pi = \lim_{R \to \infty} \int_\gamma \frac{d\zeta}{1 + \zeta^2} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} + \lim_{R \to \infty} \int_{K_R} \frac{d\zeta}{1 + \zeta^2},$$

giving

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi. \quad \square$$

Example 2.5. Prove that

$$\int_{-\infty}^{\infty} \frac{x^2\, dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}.$$

Now

$$f(z) = \frac{z^2}{1 + z^4}$$

is analytic in $\mathbb{C} \setminus \{a_1, \ldots, a_4\}$, where a_j:s are the fourth roots of -1. Making use of the same path γ as in Example 2.4, we need a_1, a_2 only;

$$a_1 = \frac{1}{\sqrt{2}}(1 + i), \qquad a_2 = \frac{1}{\sqrt{2}}(1 - i).$$

Now,

$$\begin{aligned} \operatorname{Res}(f, a_1) &= \lim_{z \to a_1} (z - a_1) f(z) = \lim_{z \to a_1} (z - a_1) \frac{z^2}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)} \\ &= \frac{a_1^2}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} = \frac{1 - i}{4\sqrt{2}}. \end{aligned}$$

Similarly,

$$\operatorname{Res}(f, a_2) = \frac{-1-i}{4\sqrt{2}}.$$

By the residue theorem,

$$\frac{1}{2\pi i}\int_\gamma f(\zeta)\,d\zeta = \operatorname{Res}(f, a_1) + \operatorname{Res}(f, a_2) = -\frac{i}{2\sqrt{2}}.$$

On the other hand,

$$\frac{1}{2\pi i}\int_\gamma f(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_{-R}^{R}\frac{x^2\,dx}{1+x^4} + \frac{1}{2\pi i}\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4}.$$

Now,

$$\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4} = \int_0^\pi \frac{R^2e^{2i\varphi}}{1+R^4e^{4i\varphi}}\cdot Rie^{i\varphi}\,d\varphi = \int_0^\pi iR^3\frac{e^{3i\varphi}\,d\varphi}{1+R^4e^{4i\varphi}}.$$

Since $|1+R^4e^{4i\varphi}| \geq R^4-1$, we get

$$\left|\int_{K_R}\frac{\zeta^2\,d\zeta}{1+\zeta^4}\right| \leq \frac{R^3}{R^4-1}\int_0^\pi d\varphi = \frac{\pi R^3}{R^4-1} \to 0 \qquad \text{as } R\to\infty$$

and so

$$-\frac{i}{2\sqrt{2}} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{x^2\,dx}{1+x^4} \implies \int_{-\infty}^{\infty}\frac{x^2\,dx}{1+x^4} = \frac{\pi}{\sqrt{2}}. \quad \square$$

Example 2.6. Compute

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi} \qquad \text{for } a>1.$$

On the unit circle $|z|=1$, $z=e^{i\varphi}$ and so $\overline{z} = e^{-i\varphi} = \frac{1}{e^{i\varphi}} = \frac{1}{z}$ and

$$\frac{z^2+2az+1}{2z} = a + \tfrac{1}{2}\left(z+\tfrac{1}{z}\right) = a + \tfrac{1}{2}(z+\overline{z}) = a + \tfrac{1}{2}(e^{i\varphi}+e^{-i\varphi}) = a+\cos\varphi.$$

Let γ be the unit circle. Observing that $\cos(-\varphi) = \cos\varphi$, we get

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi} = \tfrac{1}{2}\int_0^{2\pi}\frac{d\varphi}{a+\cos\varphi} = -i\int_\gamma \frac{dz}{z^2+2az+1} \tag{2.1}$$

Now, $z^2+2az+1 = (z-\alpha)(z-\beta)$, where

$$\alpha = -a+\sqrt{a^2-1}, \qquad \beta = -a-\sqrt{a^2-1}.$$

Since $a>1$, it is easy to see that $|\alpha|<1$, $|\beta|>1$. Therefore, by the residue theorem,

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+2az+1} &= 2\pi i\operatorname{Res}(f,\alpha) = 2\pi i\lim_{z\to a}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\\ &= 2\pi i\frac{1}{\alpha-\beta} = \frac{\pi i}{\sqrt{a^2-1}}.\end{aligned}$$

Combining with (2.1), one obtains

$$\int_0^\pi \frac{d\varphi}{a+\cos\varphi} = \frac{\pi}{\sqrt{a^2-1}}. \quad \square$$

Example 2.7. To evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx,$$

we consider

$$\begin{aligned}\int_\gamma \frac{e^{iz}}{z}\, dz &= \int_{-R}^{-\rho} \frac{e^{ix}}{x}\, dx + \int_{-\gamma_1} \frac{e^{iz}}{z}\, dz + \int_\rho^R \frac{e^{ix}}{x}\, dx + \int_{\gamma_2} \frac{e^{iz}}{z}\, dz \\ &= 2i\int_\rho^R \frac{\sin x}{x}\, dx - \int_{\gamma_1} \frac{e^{iz}}{z}\, dz + \int_{\gamma_2} \frac{e^{iz}}{z}\, dz.\end{aligned}$$

The integral $= 0$, since (1) $f(z) = e^{iz}/z$ is analytic inside of γ, (2) $e^{iz} = \cos z + i\sin z$, (3) $\cos x/x$ is an odd function and $\sin x/x$ is even.

To evaluate the integral over γ_2, we need the Jordan inequality

$$\int_0^\pi e^{-R\sin\varphi}\, d\varphi \le \frac{\pi}{R}(1-e^{-R}) \qquad (R>0).$$

To this end, consider $g(\varphi) := \sin\varphi - \varphi\cos\varphi$. Since $g(0) = 0$ and $g'(\varphi) = \cos\varphi - \cos\varphi + \varphi\sin\varphi \ge 0$, $g(\varphi) \ge 0$ for $0 \le \varphi \le \pi/2$. Therefore,

$$D\left(\frac{\sin\varphi}{\varphi}\right) = \frac{\varphi\cos\varphi - \sin\varphi}{\varphi^2} \le 0, \qquad 0 < \varphi \le \pi/2;$$

since $(\sin\varphi/\varphi)_{\varphi=\pi/2} = \frac{2}{\pi}$, we have $\sin\varphi/\varphi \ge \frac{2}{\pi}$ for $0 < \varphi \le \pi/2$. Then $e^{-R\sin\varphi} \le e^{-R\frac{2\varphi}{\pi}}$, and so

$$\int_0^\pi e^{-R\sin\varphi}\, d\varphi = 2\int_0^{\pi/2} e^{-R\sin\varphi}\, d\varphi \le 2\int_0^{\pi/2} e^{-R\cdot\frac{2\varphi}{\pi}}\, d\varphi = \frac{\pi}{R}(1-e^{-R}).$$

Therefore,

$$\begin{aligned}\left|\int_{\gamma_2} \frac{e^{iz}}{z}\, dz\right| &= \left|\int_0^\pi e^{iR(\cos\varphi + i\sin\varphi)}\cdot i\, d\varphi\right| \le \int_0^\pi |e^{iR\cos\varphi}| e^{-R\sin\varphi}\, d\varphi \\ &= \int_0^\pi e^{-R\sin\varphi}\, d\varphi \le \frac{\pi}{R}(1-e^{-R}) \to 0 \qquad \text{as } R\to\infty.\end{aligned}$$

By the Taylor expansion of e^{iz},

$$\frac{e^{iz}}{z} = \frac{1}{z} + g(z), \qquad g(z) \text{ analytic (in } \mathbb{C}).$$

So,

$$\int_{\gamma_1} \frac{e^{iz}}{z}\, dz = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_1} g(z)\, dz,$$

and now

$$\int_{\gamma_1} \frac{dz}{z} = i \int_0^\pi d\varphi = \pi i,$$

$$\left| \int_{\gamma_1} g(z)\, dz \right| \le K \int_0^\pi |\rho e^{i\varphi}|\, d\varphi = K\pi\rho \to 0 \qquad \text{as } \rho \to 0.$$

Therefore,

$$\int_{\gamma_1} \frac{e^{iz}}{z}\, dz \to \pi i \qquad \text{as } \rho \to 0.$$

Hence,

$$\begin{aligned} 0 &= 2i \int_\rho^R \frac{\sin x}{x}\, dx - \int_{\gamma_1} \frac{e^{iz}}{z}\, dz + \int_{\gamma_2} \frac{e^{iz}}{z}\, dz \\ &\to 2i \int_0^\infty \frac{\sin x}{x}\, dx - \pi i \qquad \text{as } R \to \infty \text{ and } \rho \to 0. \end{aligned}$$

This results in

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \quad \square$$

Example 2.8. Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

Consider

$$f(z) = \frac{1 + 2iz - e^{2iz}}{z^2}.$$

The only possible pole is $z = 0$. Since the power series of e^{2iz} converges for all z (e^{2iz} is entire!), $\varphi(z)$ below is bounded around $z = 0$:

$$\begin{aligned} \frac{1 + 2iz - e^{2iz}}{z^2} &= \frac{1}{z^2} + \frac{2i}{z} - \left(\frac{e^{iz}}{z}\right)^2 = \frac{1}{z^2} + \frac{2i}{z} - \left(\frac{1}{z} + i - \tfrac{1}{2}z + \cdots\right)^2 \\ &= \frac{1}{z^2} + \frac{2i}{z} - \frac{1}{z^2} - \frac{2i}{z} + \varphi(z); \end{aligned}$$

Therefore, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\varphi(z) = 0$, and so $f(z)$ has a removable singularity at $z = 0$. Since $f(z)$ is analytic in $\mathbb{C}$, by the Cauchy theorem,

$$0 = \int_\gamma f(\zeta)\, d\zeta = \int_{\widehat{\gamma}} f(\zeta) d\zeta + \int_{-R}^R \frac{1 + 2ix - e^{2ix}}{x^2}\, dx.$$

For the last integral, we get

$$\begin{aligned}\int_{-R}^{R} \frac{1+2ix-e^{2ix}}{x^2}\,dx &= \int_{-R}^{R} \frac{1-e^{2ix}}{x^2}\,dx + 2i\int_{-R}^{R}\frac{dx}{x}\\ &= \int_{-R}^{R} \frac{1-\cos 2x}{x^2}\,dx - i\int_{-R}^{R}\frac{\sin 2x}{x^2}\,dx + 2i\int_{-R}^{R}\frac{dx}{x}\\ &= 2\int_{-R}^{R}\frac{\sin^2 x}{x^2}\,dx + \text{ a purely imaginary term}\\ &= 4\int_{0}^{R}\frac{\sin^2 x}{x^2}\,dx + \text{ a purely imaginary term.}\end{aligned}$$

For the integral on $\widehat{\gamma}$,

$$\begin{aligned}\int_{\widehat{\gamma}} f(\zeta)\,d\zeta &= \int_0^{\pi} \frac{1+2iRe^{i\varphi}-e^{2iRe^{i\varphi}}}{R^2e^{2i\varphi}}\cdot iRe^{i\varphi}\,d\varphi\\ &= \int_0^{\pi}\frac{i}{R}e^{-i\varphi}\,d\varphi - 2\int_0^{\pi}d\varphi - \int_0^{\pi}\frac{i}{R}e^{-i\varphi}e^{2iRe^{i\varphi}}\,d\varphi = I_1+I_2+I_3.\end{aligned}$$

Now,

$$\begin{aligned}|I_1| &\le \frac{1}{R}\int_0^{\pi}d\varphi = \frac{\pi}{R}\to 0 \qquad \text{as } R\to\infty,\\ I_2 &= -2\pi\end{aligned}$$

and

$$\begin{aligned}|I_3| &= \left|\int_0^{\pi}\frac{i}{R}e^{-i\varphi}e^{2iR\cos\varphi}e^{-2R\sin\varphi}\,d\varphi\right|\\ &\le \frac{1}{R}\int_0^{\pi}e^{-2R\sin\varphi}\,d\varphi = \frac{2}{R}\int_0^{\pi/2}e^{-2R\sin\varphi}\,d\varphi\\ &\le \frac{2}{R}\int_0^{\pi/2}e^{-\frac{4R\varphi}{\pi}}\,d\varphi = \frac{\pi}{2R^2}(1-e^{-2R})\to 0 \qquad \text{as } R\to\infty.\end{aligned}$$

Therefore, by taking real parts,

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx = \lim_{R\to\infty}\left(-\tfrac{1}{4}\int_{\widehat{\gamma}} f(\zeta)\,d\zeta\right) = \frac{\pi}{2} + \lim_{R\to\infty}(I_1+I_3) = \frac{\pi}{2}. \quad \square$$

Example 2.9. To compute,

$$\int_0^{\infty}\frac{dx}{(x^2+1)^2},$$

denote

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}.$$

Clearly, $f(z)$ has double poles in $z = \pm i$, and no other poles. Therefore, by Theorem 2.2,

$$\operatorname{Res}(f, i) = \frac{1}{1!} g'(i),$$

where $g(z) = (z - i)^2 f(z) = \frac{1}{(z+i)^2}$. Hence,

$$\big(g'(z)\big)_{z=i} = \left(-\frac{2}{(z+i)^3}\right)_{z=i} = \frac{1}{4i}$$

and so

$$\operatorname{Res}(f, i) = \frac{1}{4i}.$$

By the residue theorem,

$$\int_\gamma \frac{d\zeta}{(\zeta^2+1)^2} = 2\pi i \operatorname{Res}(f, i) = \frac{\pi}{2}.$$

On the other hand,

$$\int_\gamma \frac{d\zeta}{(\zeta^2+1)^2} = \int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{K_R} \frac{d\zeta}{(\zeta^2+1)^2}.$$

But

$$\left| \int_{K_R} \frac{d\zeta}{(\zeta^2+1)^2} \right| \le \frac{\pi R}{(R^2-1)^2} \to 0 \qquad \text{as } R \to \infty.$$

Since $\frac{1}{(x^2+1)^2}$ is an even function,

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \tfrac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Exercises. Evaluate the following integrals by making use of the residue theorem:

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^3}$,

(2) $\displaystyle\int_0^{\pi/2} \frac{d\varphi}{a + \sin^2\varphi}$ for $a > 0$,

(3) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^3}\,dx$,

(4) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$.

Additional reading:

D. Mitrinović: Calculus of Residues
E. Saff – A. Snider: Fundamentals of Complex Analysis

3. The argument principle

3.1. The logarithm in the complex plane. The exponential function is locally injective in $\mathbb{C}$. In fact, assume

$$e^z = e^t \implies e^{z-t} = 1.$$

Denote $z - t = x + iy$, $x, y \in \mathbb{R}$. Then

$$1 = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = 1$$

$\implies$

$$\begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0. \end{cases}$$

Since

$$1 = |e^{z-t}| = e^x,$$

we see that $x = 0$. Then $\cos y = 1$, $\sin y = 0$ implies $y = n \cdot 2\pi$. Therefore, the nearest possible points z, t with $e^z = e^t$ have a distance 2π, and given any z_0, e^z is injective in $B(z_0, 2\pi)$.

So, we can locally define the inverse function $\log z$ for the exponential. Since

$$z = e^{\log z} = e^{\log z + n \cdot 2\pi i},$$

$\log z$ has infinitely many branches. Denoting $u + iv = \log z$, we get

$$z = e^{u+iv} = e^u e^{iv} \implies |z| = e^u \implies u = \log |z|$$

and

$$re^{i\varphi} = z = |z|e^{i\varphi} = e^u e^{iv}$$

and so we may take $v = \varphi = \arg z$. Hence

$$\log z = \log |z| + i \arg z + n \cdot 2\pi i$$

If γ is now a closed path in $\mathbb{C}$, and we consider $\log z$ on γ, we easily see that return to the original branch appears, if the winding number around $z = 0$ is zero; otherwise we move to another branch. So, if we have a domain $G \subset \mathbb{C} \setminus \{0\}$, then $\log z$ is uniquely determined and analytic in G. This will be applied in the proof of Theorem 3.3.

3.2. The argument principle. Assume $f(z)$ is analytic around $z = a$ and has a zero of multiplicity m at $z = a$. Then $f(z) = (z - a)^m g(z)$, $g(a) \neq 0$. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{3.1}$$

Since $g(a) \neq 0$, $g'(z)/g(z)$ is analytic around $z = a$. Similarly, if $f(z)$ has a pole of order m at $z = a$, and $f(z) = (z - a)^{-m} g(z)$, $g(a) \neq 0$, then

$$\frac{f'(z)}{f(z)} = -\frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{3.2}$$

Definition 3.1. Assume that $f\colon G \to \overline{\mathbb{C}}$ is analytic in an open set $G \subset \mathbb{C}$ except for poles. Then f is said to be *meromorphic* in G.

Theorem 3.2. *Assume that $f\colon G \to \overline{\mathbb{C}}$ is meromorphic in a convex region G except for finitely many zeros $a_1, \dots, a_n$ and poles $b_1, \dots, b_m$, each repeated according to multiplicity. If γ is a piecewise continuously differentiable closed path in G such that $a_j \notin \gamma(I)$, $j = 1, \dots, n$, and $b_j \notin \gamma(I)$, $j = 1, \dots, m$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{j=1}^{n} n(\gamma, a_j) - \sum_{j=1}^{m} n(\gamma, b_j).$$

Proof. By the same idea as in (3.1) and (3.2),

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n}\frac{1}{z-a_j} - \sum_{j=1}^{m}\frac{1}{z-b_j} + \frac{g'(z)}{g(z)},$$

where $g(z)$ is analytic and non-zero in G. Since g'/g is analytic in G, elementary integration and the Cauchy theorem result in

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_\gamma \frac{1}{\zeta-a_j}\,d\zeta - \frac{1}{2\pi i}\sum_{j=1}^{m}\int_\gamma \frac{1}{\zeta-b_j}\,d\zeta \\ &= \sum_{j=1}^{n} n(\gamma, a_j) - \sum_{j=1}^{m} n(\gamma, b_j). \quad \square\end{aligned}$$

Theorem 3.3. (Rouché). *Let f, g be meromorphic in a convex region G and let $\overline{B(a,R)} \subset G$ be a closed disc. Suppose f, g have no zeros and no poles on the circle $\gamma = \partial B(a,R) = \{\, z \in G \mid |z-a| = R \,\}$ and that $|f(z) - g(z)| < |g(z)|$ for all $z \in \gamma$. Then*

$$\mu_f - \nu_f = \mu_g - \nu_g,$$

where μ_f, μ_g, resp. ν_f, ν_g, are the number zeros, resp. poles, of f and g in $\{\, z \in G \mid |z-a| < R \,\}$, counted according to multiplicity.

Proof. By the assumption,

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \tag{3.3}$$

for all $z \in \gamma$. By the Theorem 3.2,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\big(f(\zeta)/g(\zeta)\big)'}{\big(f(\zeta)/g(\zeta)\big)}\,d\zeta &= \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}\,d\zeta \\ &= \mu_f - \nu_f - (\mu_f - \nu_g),\end{aligned}$$

since the winding number of γ for all zeros and poles in $\{\, z \in G \mid |z-a| < R \,\}$ equals to one. On the other hand, by (3.3), f/g maps γ into $B(1,1)$, and so a fixed branch of $\log(f/g)$ is a primitive of $(f/g)'/(f/g)$. Integrating over γ, the logarithm doesn't change the branch, hence $\log(f/g)$ takes the same value at $\gamma(0)$ and $\gamma(1) = \gamma(0)$ resulting in

$$\frac{1}{2\pi i}\int_\gamma \frac{\big(f(\zeta)/g(\zeta)\big)'}{\big(f(\zeta)/g(\zeta)\big)}\,d\zeta = 0.$$

The assertion now follows. $\square$

4. Infinite products

The basic idea behind of this section is the need to separate the zeros (and poles) of a meromorphic function $f(z)$ as a product component of $f(z)$. In principle, this results in an infinite product. To this end, we first prove

Theorem 4.1. *If $f(z)$ is an entire function with no zeros, then there exists another entire function $g(z)$ such that*

$$f(z) = e^{g(z)}.$$

Proof. Since $f(z) \neq 0$ for all $z \in \mathbb{C}$, then $\frac{f'(z)}{f(z)}$ is entire. Therefore,

$$\frac{f'(z)}{f(z)} = \sum_{j=0}^{\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + \cdots .$$

is a power series representation converging in the whole $\mathbb{C}$. Consider

$$h(z) = a_0 z + \tfrac{1}{2} a_1 z^2 + \tfrac{1}{3} a_2 z^3 + \cdots = z(a_0 + \tfrac{1}{2} a_1 z + \tfrac{1}{3} a_2 z^2 + \cdots). \tag{4.1}$$

Since

$$\limsup_{j\to\infty} \sqrt[j]{\frac{1}{j+1}|a_j|} = \limsup_{j\to\infty} \frac{1}{\sqrt[j]{j+1}} \sqrt[j]{|a_j|} = \limsup_{j\to\infty} \sqrt[j]{|a_j|} = 0,$$

the power series (4.1) has radius of convergence $= \infty$. Therefore, (4.1) determines an entire function. Differentiating term by term, as we may do for a converging power series, we get

$$h'(z) = \frac{f'(z)}{f(z)}.$$

Define now

$$\varphi(z) := f(z) e^{-h(z)},$$

hence

$$\varphi'(z) = f'(z)e^{-h(z)} - f(z)h'(z)e^{-h(z)} = e^{-h(z)}\big(f'(z) - f(z)h'(z)\big) \equiv 0.$$

Therefore, $\varphi(z)$ is constant, say $\varphi(z) \equiv e^a$, $a \in \mathbb{C}$. Note that $\varphi(z) \neq 0$ for all $z \in \mathbb{C}$. So,

$$f(z)e^{-h(z)} = e^a \implies f(z) = e^{a+h(z)}.$$

Defining $g(z) := a + h(z)$, we have the assertion. □

Definition 4.2. The infinite product $\prod_{j=1}^{\infty} b_j$ of complex numbers b_j *converges*, if there exists

$$\lim_{n\to\infty} \prod_{j=1}^{n} b_j \neq 0.$$

Remark. Define $P_n := \prod_{j=1}^n b_j$. Clearly, $\prod_{j=1}^\infty b_j$ converges if and only if (P_n) converges and $\lim_{n\to\infty} P_n \neq 0$. Then $b_n = P_n/P_{n-1}$ and there exists

$$\lim_{n\to\infty} b_n = \frac{\lim_{n\to\infty} P_n}{\lim_{n\to\infty} P_{n-1}} = 1. \tag{4.2}$$

Therefore, it is customary to use the notation

$$b_n = 1 + a_n;$$

then $\lim_{n\to\infty} a_n = 0$ by (4.2).

Theorem 4.3. *If $a_j \geq 0$ for all $j \in \mathbb{N}$, then $\prod_{j=1}^\infty (1+a_j)$ converges if and only if $\sum_{j=1}^\infty a_j$ converges.*

Proof. Observe first that $P_n := \prod_{j=1}^n (1+a_j)$ is a non-decreasing sequence, since $a_j \geq 0$. Therefore, (P_n) either converges to a finite (real) value, or to $+\infty$. Clearly,

$$a_1 + a_2 + \cdots + a_n \leq (1+a_1)(1+a_2)\cdots(1+a_n).$$

On the other hand,

$$(1+a_1)\cdots(1+a_n) \leq e^{a_1}\cdots e^{a_n} = e^{a_1+\cdots+a_n},$$

since $e^x \geq 1 + x$ for every $x \geq 0$. So, we have

$$\sum_{j=1}^n a_j \leq \prod_{j=1}^n (1+a_j) \leq e^{\sum_{j=1}^n a_j}. \tag{4.3}$$

If $(\sum_{j=1}^n a_j)_{n\in\mathbb{N}}$ converges, then $(e^{\sum_{j=1}^n a_j})_{n\in\mathbb{N}}$ converges by the continuity of the exponential function. This implies that the increasing sequence $\left(\prod_{j=1}^n (1+a_j)\right)_{n\in\mathbb{N}}$ converges to a non-zero limit by (4.3). If $\left(\prod_{j=1}^n (1+a_j)\right)_{n\in\mathbb{N}}$ converges, then the increasing sequence $(\sum_{j=1}^n a_j)_{n\in\mathbb{N}}$ converges, again by (4.3). $\square$

Theorem 4.4. *If $a_j \geq 0$, $a_j \neq 1$, for all $j \in \mathbb{N}$, then $\prod_{j=1}^\infty (1-a_j)$ converges if and only if $\sum_{j=1}^\infty a_j$ converges.*

Proof. (1) Assume $\sum_{j=1}^\infty a_j$ converges. By the Cauchy criterium,

$$\sum_{j=N}^\infty a_j < \tfrac{1}{2}$$

for N sufficiently large; then also $a_j < 1$, $j \geq N$. Observe that

$$\begin{aligned}(1-a_N)(1-a_{N+1}) &= 1 - a_N - a_{N+1} + a_N a_{N+1} \\ &\geq 1 - a_N - a_{N+1} \quad \left(= 1 - (a_N + a_{N+1}) > \tfrac{1}{2}\right).\end{aligned}$$

Assume we have proved

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n) \geq 1-a_N-a_{N+1}-\cdots-a_n. \tag{4.4}$$

Then

$$\begin{aligned}
&(1-a_N)(1-a_{N+1})\cdots(1-a_n)(1-a_{n+1}) \\
&\quad \geq (1-a_N-a_{N+1}-\cdots-a_n)(1-a_{n+1}) \\
&\quad = 1-a_N-a_{N+1}-\cdots-a_n-a_{n+1}+(a_N+\cdots+a_n)a_{n+1} \\
&\quad \geq 1-a_N-a_{N+1}-\cdots-a_{n+1},
\end{aligned}$$

and so (4.4) is true for all $n \geq N$. Therefore

$$(1-a_N)(1-a_{N+1})\cdots(1-a_n) \geq 1-(a_N+\cdots+a_n) > \tfrac{1}{2}.$$

This implies that the decreasing sequence $\prod_{j=N}^{\infty}(1-a_j)$ converges to a limit $P \geq \frac{1}{2}$. If N is sufficiently large, then $0 < 1-a_j < 1$ and so $P \leq 1$. Writing, for $n > N$,

$$P_n = \prod_{j=1}^{n}(1-a_j) = P_{N-1} \cdot \prod_{j=N}^{n}(1-a_j),$$

we get

$$\lim_{n\to\infty} P_n = P_{N-1} \cdot \lim_{n\to\infty} \prod_{j=N}^{n}(1-a_j) = P_{N-1} \cdot P = (1-a_1)\cdots(1-a_{N-1})P \neq 0,$$

so $\prod_{j=1}^{\infty}(1-a_j)$ converges.

(2) Assume now that $\sum_{j=1}^{\infty} a_j$ diverges. If a_j does not converge to zero, then $1-a_j$ does not converge to one. By the Remark after Definition 4.2, $\prod_{j=1}^{\infty}(1-a_j)$ diverges.

So, we may assume that $\lim_{j\to\infty} a_j = 0$. Let N be sufficiently large so that $0 \leq a_j < 1$ for $j \geq N$. Since $1-x \leq e^{-x}$ for $0 \leq x < 1$, we have

$$1-a_j \leq e^{-a_j}, \qquad j \geq N.$$

Therefore,

$$0 \leq \prod_{j=N}^{n}(1-a_j) \leq \prod_{j=N}^{n} e^{-a_j} = e^{-\sum_{j=N}^{n} a_j}, \qquad n > N.$$

Since $\sum_{j=N}^{\infty} a_j$ diverges, $\lim_{n\to\infty} \sum_{j=N}^{n} a_j = +\infty$, and so $\lim_{n\to\infty} e^{-\sum_{j=N}^{n} a_j} = 0$, implying that

$$\lim_{n\to\infty} \prod_{j=1}^{n}(1-a_j) = 0.$$

By Definition 4.2, $\prod_{j=1}^{\infty}(1-a_j)$ diverges. $\square$

Definition 4.5. The infinite product $\prod_{j=1}^{\infty}(1+a_j)$ is *absolutely convergent*, if $\prod_{j=1}^{\infty}(1+|a_j|)$ converges.

Remark. By Theorem 4.3, this is the case if and only if $\sum_{j=1}^{\infty}|a_j|$ converges.

Theorem 4.6. *An absolutely convergent infinite product is convergent.*

Proof. Denote

$$P_n = \prod_{j=1}^{n}(1+a_j) \quad \text{and} \quad Q_n := \prod_{j=1}^{n}(1+|a_j|).$$

Then

$$\begin{aligned} P_n - P_{n-1} &= \prod_{j=1}^{n}(1+a_j) - \prod_{j=1}^{n-1}(1+a_j) \\ &= \Big(\prod_{j=1}^{n-1}(1+a_j)\Big)(1+a_n-1) = a_n\prod_{j=1}^{n-1}(1+a_j) \end{aligned}$$

and, similarly,

$$Q_n - Q_{n-1} = |a_n|\prod_{j=1}^{n-1}(1+|a_j|).$$

Clearly,

$$|P_n - P_{n-1}| \leq Q_n - Q_{n-1}.$$

Since $\prod_{j=1}^{\infty}(1+|a_j|)$ converges, $\lim_{n\to\infty} Q_n$ exists. Therefore, $\sum_{j=1}^{\infty}(Q_j - Q_{j-1})$ converges, and so by the standard majorant principle, $\sum_{j=1}^{\infty}(P_n - P_{n-1})$ converges, implying that $\lim_{n\to\infty} P_n$ exists.

It remains to show that this limit is non-zero. Since $\sum_{j=1}^{\infty}|a_j|$ converges, $\lim_{n\to\infty} a_n = 0$, and so $\lim_{n\to\infty}(1+a_n) = 1$. Therefore, $\sum_{j=1}^{\infty}|\frac{a_j}{1+a_j}|$ converges by the majorant principle, since $|1+a_j| \geq \frac{1}{2}$ for j large enough and so $|\frac{a_j}{1+a_j}| \leq 2|a_j|$. Therefore

$$\prod_{j=1}^{\infty}\left(1 - \frac{a_j}{1+a_j}\right)$$

is absolutely convergent. By the preceding part of the proof, a finite limit

$$\lim_{n\to\infty}\prod_{j=1}^{n}\left(1 - \frac{a_j}{1+a_j}\right)$$

exists. But

$$\prod_{j=1}^{n}\left(1 - \frac{a_j}{1+a_j}\right) = \prod_{j=1}^{n}\frac{1}{1+a_j} = \frac{1}{\prod_{j=1}^{n}(1+a_j)} = \frac{1}{P_n},$$

and so $\lim_{n\to\infty} P_n \neq 0$. $\square$

Consider finally a sequence $\big(f_j(z)\big)_{j\in\mathbb{N}}$ of analytic functions in a domain $G \subset \mathbb{C}$. Similarly as to Definition 4.2, we say that

$$\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$$

converges in G, if

$$\lim_{n\to\infty}\prod_{j=1}^{n}\big(1+f_j(z)\big)\neq 0$$

exists for each $z \in G$.

Theorem 4.7. *An infinite product $\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$ is (locally) uniformly convergent in G, if the series $\sum_{j=1}^{\infty}|f_j(z)|$ converges (locally) uniformly in G.*

Proof. Assume the uniform convergence in $D \subset G$. Then

$$\sum_{j=1}^{\infty}|f_j(z)| < M (<\infty)$$

for all $z \in D$. Then by (4.3),

$$\big(1+|f_1(z)|\big)\cdots\big(1+|f_n(z)|\big) \leq e^{|f_1(z)|+\cdots+|f_n(z)|} \leq e^M.$$

Denote

$$P_n(z) := \prod_{j=1}^{n}\big(1+|f_j(z)|\big).$$

Then

$$P_n(z) - P_{n-1}(z) = |f_n(z)|\big(1+|f_1(z)|\big)\cdots\big(1+|f_{n-1}(z)|\big) \leq e^M|f_n(z)|.$$

Since

$$\sum_{j=2}^{\infty}\big(P_n(z)-P_{n-1}(z)\big) \leq e^M\sum_{j=2}^{\infty}|f_j(z)| \leq e^M\sum_{j=1}^{\infty}|f_j(z)|,$$

$\sum_{j=2}^{\infty}\big(P_n(z)-P_{n-1}(z)\big)$ converges uniformly, and so (P_n) as well. This means that $\prod_{j=1}^{\infty}\big(1+f_j(z)\big)$ is absolutely (uniformly) convergent, hence (uniformly) convergent by Theorem 4.6. $\square$

Exercises.

(1) Show that $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{2}{(n+1)(n+2)}\right)=\tfrac{1}{3}$.

(2) Show that $\displaystyle\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}=\tfrac{1}{4}$.

(3) Show that $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$ converges.

(4) Determine whether or not $\displaystyle\prod_{n=k}^{\infty}(1-2^{-n})$ is convergent, for $k=0$ and for $k=1$.

(5) Prove that $\displaystyle\prod_{k=0}^{\infty}\left(1+\frac{z^k}{k!}\right)$ defines an entire function.

(6) Prove that $\displaystyle\prod_{k=0}^{\infty}(1+z^{2^k})=\frac{1}{1-z}$ for all z in the unit disc $|z|<1$.

5. Weierstrass factorization theorem

Consider a polynomial $P(z)$ with (all) zeros $z_1, \dots, z_n$. Then

$$\begin{aligned} P(z) &= C(z_1 - z)\cdots(z_n - z) \qquad (C \text{ constant}) \\ &= Cz_1\cdots z_n\left(1-\frac{z}{z_1}\right)\cdots\left(1-\frac{z}{z_n}\right) \\ &= P(0)\left(1-\frac{z}{z_1}\right)\cdots\left(1-\frac{z}{z_n}\right). \end{aligned}$$

Let now $f(z)$ be an entire function with zeros $z_1, z_2, \dots, z_n, \dots$ arranged by increasing moduli, i.e.,

$$0 \le |z_1| \le |z_2| \le \cdots \le |z_n| \le \cdots .$$

By the uniqueness theorem of analytic functions, $\lim_{n\to\infty} |z_m| = \infty$. Assume $z_1 \neq 0$. Then a factorization similar to the polynomial case above is not immediate, since

$$\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)$$

may diverge. Therefore, we must somehow modify the situation to ensure the convergence. This may be done by the following

Theorem 5.1. (Weierstraß). *Let $(z_m)_{n\in\mathbb{N}}$ be an arbitrary sequence of complex numbers different from zero, arranged by increasing moduli and $\lim_{n\to\infty} |z_n| = \infty$. Let $m \in \mathbb{N}\cup\{0\}$. Then there exist $\nu \in \mathbb{N}\cup\{0\}$, $\nu = \nu(j)$, such that $\sum_{j=1}^{\infty} |z_j|^{-(\nu+1)}$ converges in $\mathbb{C}$ and that for the polynomial*

$$Q_\nu(z) := z + \tfrac{1}{2}z^2 + \cdots + \tfrac{1}{\nu}z^\nu, \qquad \nu \ge 1; \quad Q_0(z) \equiv 0,$$

and for an arbitrary entire function $g(z)$,

$$G(z) := e^{g(z)} z^m \prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right) e^{Q_\nu\left(\frac{z}{z_j}\right)} \tag{5.1}$$

is an entire function with a zero of multiplicity m at $z = 0$ and with the other zeros exactly at (z_n).

Remark. The sequence (z_n) is *not necessarily* formed by distinct points. A repeated z_n represents a multiple zero of $G(z)$.

Before proceeding to prove Theorem 5.1, we consider the function (entire)

$$E_\nu(z) := (1-z)e^{Q_\nu(z)}, \qquad \nu \ge 1; \quad E_0(z) := 1 - z,$$

usually called as the Weierstraß factor.

We first prove three basic properties for $E_\nu(z)$:

(1) $E'_\nu(z) = -z^\nu e^{Q_\nu(z)}$ for $\nu \geq 1$:

$$\begin{aligned} E'_\nu(z) &= -e^{Q_\nu(z)} + (1-z)(1+z+\cdots+z^{\nu-1})e^{Q_\nu(z)} \\ &= e^{Q_\nu(z)}(-1+1+\cdots+z^{\nu-1}-z-z^2-\cdots-z^\nu) = -z^\nu e^{Q_\nu(z)}. \end{aligned}$$

(2) $E_\nu(z) = 1 + \sum_{j>\nu} a_j z^j$ with $\sum_{j>\nu} |a_j| = 1$ for $\nu \geq 0$.

For $\nu = 0$, this is trivial. Since $E_\nu(z)$ is entire, we may consider its Taylor expansion around $z = 0$:

$$E_\nu(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Differentiating, we get

$$\sum_{j=1}^{\infty} j a_j z^{j-1} = E'_\nu(z) = -z^\nu e^{Q_\nu(z)}.$$

Expanding the right hand around $z = 0$, we get $-z^\nu \sum_{j=0}^{\infty} \beta_j z^j$ with $\beta_j \geq 0$ for all j. Therefore $a_1 = a_2 = \cdots = a_\nu = 0$ and $a_j \leq 0$ for $j > \nu$, hence $|a_j| = -a_j$ for $j > \nu$. Moreover, $a_0 = E_\nu(0) = 1$ and

$$0 = E_\nu(1) = 1 + \sum_{j>\nu} a_j;$$

thus

$$\sum_{j>\nu} a_j = -\sum_{j>\nu} |a_j| = -1,$$

resulting in the assertion.

(3) If $|z| \leq 1$, then $|E_\nu(z) - 1| \leq |z|^{\nu+1}$, $\nu \geq 0$. By (2),

$$\begin{aligned} |E_\nu(z) - 1| &= \Big| \sum_{j=\nu+1}^{\infty} a_j z^j \Big| \leq \sum_{j=\nu+1}^{\infty} |a_j||z|^j \\ &= |z|^{\nu+1} \sum_{j=\nu+1}^{\infty} |a_j||z|^{j-(\nu+1)} \leq |z|^{\nu+1} \sum_{j>\nu} |a_j| = |z|^{\nu+1}. \end{aligned}$$

Proof of Theorem 5.1. We consider $E_\nu(\frac{z}{z_j})$ for $j \in \mathbb{N}$. The idea is to determine ν so that $\prod_{j=1}^{\infty} E_\nu(\frac{z}{z_j})$ converges absolutely and uniformly for $|z| \leq R$, R large enough. To this end, fix $R > 1$ and $0 < \alpha < 1$. Since $\lim_{n\to\infty} |z_m| = \infty$, we find q such that $|z_q| \leq \frac{R}{\alpha}$, while $|z_{q+1}| > \frac{R}{\alpha}$. Then $\prod_{j=1}^{q} E_\nu(\frac{z}{z_j})$ is an entire function as a finite product of entire functions. Consider now the remainder term

$$\prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

in the disc $|z| \leq R$. Since $j > q$, $|z_j| > \frac{R}{\alpha}$ and so

$$|z/z_j| < \alpha < 1.$$

Writing

$$E_\nu\left(\frac{z}{z_j}\right) = \left(1 - \frac{z}{z_j}\right) e^{Q_\nu(\frac{z}{z_j})} = 1 + U_j(z),$$

we proceed to estimate $U_j(z)$. Since $j > q$, and $|z/z_j| < 1$, (3) above implies

$$|U_j(z)| = \left|E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1}. \tag{5.2}$$

We now divide our consideration in two cases:

Case I: There exists $p \in \mathbb{N}$ such that $\sum_{j=1}^{\infty} |z_j|^{-p} < \infty$. In this case, we define $\nu := p - 1$. From (5.2), we obtain

$$|U_j(z)| \leq R^p |z_j|^{-p},$$

since $|z| \leq R$. Therefore,

$$\sum_{j=1}^{\infty} |U_j(z)| \leq R^p \sum_{j=1}^{\infty} |z_j|^{-p} < \infty$$

for $|z| \leq R$. By Theorem 4.3, Definition 4.5 and Theorem 4.7,

$$\prod_{j=q+1}^{\infty} \left(1 + U_j(z)\right) = \prod_{j=q+1}^{\infty} E_\nu\left(\frac{z}{z_j}\right)$$

converges absolutely and uniformly.

Case II: For all $p \in \mathbb{N}$, $\sum_{j=1}^{\infty} |z_j|^{-p} = \infty$. In this case, we take $\nu = j - 1$, so ν depends on j. Then, by (5.2) again

$$|U_j(z)| \leq \left|\frac{z}{z_j}\right|^j$$

provided $j > q$ (which means $|\frac{z}{z_j}| < \alpha < 1$) and $|z| \leq R$. Since $|z/z_j| < \alpha < 1$, we have

$$\limsup_{j\to\infty} \sqrt[j]{\left|\frac{z}{z_j}\right|^j} \leq \alpha < 1,$$

and therefore, by the root test, which carries over from the (real) analysis word by word, $\sum_{j=q+1}^{\infty} |U_j(z)|$ converges. As above, we get that $\prod_{j=q+1}^{\infty} E_\nu(\frac{z}{z_j})$ converges absolutely and uniformly for $|z| \leq R$. *If we now have proved* that $\prod_{j=1}^{\infty} E_\nu(\frac{z}{z_j})$ is analytic in $\mathbb{C}$, then $G(z)$ is entire and has exactly the desired zeros. Therefore, it remains to prove

Theorem 5.2. *If $\big(f_n(z)\big)$ is a sequence of analytic functions in a domain G and if there exists*

$$\lim_{n\to\infty} f_n(z) = f(z) \tag{5.3}$$

uniformly in closed subdomains of G, then $f(z)$ is analytic and $f'(z) = \lim_{n\to\infty} f_n'(z)$.

Proof. This is a consequence of the Cauchy integral formula. In fact, fix $z \in G$ arbitrarily and let $B(z_0, r)$ be a disc s.th. $\overline{B(z_0, r)} \subset G$. By the Cauchy integral formula, and the fact that $z \in B(z_0, r)$,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta, \qquad n \in \mathbb{N}. \tag{5.4}$$

Since the convergence is uniform on ∂B,

$$|f_n(\zeta) - f(\zeta)| < \varepsilon$$

for $n \geq n_\varepsilon$ and for all $\zeta \in \partial B$. Therefore, since $|\zeta - z| \geq \beta r$ for all $\zeta \in \partial B$, $0 < \beta \leq 1$, (β depends on z).

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right| \\ \leq \frac{1}{2\pi}\int_{\partial B} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|}\, |d\zeta| \leq \frac{\varepsilon \cdot 2\pi r}{2\pi\beta r} = \frac{\varepsilon}{\beta},\end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

By (5.3) and (5.4),

$$f(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Now, $f'(z)$ exists, since

$$\begin{aligned}\frac{1}{h}[f(z+h) - f(z)] &= \frac{1}{2\pi h i}\int_{\partial B} \left(\frac{f(\zeta)}{\zeta - (z+h)} - \frac{f(\zeta)}{\zeta - z}\right) d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)(\zeta - (z+h))}\, d\zeta \to \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta,\end{aligned}$$

provided z, $z + h \in B(z_0, r)$. Therefore, $f(z)$ is analytic. Since the limit (5.3) is uniform in ∂B, we get

$$\begin{aligned}f'(z) &= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = \frac{1}{2\pi i}\int_{\partial B} \big(\lim_{n\to\infty} f_n(\zeta)\big)\frac{d\zeta}{(\zeta - z)^2} \\ &= \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta. \quad \square\end{aligned}$$

Theorem 5.3. (Weierstraß product theorem). *Let $f(z)$ be entire with a zero of multiplicity $m \in \mathbb{N} \cup \{0\}$ at $z = 0$ and the zeros $z_j \neq 0$ s.th. $0 < |z_1| \leq |z_2| \leq \cdots$, possibly including repeated points. Let $H(z)$ denote the Weierstraß product* (5.1) *with $g(z) \equiv 0$. Then there exists an entire function $h(z)$ s.th.*

$$f(z) = H(z)e^{h(z)}. \tag{5.5}$$

Proof. Since $f(z)$ and $H(z)$ have exactly the same zeros, it is clear that $f(z)/H(z)$ is entire with no zeros. Applying Theorem 4.1 results in (5.5).

Remark. A possible zero at $z = 0$, i.e. $m > 0$, corresponding to z^m in Theorem 5.1, is contained in $H(z)$.

Observe that Theorem 5.1 may be expressed as

Theorem 5.4. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence the distinct complex numbers having no finite accumulation points, and let a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers be given. Then there exists an entire function having roots of multiplicity k_n at z_n for all $n \in \mathbb{N}$, and nowhere else.*

Example. As an example, we construct the classical product representation $\sin \pi z =: f(z)$. Clearly, $f(z)$ has simple zeros exactly at $z = n$, $n \in \mathbb{Z}$. Since $\sum_{j=1}^{\infty} n^{-p}$ converges for $p = 2$ and diverges for $p = 1$, we may take $\nu = 1$ in Theorem 5.1, see Case I of the proof. By Theorem 5.3,

$$f(z) = ze^{z+h(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right),$$

which we may write as

$$f(z) = \sin \pi z = \pi z e^{g(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right),$$

where g is entire. By logarithmic differentiation,

$$\frac{f'(z)}{f(z)} = \pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z) + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}.$$

The problem now is to determine $g(z)$. To this end, we consider

$$h(z) := \lim_{n\to\infty} \sum_{j=-n}^{n} \frac{1}{z+j} = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}.$$

The function $h(z)$ is a meromorphic function, with simple pole at $z = n$, $n \in \mathbb{Z}$, with residue $= 1$ at each pole that means, exactly the same poles (and same residues) as $\pi \cot \pi z$, see Exercises. Also, we leave as an exercise to show that

$$2\pi \cot 2\pi z = \pi \cot \pi z + \pi \cot\big(\pi(z + \tfrac{1}{2})\big). \tag{$**$}$$

Lemma 5.5. *Let $g(z)$ be analytic in $\mathbb{C} \setminus \mathbb{Z}$, with simple poles with residue $= 1$ at $z = n$, $n \in \mathbb{Z}$. Moreover, suppose that $g(z)$ is odd, i.e. $g(-z) = -g(z)$, and satisfies*

$$2g(2z) = g(z) + g(z + \tfrac{1}{2}).$$

Then $g(z) = \pi \cot \pi z$.

Proof. Clearly, $H(z) := g(z) - \pi \cot \pi z$ is entire, odd, $H(0) = 0$ and

$$2H(2z) = H(z) + H(z + \tfrac{1}{2}). \qquad (*)$$

Suppose $H(z)$ does not vanish identically. Consider the (closed) disc $\overline{B(0,2)}$. By the maximum principle, we find $c \in \partial B(0,2)$ such that $|H(z)| < |H(c)|$ for all $z \in B(0,2)$. Now, $c/2$ and $(c+1)/2$ are both in $B(0,2)$, and therefore

$$|H(\tfrac{c}{2}) + H(\tfrac{c}{2} + \tfrac{1}{2})| \leq |H(\tfrac{c}{2})| + |H(\tfrac{c+1}{2})| < 2|H(c)|,$$

contradicting $(*)$. Hence, $H(z) \equiv 0$, and we are done. $\square$

Now, it is immediate to see that $h(z)$ is an odd function.
It remains to prove that

$$2h(2z) = h(z) + h(z + \tfrac{1}{2}).$$

We temporarily use the notation

$$s_n(z) := \frac{1}{z} + \sum_{j=1}^{n} \left(\frac{1}{z+j} + \frac{1}{z-j} \right)$$

and proceed to prove

$$2s_{2n}(2z) = s_n(z) + s_n(z + \tfrac{1}{2})$$

for all $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}
2s_{2n}(2z) - s_n(z) - s_n(z + \tfrac{1}{2}) &= \sum_{j=1}^{2n} \left(\frac{2}{2z+j} + \frac{2}{2z-j} \right) - \sum_{j=1}^{n} \left(\frac{1}{z+j} + \frac{1}{z-j} \right) \\
&\quad - \frac{2}{2z+1} - \sum_{j=1}^{n} \left(\frac{2}{2z+1+2j} + \frac{2}{2z+1-2j} \right) \\
&= -\frac{2}{2z+1} + \sum_{j=1}^{2n} \frac{2}{2z+j} - \sum_{j=1}^{n} \frac{2}{2z+1+2j} \\
&\quad + \sum_{j=1}^{2n} \frac{2}{2z-j} - \sum_{j=1}^{n} \frac{2}{2z+1-2j} - \sum_{j=1}^{n} \frac{1}{z+j} - \sum_{j=1}^{n} \frac{1}{z-j} \\
&= -\frac{2}{2z+1} + \frac{2}{2z+1} - \frac{2}{2z+2n+1} + \sum_{j=1}^{n} \frac{2}{2z+2j} + \sum_{j=1}^{n} \frac{2}{2z-2j} \\
&\quad - \sum_{j=1}^{n} \frac{1}{z+j} - \sum_{j=1}^{n} \frac{1}{z-j} \\
&= -\frac{2}{2z+2n+1}.
\end{aligned}$$

Therefore, letting $n \to \infty$, we get

$$2h(2z) = 2s_\infty(2z) = s_\infty(z) + s_\infty(z + \tfrac{1}{2}) = h(z) + h(z + \tfrac{1}{2}).$$

By Lemma 5.5, we get the identity

$$h(z) = \pi \cot \pi z.$$

Equating the expansions of these two functions, we obtain

$$\frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2} = \frac{1}{z} + g'(z) + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}.$$

Thus, $g' \equiv 0$ and so g is a constant. But then

$$e^{g(0)} = \frac{\sin \pi z}{\pi z} \to 1$$

as $z \to 0$, and hence $g(0) = 0$ meaning that $g(z) \equiv 0$. This implies now the product representation

$$\sin \pi z = \pi z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

This further implies, as an application, that

$$\prod_{j=1}^{\infty} \left(1 + \frac{1}{j^2}\right) = \prod_{j=1}^{\infty} \left(1 - \frac{i^2}{j^2}\right) = \frac{\sin \pi i}{\pi i} = -\frac{1}{2\pi}(e^{\pi i^2} - e^{-\pi i^2}) = \frac{e^{\pi} - e^{-\pi}}{2\pi}.$$

Exercises:

(1) Prove that $\pi \cot \pi z$ has simple poles exactly at $z = n \in \mathbb{Z}$, with residue $= 1$ at each pole.
(2) Prove the identity $(*)$.
(3) Compute $\displaystyle\prod_{j=1}^{\infty} \left(1 - \frac{1}{(2j)^2}\right)$.

6. Complex interpolation

This section is entirely devoted to proving the following interpolation theorem for analytic functions:

Theorem 6.1. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of distinct points in $\mathbb{C}$ having no finite accumulation points and $(\zeta_n)_{n\in\mathbb{N}}$ a sequence of complex numbers, not necessarily distinct. Then there exists an entire function $f(z)$ such that $f(z_n) = \zeta_n$ for all $n \in \mathbb{N}$.*

To prove this result, we first need to prove the following Mittag-Leffler theorem. To this end, recall that Definition 3.1 for a meromorphic function f. By this definition, the Laurent expansion of f around $a \in \mathbb{C}$ must be of the form

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-a)^j,$$

where $m = m(a) \in \mathbb{Z}$. If $m > 0$, the finite part

$$\sum_{j=-m}^{-1} a_j (z-a)^j$$

is called *the singular part* of f at $z = a$.

Theorem 6.2. (Mittag-Leffler). *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of distinct points in $\mathbb{C}$ having no finite accumulation points, and let $\big(P_n(z)\big)_{n\in\mathbb{N}}$ be a sequence of polynomials such that $P_n(0) = 0$. Then there exists a meromorphic function $f(z)$ having the singular part*

$$P_n\left(\frac{1}{z-z_n}\right)$$

at $z = z_n$, and no other poles in $\mathbb{C}$.

Proof. We may assume that $|z_1| \le |z_2| \le \cdots$. Moreover, we assume, temporarily, that $z_1 \neq 0$. Next, let $\sum_{n=1}^{\infty} c_n$ be a convergent series of strictly positive real numbers. As $P_n(z)$ is a polynomial, $P_n(\frac{1}{z-z_n})$ must be analytic in $B(0, |z_n|)$; therefore we may take its Taylor expansion

$$P_n\left(\frac{1}{z-z_n}\right) = \sum_{j=0}^{\infty} a_j^{(n)} z^j \tag{6.1}$$

in $B(0, |z_n|)$. By elementary facts of (complex) power series, (6.1) converges absolutely and uniformly in $B(0, \rho)$, where $|z_n|/2 < \rho < |z_n|$. Denote now

$$Q_n(z) := \sum_{j=0}^{k_n} a_j^{(n)} z^j, \tag{6.2}$$

where k_n has been chosen large enough to satisfy

$$\sup_{z\in \overline{B(0,\frac{|z_n|}{2})}} \left| P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right| < c_n. \tag{6.3}$$

We now proceed to consider the series

$$\sum_{n=1}^{\infty} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right). \tag{6.4}$$

Take an arbitrary $R > 0$. Clearly, only those singular parts $P_n\bigl(1/(z-z_n)\bigr)$ with $z_n \in B(0,R)$ contribute poles to the sum (6.4). We now break the sum (6.4) in two parts:

$$\sum_{|z_n|\le 2R} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right), \qquad \sum_{|z_n|>2R} \left(P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right). \tag{6.5}$$

The second (infinite) part has no poles in $B(0,R)$. Moreover, in this part, $R < |z_n|/2$, and so, by (6.3),

$$\sup_{z\in \overline{B(0,R)}} \left| P_n\left(\frac{1}{z-z_n}\right) - Q_n(z)\right| < c_n.$$

By the standard majorant principle, the infinite part of (6.5) converges absolutely and uniformly in $\overline{B(0,R)}$, and therefore it defines an analytic function in $B(0,R)$ by Theorem 5.2. The first part in (6.5) is a rational function with prescribed behavior of poles exactly at $z = z_n \in B(0,R)$.

Now, since R is arbitrary, the series (6.4) converges locally uniformly in $\mathbb{C} \setminus \bigcup_{n=1}^{\infty}\{z_n\}$, having prescribed behavior of poles in $\mathbb{C}$ except perhaps at $z = 0$. Adding one singular part, say $P_0(1/z)$, for $z = 0$, we obtain a function with the asserted properties.

Proof of Theorem 6.1. By Theorem 5.4 (or Theorem 5.1), construct an entire function $g(z)$ with simple zeros only, exactly at each z_n. Then $g'(z_n) \neq 0$ for all $n \in \mathbb{N}$. By the Mittag-Leffler theorem, there exists a meromorphic function $h(z)$ with simple poles only exactly at each z_n, with residue $\zeta_n/g'(z_n)$ at each z_n. Consider $f(z) := h(z)g(z)$, analytic except perhaps at the points z_n. But near $z = z_n$,

$$g(z) = g'(z_n)(z-z_n) + \cdots = (z-z_n)g_n(z), \qquad g_n(z_n) = g'(z_n)$$
$$h(z) = \frac{\zeta_n}{g'(z_n)} \cdot \frac{1}{z-z_n} + \cdots = \frac{h_n(z)}{z-z_n}, \qquad h_n(z_n) = \frac{\zeta_n}{g'(z_n)},$$

where $g_n(z)$, $h_n(z)$ are analytic at $z = z_n$. Therefore, $f(z) = g_n(z)h_n(z)$ near $z = z_n$, and so analytic. Moreover,

$$f(z_n) = g_n(z_n)h_n(z_n) = g'(z_n) \cdot \frac{\zeta_n}{g'(z_n)} = \zeta_n$$

for each z_n. $\square$

7. Growth of entire functions

Definition 7.1. For an entire function $f(z)$,

$$M(r,f) = \max_{|z|\le r} |f(z)|$$

is the *maximum modulus* of f.

Remark. By the maximum principle,

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Lemma 7.2. *Let $P(z) = a_n z^n + \cdots + a_0$, $a_n \ne 0$, be a polynomial. Given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ s.th.*

$$(1-\varepsilon)|a_n|r^n \le |P(z)| \le (1+\varepsilon)|a_n|r^n$$

whenever $r = |z| > r_\varepsilon$.

Proof. Clearly, $|P(z)| = |a_n||z|^n \left|1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right|$. Denote

$$r_n(z) = \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}.$$

Obviously, $|r_n(z)| < \varepsilon$, if $|z| > r_\varepsilon$ for some $\varepsilon > 0$. This means that

$$\begin{aligned}(1-\varepsilon)|a_n|r^n &\le \big(1 - |r_n(z)|\big)|a_n|r^n \\ &= |P(z)| \le \big(1 + |r_n(z)|\big)|a_n|r^n \le (1+\varepsilon)|a_n|r^n. \quad \square\end{aligned}$$

Definition 7.3. For an entire function $f(z)$, the *order*, resp. *lower order*, is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \qquad \text{resp.} \quad \mu(f) := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Remark. By the Liouville theorem, $\rho(f) \ge 0$ and $\mu(f) \ge 0$.

Examples. (1) Show that $\rho(e^z) = 1 = \mu(e^z)$.

(2) For a polynomial $P(z)$, show that $\rho(P) = \mu(P) = 0$.

(3) Determine $\rho(\cos z)$.

(4) Consider

$$f(z) = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots \qquad (= \cos\sqrt{z}).$$

Show that f is entire and determine $\rho(f)$.

Now, let $f(z)$ be an entire function of finite order $\rho < +\infty$. By the definition of the order, this means that for some r_ε,

$$\frac{\log\log M(r,f)}{\log r} < \rho + \varepsilon, \qquad \text{for all } r \ge r_\varepsilon,$$

hence

$$\log\log M(r,f) < (\rho+\varepsilon)\log r = \log r^{\rho+\varepsilon}$$

and so

$$|f(z)| \le M(r,f) \le e^{r^{\rho+\varepsilon}} \qquad \text{for all } |z| \le r. \tag{7.1}$$

Lemma 7.4. *Defining*

$$\alpha := \inf\{\, \lambda > 0 \mid M(r,f) \le e^{r^\lambda} \textit{ for all } r \textit{ suff. large}\},$$

the order of f satisfies $\rho(f) = \alpha$.

Proof. By (7.1), $\alpha \le \rho(f) + \varepsilon$ for all $\varepsilon > 0$, so $\alpha \le \rho(f)$. On the other hand, given any $\lambda > 0$ such that the condition is satisfied, we get

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \le \limsup_{r\to\infty} \frac{\log\log e^{r^\lambda}}{\log r} = \lambda$$

and so $\rho(f) \le \alpha$. $\square$

Theorem 7.5. *Let $f_1(z)$, $f_2(z)$ be two entire functions. Then*

(1) $\rho(f_1 + f_2) \le \max\big(\rho(f_1), \rho(f_2)\big)$,
(2) $\rho(f_1 f_2) \le \max\big(\rho(f_1), \rho(f_2)\big)$.

Moreover, if $\rho(f_1) < \rho(f_2)$, then

(3) $\rho(f_1 + f_2) = \rho(f_2)$,

Proof. (1) Assume therefore that $\rho(f_1)\rho_1$, $\rho(f_2) = \rho_2$. By Lemma 7.4, for r sufficiently large,

$$M(r,f_1) \le e^{r^{\rho_1+\varepsilon}}, \qquad M(r,f_2) \le e^{r^{\rho_2+\varepsilon}}.$$

By elementary estimates, for r sufficiently large,

$$\begin{aligned} M(r, f_1 + f_2) &= \max_{|z|=r} |f(z_1) + f(z_2)| \le \max_{|z|=r} |f(z_1)| + \max_{|z|=r} |f(z_2)| \\ &= M(r,f_1) + M(r,f_2) \le e^{r^{\rho_1+\varepsilon}} + e^{r^{\rho_2+\varepsilon}} \le 2e^{r^{\max(\rho_1,\rho_2)+\varepsilon}} \\ &\le e^{r^{\max(\rho_1,\rho_2)+2\varepsilon}}. \end{aligned}$$

By Lemma 7.4 again, $\rho(f_1+f_2) \le \max(\rho_1,\rho_2)+2\varepsilon$ and so $\rho(f_1+f_2) \le \max(\rho_1,\rho_2)$.

(2) Similarly, for $\rho_1 = \rho(f_1)$, $\rho_2 = \rho(f_2)$,

$$\begin{aligned} M(r, f_1 f_2) &= \max_{|z|=r} |f_1(z) f_2(z)| \le \big(\max_{|z|=r} |f_1(z)|\big)\big(\max_{|z|=r} |f_2(z)|\big) \\ &= M(r,f_1) M(r,f_2) \le e^{r^{\rho_1+\varepsilon}} \cdot e^{r^{\rho_2+\varepsilon}} \le e^{2r^{\max(\rho_1,\rho_2)+\varepsilon}} \le e^{r^{\max(\rho_1,\rho_2)+2\varepsilon}} \end{aligned}$$

and we obtain $\rho(f_1 f_2) \le \max\big(\rho(f_1), \rho(f_2)\big)$ by taking logarithms twice.

(3) We now assume $\rho(f_1) < \rho(f_2) = \rho$. The inequality in (1) is immediate:

$$M(r, f_1 + f_2) \le M(r,f_1) + M(r,f_2) \le e^{r^{\rho(f_1)+\varepsilon}} + e^{r^{\rho+\varepsilon}} \le 2e^{r^{\rho+\varepsilon}} \le e^{r^{\rho+2\varepsilon}}.$$

Therefore, it remains to prove that for any $\varepsilon > 0$,

$$\rho(f_1 + f_2) \ge \rho - \varepsilon.$$

Now, we again have $M(r, f_1) \le e^{r^{\rho(f_1)+\varepsilon}}$ for all r sufficiently large and, by the definition of lim sup,

$$M(r, f_2) \ge e^{r_n^{\rho-\varepsilon}} \tag{7.2}$$

for a sequence (r_n) such that $r_n \to \infty$ as $n \to \infty$. Now, given r_n, since f_2 is continuous and $|z| = r_n$ is compact, we find z_n such that $|z_n| = r_n$ and that $|f(z_n)| = M(r_n, f_2) \ge \exp(r_n^{\rho-\varepsilon})$ by (7.2). Therefore

$$|(f_1 + f_2)(z_n)| = |f_1(z_n) + f_2(z_n)| \ge |f_2(z_n)| - |f_1(z_n)| \ge e^{r_n^{\rho-\varepsilon}} - e^{r_n^{\rho(f_1)+\varepsilon}}.$$

To estimate further, take $\varepsilon > 0$ so that $\rho - \varepsilon > \rho(f_1) + \varepsilon > 0$. Then

$$r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon} = r_n^{\rho-\varepsilon}(r_n^{\rho(f_1)-\rho+2\varepsilon} - 1) \to -\infty$$

as $n \to \infty$, since $\rho(f_1) - \rho < 0$. Therefore,

$$\begin{aligned} M(r_n, f_1 + f_2) \ge |(f_1 + f_2)(z_n)| &\ge e^{r_n^{\rho-\varepsilon}} - e^{r_n^{\rho(f_1)+\varepsilon}} \\ &= e^{r_n^{\rho-\varepsilon}}(1 - e^{r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon}}) \ge \tfrac{1}{2} e^{r_n^{\rho-\varepsilon}} \end{aligned}$$

for n sufficiently large, since $e^{r_n^{\rho(f_1)+\varepsilon} - r_n^{\rho-\varepsilon}} \to 0$ as $n \to \infty$. □

Remark. If $\rho(f_1) < \rho(f_2)$, then $\rho(f_1 f_2) = \rho(f_2)$ also holds. This can be proved with some more knowledge on meromorphic functions. In fact, since $1/f_1$ is meromorphic and non-entire in general, we cannot directly apply the above reasoning.

Definition 7.6. Given an entire function $f(z)$, define

$$A(r, f) := \max_{|z|=r} \operatorname{Re} f(z).$$

Theorem 7.7. *For an entire function $f(z) = \sum_{j=0}^{\infty} a_j z^j$,*

$$|a_j| r^j \le \max[0, 4A(r, f)] - 2 \operatorname{Re} f(0), \tag{7.3}$$

for all $j \in \mathbb{N}$.

Proof. For $r = 0$, the assertion is trivial. So, assume $r > 0$, and denote $z = re^{i\varphi}$, $a_n = \alpha_n + i\beta_n$. Then

$$\begin{aligned} \operatorname{Re} f(re^{i\varphi}) &= \operatorname{Re} \sum_{j=0}^{\infty} (\alpha_j + i\beta_j) r^j (\cos\varphi + i \sin\varphi)^j \\ &= \operatorname{Re} \sum_{j=0}^{\infty} (\alpha_j + i\beta_j)(\cos j\varphi + i \sin j\varphi) r^j \\ &= \sum_{j=0}^{\infty} (\alpha_j \cos j\varphi - \beta_j \sin j\varphi) r^j. \end{aligned}$$

Multiply now by $\cos n\varphi$, resp. by $\sin n\varphi$, and integrate term by term. This results in

$$\alpha_n r^n = \frac{1}{\pi}\int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)\cos n\varphi\, d\varphi, \qquad n > 0,$$
$$-\beta_n r^n = \frac{1}{\pi}\int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)\sin n\varphi\, d\varphi, \qquad n > 0,$$
$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)\, d\varphi, \qquad \beta_0 = 0.$$

Subtracting for $n > 0$, we obtain

$$\begin{aligned} a_n r^n &= (\alpha_n + i\beta_n) r^n \\ &= \frac{1}{\pi}\int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)(\cos n\varphi - i\sin n\varphi)\, d\varphi \\ &= \frac{1}{\pi}\int_0^{2\pi} \big(\operatorname{Re} f(re^{i\varphi})\big)e^{-in\varphi}\, d\varphi, \end{aligned}$$

and so

$$|a_n| r^n \le \frac{1}{\pi}\int_0^{2\pi} |\operatorname{Re} f(re^{i\varphi})|\, d\varphi,$$
$$|a_n| r^n + 2\alpha_0 \le \frac{1}{\pi}\int_0^{2\pi} \big(|\operatorname{Re} f(re^{i\varphi})| + \operatorname{Re} f(re^{i\varphi})\big)\, d\varphi. \tag{7.4}$$

If $A(r,f) < 0$, then $|\operatorname{Re} f(re^{i\varphi})| + \operatorname{Re} f(re^{i\varphi}) = 0$, and (7.3) is an immediate consequence of (7.4). If $A(r,f) \ge 0$, then

$$|a_n| r^n + 2\alpha_0 \le \frac{1}{\pi}\int_0^{2\pi} 2A(r,f)\, d\varphi = 4A(r,f);$$

the proof is now complete. □

Theorem 7.8. (Hadamard). *If $f(z)$ is entire and*

$$L := \liminf_{r\to\infty} A(r,f) r^{-s} < \infty$$

for some $s \ge 0$, then $f(z)$ is a polynomial of degree $\deg f \le s$.

Proof. By assumption, there is a sequence $r_n \to \infty$ such that $A(r_n, f) \le (L+1) r_n^s \le (|L|+1) r_n^s$. If now $j > s$, then

$$|a_j| r_n^j \le 4(|L|+1) r_n^s - 2\operatorname{Re} f(0)$$

by Theorem 7.7. Therefore

$$|a_j| \le \frac{4(|L|+1)}{r_n^{j-s}} - \frac{2\operatorname{Re} f(0)}{r_n^j} \to 0 \qquad \text{as } r_n \to \infty.$$

So, $a_j = 0$ for all $j > s$. □

Theorem 7.9. *Let $f(z)$ be entire with no zeros and such that its lower order $\mu(f) < \infty$. Then $f(z) = e^{P(z)}$ for a polynomial*

$$P(z) = a_m z^m + \cdots + a_0, \qquad a_n \neq 0,$$

such that $m = \mu(f) = \rho(f)$.

Proof. By Theorem 4.1, $f(z) = e^{g(z)}$ for an entire function $g(z)$. Now, given $\varepsilon > 0$, there is a sequence $r_n \to \infty$ such that for any z with $|z| = r_n$,

$$e^{\operatorname{Re} g(z)} = |e^{g(z)}| = |f(z)| \leq M(r_n, f) \leq e^{r_n^{\mu(f)+\varepsilon}}. \tag{7.5}$$

Indeed, from the definition of lower order,

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \mu(f),$$

it follows that

$$\log\log M(r_n, f) \leq \big(\mu(f) + \varepsilon\big) \log r_n,$$

and so

$$M(r_n, f) \leq e^{r_n^{\mu(f)+\varepsilon}}.$$

By (7.5), $\operatorname{Re} g(z) \leq r_n^{\mu(f)+\varepsilon}$ for all $|z| = r_n$, hence

$$A(r_n, g) \leq r_n^{\mu(f)+\varepsilon}.$$

By Theorem 7.8, since

$$\liminf_{r\to\infty} A(r,g) r^{-(\mu(f)+\varepsilon)} \leq 1 < \infty.$$

Hence, g must be a polynomial of degree $\leq \mu(f) + \varepsilon$, hence $\leq \mu(f)$.

We still have to prove that $\mu(f) = \rho(f) = m$ for $f(z) = e^{P(z)}$, if $P(z) = a_m z^m + \cdots + a_0$, $a_m \neq 0$.

To this end, we first observe, by Lemma 7.2, that

$$|f(z)| = |e^{P(z)}| = e^{\operatorname{Re} P(z)} \leq e^{|P(z)|} \leq e^{2|a_m| r^m}$$

for every $|z| = r$, r sufficiently large. Therefore,

$$\log M(r,f) \leq 2|a_m| r^m,$$
$$\log\log M(r,f) \leq m \log r + \log(2|a_m|)$$

and so

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \leq \limsup_{r\to\infty} \frac{m\log r + \log(2|a_m|)}{\log r} = m.$$

So,

$$\rho(f) \leq m = \deg P \leq \mu(f) \leq \rho(f),$$

and we are done. □

Considering an entire function f with the Taylor expansion

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

it is possible to determine its order by the coefficients a_j.

Theorem 7.10. *Defining*

$$b_j := \begin{cases} 0, & \text{if } a_j = 0 \\ \frac{j \log j}{\log \frac{1}{|a_j|}}, & \text{if } a_j \neq 0, \end{cases}$$

the order $\rho(f)$ of f is determined by

$$\rho(f) = \limsup_{j \to \infty} b_j.$$

Proof. Denote $\mu := \limsup_{j \to \infty} b_j$.

1) We first prove that $\rho(f) \geq \mu$. If $\mu = 0$, this inequality is trivial. So, we may assume $\mu > 0$. Recall first Cauchy inequalities:

$$\begin{aligned} |a_j| &= \left| \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta^{j+1}} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta|^{j+1}} r\, d\varphi \\ &\leq \frac{M(r,f)}{2\pi} \int_0^{2\pi} r^{-j}\, d\varphi = \frac{M(r,f)}{r^j}, \qquad \text{for all } j \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Take now $\sigma \in \mathbb{R}$ such that $0 < \sigma < \mu$, and proceed to prove that $\rho(f) \geq \sigma$. Since σ is arbitrary, this means that $\rho(f) \geq \mu$. By the definition of σ and μ, there exist infinitely many natural numbers j such that

$$j \log j \geq \sigma \log \frac{1}{|a_j|} = -\sigma \log |a_j|$$

hence

$$\log |a_j| \geq -\frac{1}{\sigma} j \log j.$$

By the Cauchy inequalities,

$$\log M(r,f) \geq \log(r^j |a_j|) = j \log r + \log |a_j| \geq j \log r - \frac{1}{\sigma} j \log j.$$

The above infinitely many j:s will be used to determine a sequence of r-values as follows:

$$r_j := (ej)^{1/\sigma}, \qquad \text{hence } j = \frac{1}{e} r_j^\sigma.$$

Then

$$\log M(r_j, f) \geq j \cdot \frac{1}{\sigma} \log(ej) - \frac{1}{\sigma} j \log j = \frac{1}{\sigma} j = \frac{1}{\sigma e} r_j^\sigma,$$

hence

$$\log \log M(r_j, f) \geq \sigma \log r_j + \log \frac{1}{\sigma e}$$

and finally

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \geq \limsup_{r_j\to\infty} \frac{\log\log M(r_j,f)}{\log r_j}$$
$$\geq \limsup_{r_j\to\infty} \frac{\sigma \log r_j + \log \frac{1}{\sigma e}}{\log r_j} = \sigma.$$

2) To prove that $\rho(f) \leq \mu$, we may now assume that $\mu < +\infty$. Fix $\varepsilon > 0$. Then, for all sufficiently large j, such that $a_j \neq 0$,

$$0 \leq \frac{j \log j}{\log \frac{1}{|a_j|}} \leq \mu + \varepsilon.$$

Therefore,

$$\frac{j}{\mu+\varepsilon} \log j \leq \log \frac{1}{|a_j|} = -\log |a_j|$$

and so

$$\log |a_j| \leq -\frac{j}{\mu+\varepsilon} \log j = \log(j^{-\frac{j}{\mu+\varepsilon}}).$$

By monotonicity of the logarithm,

$$|a_j| \leq j^{-j/(\mu+\varepsilon)}.$$

Now,

$$M(r,f) = \max_{|z|=r} \Big| \sum_{j=0}^{\infty} a_j z^j \Big| \leq |a_0| + \sum_{j=1}^{\infty} |a_j| r^j \leq |a_0| + \sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}} r^j$$
$$= |a_0| + \sum_{0\neq j<(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j + \sum_{j\geq(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j$$
$$= S_1 + S_2 + |a_0|.$$

Since $(2r)^{\mu+\varepsilon} \leq j$ in the sum S_2, we get

$$2r \leq j^{\frac{1}{\mu+\varepsilon}}.$$

Hence $rj^{-\frac{1}{\mu+\varepsilon}} \leq \frac{1}{2}$, and so

$$S_2 = \sum_{j\geq(2r)^{\mu+\varepsilon}} (rj^{-\frac{1}{\mu+\varepsilon}})^j \leq \sum_{j\geq(2r)^{\mu+\varepsilon}} \left(\tfrac{1}{2}\right)^j \leq \sum_{j=1}^{\infty} \left(\tfrac{1}{2}\right)^j \leq 1.$$

For S_1, we obtain

$$S_1 = \sum_{0\neq j<(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^j \leq \sum_{0\neq j<(2r)^{\mu+\varepsilon}} j^{-\frac{j}{\mu+\varepsilon}} r^{(2r)^{\mu+\varepsilon}}$$
$$\leq r^{(2r)^{\mu+\varepsilon}} \sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}} = K r^{(2r)^{\mu+\varepsilon}}, \qquad K < \infty.$$

In fact, since

$$j^{-\frac{j}{\mu+\varepsilon}} \le \frac{1}{j^2}$$

for all j sufficiently large, the sum $\sum_{j=1}^{\infty} j^{-\frac{j}{\mu+\varepsilon}}$ converges. Therefore,

$$\begin{aligned}
\rho(f) &= \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \le \limsup_{r\to\infty} \frac{\log\log(S_1+S_2+|a_0|)}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log\log S_1}{\log r} \le \limsup_{r\to\infty} \frac{\log\log(Kr^{(2r)^{\mu+\varepsilon}})}{\log r} \\
&\le \mu + 2\varepsilon
\end{aligned}$$

and so

$$\rho(f) \le \mu. \quad \square$$

Example. Consider

$$f(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j,$$

and recall the Stirling formula

$$\lim_{j\to\infty} \left(j!/\sqrt{2\pi j}\, e^{-j} j^j)\right) = 1.$$

Now,

$$\frac{1}{b_j} = \frac{\log(j!)}{j\log j} \sim \frac{j\log j - j + \log\sqrt{2\pi j}}{j\log j} \to 1$$

and so $\rho(e^z) = \limsup_{j\to\infty} b_j = 1$, as already known.

Definition 7.11. For an entire function $f(z)$ of order ρ such that $0 < \rho < \infty$, its type τ is defined by

$$\tau = \tau(f) := \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho}.$$

The next lemma is a counterpart to Lemma 7.4:

Lemma 7.12. *Define*

$$\beta := \inf\{\, K > 0 \mid M(r,f) \le e^{Kr^\rho} \text{ for all } r \text{ sufficiently large}\},$$

where f is entire and $\rho = \rho(f)$, $\rho \in (0, +\infty)$. Then $\tau(f) = \beta$.

Proof. Observe that we understand, as usually, that $\inf \emptyset = +\infty$.

1) If $\tau(f) = +\infty$, then for all $K > 0$, there is a sequence $r_n \to \infty$ such that

$$\log M(r_n, f) \geq K r_n^\rho$$

and so

$$M(r_n, f) \geq \exp(K r_n^\rho).$$

Therefore, there is no $K > 0$ such that

$$M(r, f) \leq e^{K r^\rho}$$

for all r sufficiently large, implying that

$$\beta = +\infty.$$

Conversely, if $\beta = +\infty$, then $\{\, K > 0 \mid M(r, f) \leq e^{K r^\rho}$ for all r sufficiently large$\,\}$ $= \emptyset$. So, for all $K > 0$, we find a sequence $r_n \to +\infty$ such that $M(r_n, f) > \exp(K r_n^\rho)$. Therefore $\tau(f) = +\infty$.

2) Take now K ($\geq \beta$) such that $M(r, f) \leq e^{K r^\rho}$ for all r sufficiently large. But then

$$\frac{\log M(r, f)}{r^\rho} \leq \frac{K r^\rho}{r^\rho} = K$$

for all r sufficiently large. This results in

$$\tau(f) = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho} \leq K.$$

Since $K \geq \beta$ is arbitrary, we conclude that $\tau(f) \leq \beta$.

3) To prove that $\tau(f) \geq \beta$, observe, by the definition of $\tau(f)$, that given $\varepsilon > 0$,

$$\frac{\log M(r, f)}{r^\rho} \leq \tau(f) + \varepsilon$$

for all r sufficiently large. Then

$$\log M(r, f) \leq \big(\tau(f) + \varepsilon\big) r^\rho$$

and so

$$M(r, f) \leq \exp\big(\big(\tau(f) + \varepsilon\big) r^\rho\big).$$

This implies

$$\beta \leq \tau(f) + \varepsilon,$$

hence

$$\beta \leq \tau(f). \quad \square$$

Lemma 7.13. *Let $f(z)$ be analytic in a neighborhood of $z = 0$ with the Taylor expansion*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j. \tag{7.6}$$

Suppose there exist $\lambda > 0$, $\mu > 0$ and a natural number $N = N(\mu, \lambda) > 0$ such that

$$|a_j| \leq (e\mu\lambda/j)^{j/\mu} \tag{7.7}$$

for all $j > N$. Then the Taylor expansion converges in the whole complex plane, and therefore $f(z)$ is entire. Moreover, for every $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$M(r, f) \leq e^{(\lambda+\varepsilon)r^\mu}$$

for all $r > R$.

Proof. By (7.7),

$$\sqrt[j]{|a_j|} \leq \left(\frac{e\mu\lambda}{j}\right)^{1/\mu} \to 0 \qquad \text{as } j \to \infty.$$

Therefore, the radius of convergence R for the power series (7.8) is $R = +\infty$, since

$$\frac{1}{R} = \limsup_{j\to\infty} \sqrt[j]{|a_j|} = 0.$$

Therefore, (7.6) determines an entire function.

To prepare the subsequent estimate for $M(r, f)$, observe first (exercise!) that the maximum of

$$\left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x$$

for $x \geq 0$ will be achieved as $x = \mu\lambda r^\mu$. Therefore,

$$\left(\frac{e\mu\lambda}{x}\right)^{x/\mu} r^x \leq e^{\lambda r^\mu}.$$

Moreover, if $j > N(r) := \max(N, 2^\mu e\mu\lambda r^\mu)$, then

$$\sqrt[j]{|a_j| r^j} < \left(\frac{e\mu\lambda}{j}\right)^{1/\mu} r < \tfrac{1}{2},$$

and so

$$|a_j| r^j < \frac{1}{2^j} \qquad \text{for } j > N(r).$$

For the maximum modulus of f, we now obtain, for $r > 1$,

$$\begin{aligned}
M(r,f) &= \max_{|z|=r}\Big|\sum_{j=0}^{\infty} a_j z^j\Big| \le \sum_{j=0}^{\infty}|a_j|r^j \\
&= \sum_{j=0}^{N}|a_j|r^j + \sum_{j=N+1}^{N(r)}|a_j|r^j + \sum_{j=N(r)+1}^{\infty}|a_j|r^j \\
&\le r^N\Big(\sum_{j=0}^{N}|a_j|\Big) + \big(N(r)-N\big)\max_{N+1\le j\le N(r)}|a_j|r^j + \sum_{j=1}^{\infty}\frac{1}{2^j} \\
&\le r^N\Big(\sum_{j=0}^{N}|a_j|\Big) + \big(N(r)-N\big)\max_{j>N}(|a_j|r^j) + 1 \\
&\le r^N\underbrace{\Big(\sum_{j=0}^{N}|a_j|\Big)}_{=:b} + \big(N(r)-N\big)\max_{j\ge N}\left(\left(\frac{e\mu\lambda}{j}\right)^{j/\mu} r^j\right) + 1 \\
&\le 1 + br^N + \max(0, 2^{\mu}e\mu\lambda r^{\mu} - N)e^{\lambda r^{\mu}} \le e^{(\lambda+\varepsilon)r^{\mu}},
\end{aligned}$$

provided r is sufficiently large. □

Theorem 7.14. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function of finite order $\rho > 0$ and of type $\tau = \tau(f)$. Then*

$$\tau = \frac{1}{e\rho}\limsup_{j\to\infty}(j|a_j|^{\rho/j}).$$

Proof. Denoting $\nu := \limsup_{j\to\infty}(j|a_j|^{\rho/j})$, we have to prove that $\tau = \frac{\nu}{e\rho}$.

1) We first prove that $\tau \le \nu/e\rho$. If $\nu = +\infty$, this is trivial. Therefore, we may assume that $(0 \le)\nu < +\infty$. Take any $K > \nu/e\rho$, i.e. $e\rho K > \nu$. By the definition of ν,

$$j|a_j|^{\rho/j} < e\rho K$$

for j sufficiently large. Hence,

$$|a_j| < \left(\frac{e\rho K}{j}\right)^{j/\rho}.$$

By Lemma 7.13, for each $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that

$$M(r,f) \le e^{(K+\varepsilon)r^{\rho}}$$

whenever $r > R(\varepsilon)$. By Definition 7.11, $\tau \le K + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\tau \le K$ and since $K > \nu/e\rho$ is arbitrary,

$$(0 \le)\tau \le \nu/e\rho. \tag{7.8}$$

2) To prove the reversed inequality, we first observe that $\nu = 0$ implies $\tau = 0$ by (7.8), so we may now assume that $0 < \nu \le +\infty$. Take β such that $0 < \beta < \nu$. By the definition of ν again, there is a sequence of j:s ($\to \infty$) such that

$$j|a_j|^{\rho/j} \ge \beta$$

and so

$$|a_j| \ge (\beta/j)^{j/\rho}.$$

Corresponding to these j:s define a sequence r_j by

$$(r_j)^\rho = je/\beta \to \infty \qquad \text{as } j \to \infty. \tag{7.9}$$

By the Cauchy inequalities $|a_j| \le \frac{M(r,f)}{r^j}$, we obtain by (7.9)

$$M(r_j, f) \ge |a_j|(r_j)^j \ge \left(\frac{\beta}{j}\right)^{j/\rho} \left(\frac{je}{\beta}\right)^{j/\rho} = e^{j/\rho} = e^{\frac{1}{\rho}\frac{\beta}{e}(r_j)^\rho}. \tag{7.10}$$

Therefore,

$$\tau = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho} \ge \limsup_{j\to\infty} \frac{\log M(r_j,f)}{r_j^\rho} \ge \limsup_{j\to\infty} \frac{1}{\rho}\frac{\beta}{e}\frac{(r_j)^\rho}{(r_j)^\rho} = \frac{\beta}{\rho e}.$$

Since $\beta < \nu$ is arbitrary, this implies $\tau \ge \nu/\rho e$. $\square$

8. Phragmén-Lindelöf theorems

Theorem 8.1. *Suppose $f(z)$ is analytic inside a sectorial domain, centred at the origin, of opening π/α, where $\alpha > 1$. Moreover, assume that $f(z)$ is continuous on the closure of the sectorial domain. If $|f(z)| \leq M$ on the boundary of the domain and*

$$|f(z)| \leq K e^{|z|^\beta}$$

inside of the domain for some constant $\beta < \alpha$, then $|f(z)| \leq M$ inside of the domain.

Proof. By a rotation, we may assume that the domain in question is $\{ z \neq 0 \mid |\arg z| < \pi/2\alpha \}$. Choose now $\varepsilon > 0$ and γ such that $\beta < \gamma < \alpha$, and consider

$$F(z) := e^{-\varepsilon z^\gamma} f(z),$$

where $z^\gamma = (re^{i\varphi})^\gamma = r^\gamma e^{i\gamma\varphi}$.

Since

$$\mathrm{Re}(z^\gamma) = \mathrm{Re}(r^\gamma e^{i\gamma\varphi}) = \mathrm{Re}\Big(r^\gamma\big(\cos(\gamma\varphi) + i\sin(\gamma\varphi)\big)\Big) = r^\gamma \cos(\gamma\varphi),$$

we observe that

$$|F(z)| = |F(re^{i\varphi})| = e^{\mathrm{Re}(-\varepsilon z^\gamma)}|f(z)| = e^{-\varepsilon r^\gamma \cos(\gamma\varphi)}|f(z)|.$$

Since $|\gamma\varphi| < \frac{\gamma\pi}{2\alpha} < \frac{\pi}{2}$ for the closed sectorial domain, $\cos(\gamma\varphi) > 0$ and so $\exp\big(-\varepsilon r^\gamma \cos(\gamma\varphi)\big) < 1$, hence

$$|F(z)| \leq |f(z)|$$

in the closed domain. In particular, $|F(z)| \leq M$ on the boundary of the domain. In the open sector,

$$|F(re^{i\varphi})| = e^{-\varepsilon r^\gamma \cos(\gamma\varphi)}|f(z)| \leq K e^{r^\beta - \varepsilon r^\gamma \cos(\gamma\varphi)}.$$

Since $\gamma > \beta$, $r^\beta - \varepsilon r^\gamma \cos(\gamma\varphi) \to -\infty$ as $r \to +\infty$; therefore

$$|F(re^{i\varphi})| \leq M$$

for r large enough. Therefore, by the maximum principle, applied for the shaded domain in the adjacent figure, $|F(z)| \leq M$ in the whole shaded domain. Since r may be taken arbitrarily large, the inequality holds $|F(z)| \leq M$ in the whole open sector. Therefore,

$$|f(z)| \leq M e^{\varepsilon r^\gamma \cos(\gamma\varphi)} \leq M e^{\varepsilon r^\gamma}.$$

Letting $\varepsilon \to 0$, we get the assertion. □

Theorem 8.2. *Change the estimate for f in the open sector to*

$$|f(z)| \leq Ke^{\delta|z|^\alpha} = K(\delta)e^{\delta|z|^\alpha}$$

for every $\delta > 0$, and keep the remaining assumptions unchanged. Then the same conclusion holds.

Proof. Again, we may assume the sector to be $|\varphi| \leq \frac{\pi}{2\alpha}$. Given $\varepsilon > 0$, define

$$F(z) := e^{-\varepsilon z^\alpha} f(z).$$

If $\delta < \varepsilon$, then we get on the real axis

$$|f(x)| \leq Ke^{\delta x^\alpha}$$

and

$$|F(x)| \leq Ke^{-\varepsilon x^\alpha} e^{\delta x^\alpha} = Ke^{(\delta-\varepsilon)x^\alpha} \to 0 \qquad \text{as } x \to \infty.$$

Since $|F(x)| \geq 0$ is continuous, we get, for a finite M',

$$|F(x)| \leq M' := \max\{\, |F(t)| \mid t \geq 0 \,\}$$

for all $x \geq 0$. Consider now $F(z)$ in the upper and lower half-sectors. Defining $M'' := \max(M, M')$, we see that the inequality $|F(z)| \leq M''$ holds on the boundaries of both half-sectors and $|F(z)| \leq Ke^{\delta r^\alpha}$ inside of the half-sectors. For φ such that $|\varphi| \leq \frac{\pi}{2\alpha}$, obviously

$$e^{-\varepsilon r^\alpha \cos(\varphi\alpha)} \leq e^{+\varepsilon r^\alpha}$$

and so, for some $K' > 0$,

$$|F(z)| = |e^{-\varepsilon z^\alpha}||f(z)| \leq Ke^{-\varepsilon r^\alpha \cos(\varphi\alpha)} e^{\delta r^\alpha} \leq Ke^{(\delta+\varepsilon)r^\alpha} \leq K'e^{r^\beta}$$

for any β such that $\alpha < \beta < 2\alpha$. By Theorem 8.1, $|F(z)| \leq M''$ in both half-sectors, and therefore in the whole sector $|\varphi| \leq \frac{\pi}{\alpha}$.

Assume now that $M' > M$, hence $M'' = M' > M$. Since $F(x) \to 0$ as $x \to \infty$ and $|F(0)| \leq M$, there must exist a point $x_0 \in (0, +\infty)$ such that $|F(x_0)| = M' = M''$. By the maximum principle, F must be identically equal to the constant M', a contradiction. Therefore, we must have $M' \leq M$ and so $M'' = M$. This implies that $|F(z)| \leq M$ in the whole sector. But this means that

$$|f(z)| \leq M|e^{\varepsilon z^\alpha}|.$$

Letting now $\varepsilon \to 0$, the assertion follows. □

Theorem 8.3. *Suppose $f(z) \to a$ as $z \to \infty$ along two half-lines starting from the origin, and assume that $f(z)$ is analytic and bounded in one of the sectors between these two half-lines. Then $f(z) \to a$ uniformly as $r \to \infty$ in that sector.*

Proof. Considering $f(z) - a$, if needed, we may assume that $a = 0$. Moreover, if needed, we may consider $g(\zeta) = f(\zeta^2)$ to achieve that the sector to be treated is $< \pi$. Finally, we may restrict us to considering the case of two half-lines $\pm\varphi$, $\varphi < \frac{\pi}{2}$, by an additional rotation.

Take now an arbitrary $\varepsilon > 0$. Clearly, we may assume that $|f(z)| \le M$ in the closed sector, while on the boundary half-lines, $|f(z)| < \varepsilon$ for all $r > r_1 = r_1(\varepsilon)$. Denote now $\lambda = \frac{r_1 M}{\varepsilon} > 0$ and define

$$F(z) = \frac{z}{z+\lambda} f(z).$$

Then

$$|F(z)| = \frac{r}{(r^2 + 2\lambda \operatorname{Re} z + \lambda^2)^{1/2}} |f(z)| < \frac{r}{(r^2+\lambda^2)^{1/2}} |f(z)|.$$

Now, for $r \le r_1$,

$$|F(z)| < \frac{r|f(z)|}{(r^2+\lambda^2)^{1/2}} \le \frac{rM}{\lambda} \le \frac{r_1 M}{\lambda} = \varepsilon$$

and on the boundary half-lines

$$|F(z)| < |f(z)| < \varepsilon,$$

provided $r > r_1$. Inside of the open sector, uniformly as $r \to \infty$,

$$|F(z)| < |f(z)| \le M \le Me^r \le Me^{r^\beta} \le Me^{r^\alpha}$$

for any α, β such that $1 < \beta < \alpha$. Since the opening of the sector is $< \pi$, we may take some $\alpha > 1$ such that the opening equals to $\frac{\pi}{\alpha}$. By Theorem 8.1, $|F(z)| \le \varepsilon$ in the closed sector. Therefore,

$$|f(z)| = \left|1 + \frac{\lambda}{z}\right| |F(z)| \le \left(1 + \frac{\lambda}{r}\right) |F(z)| \le 2\varepsilon$$

for all $r > \lambda$. Since $\varepsilon > 0$ is arbitrary, $f(z) \to 0$ uniformly as $r \to \infty$ inside of the sector. □

Theorem 8.4. *Suppose $f(z) \to a$ along a half-line starting from the origin and $f(z) \to b$ along a second half-line, again starting from the origin. Moreover, suppose that f is analytic and bounded in one of the two sectors between these half-lines. Then $a = b$ and $f(z) \to a$ uniformly in that sector as $r \to \infty$.*

Proof. Suppose that $f(z) \to a$ along $\varphi = \alpha$ and $f(z) \to b$ along $\varphi = \beta$, and that $\alpha < \beta$. Consider now, instead of f, the function

$$g(z) := \left(f(z) - \frac{a+b}{2}\right)^2.$$

It is now immediate to observe that

$$g(z) \to \left(a - \frac{a+b}{2}\right)^2 = \tfrac{1}{4}(a-b)^2$$

on $\varphi = \alpha$ and

$$g(z) \to \left(b - \frac{a+b}{2}\right)^2 = \tfrac{1}{4}(a-b)^2.$$

By Theorem 8.3, $g(z) \to \frac{1}{4}(a-b)^2$ uniformly in the sector as $r \to \infty$. Therefore,

$$g(z) - \tfrac{1}{4}(a-b)^2 = \big(f(z) - \tfrac{1}{2}(a+b)\big)^2 - \tfrac{1}{4}(a-b)^2 = \big(f(z)-a\big)\big(f(z)-b\big) \to 0$$

in the whole sector, uniformly as $r \to \infty$. Take now a circular arc, centred at the origin, such that

$$|f(z)-a||f(z)-b| \le \varepsilon$$

along this arc, inside of the closed sector. Then, at every point of this arc,

$$|f(z)-a| \le \sqrt{\varepsilon} \quad \text{or} \quad |f(z)-b| \le \sqrt{\varepsilon}.$$

If one of these inequalities holds on the whole arc, say $|f(z)-a| \le \sqrt{\varepsilon}$, and assuming that this circular arc has a radius large enough, then at the endpoint with $\varphi = \beta$, we get

$$|a-b| \le |f(z)-a| + |f(z)-b| \le 2\sqrt{\varepsilon}.$$

If this is not the case, then denote the two non-empty parts of the arc as $\Gamma_a = \{\, z \mid |f(z)-a| \le \sqrt{\varepsilon}\,\}$ and $\Gamma_b = \{\, z \mid |f(z)-b| \le \sqrt{\varepsilon}\,\}$. These are now closed sets and their union clearly equals to the whole circular arc. If their intersection would be empty, then, by elementary topology, one of these sets had to be empty, reducing to the previous case. Therefore, we may take a point z_0 from the intersection. Then

$$|a-b| \le |f(z_0)-a| + |f(z_0)-b| \le 2\sqrt{\varepsilon}.$$

Letting now $\varepsilon \to 0$, we get $a = b$. By Theorem 8.3, we get the assertion. □

Remark. Several variants of the Phragmén-Lindelöf theorems can be found in the literature, including also various regions, instead of sectors only.

9. Zeros of entire functions

Let $f(z)$ be an entire function and consider a disk $|z| \le r$ centred at $z = 0$. If r is large enough and $f(z)$ is a polynomial of degree n, then $f(z) = \alpha$ has n roots in $|z| \le r$. Moreover $M(r, f) \sim r^n$ on the boundary of the disk. This connection between the number of a-points and the maximum modulus carries over to transcendental entire functions. This is a deep property; moreover, some exceptional values α may appear.

Definition 9.1. Let (r_j) be a sequence of real numbers such that $0 < r_1 \le r_2 \le \cdots$. The *convergence exponent* λ for (r_j) will be defined by setting

$$\lambda = \inf\Big\{ \alpha > 0 \;\Big|\; \sum_{j=1}^{\infty} (r_j)^{-\alpha} \text{ converges } \Big\}.$$

Remark. If $\sum_{j=1}^{\infty} r_j^{-\alpha}$ diverges for all $\alpha > 0$, then $\lambda = +\infty$ as the infimum of an empty set.

Definition 9.2. Let $f(z)$ be entire and let (z_n) be the zero-sequence of $f(z)$, deleting the possible zero at $z = 0$, every zero $\ne 0$ repeated according to its multiplicity, and arranged according to increasing moduli, i.e. $0 < |z_1| \le |z_2| \le \cdots$. The convergence exponent $\lambda(f)$ (for the zero-sequence of f) is now

$$\lambda(f) := \inf\Big\{ \alpha > 0 \;\Big|\; \sum_{j=1}^{\infty} |z_j|^{-\alpha} \text{ converges } \Big\}.$$

Definition 9.3. Denote by $n(t) = n(t, \frac{1}{f})$ the number of zeros of $f(z)$ in $|z| \le t$, each zero counted according to its multiplicity.

Remark. In what follows, we assume that $f(0) \ne 0$. This is no essential restriction, since we may always replace $n(t)$ by $n(t) - n(0)$ below, if $f(0) = 0$.

Lemma 9.4. *The series $\sum_{j=1}^{\infty} |z_j|^{-\alpha}$ converges if and only if $\int_0^{\infty} n(t) t^{-(\alpha+1)}\, dt$ converges.*

Proof. Observe that $n(t)$ is a step function: zeros of $f(z)$ are situated on countably many circles centred at $z = 0$. Between these radii, $n(t)$ is constant and so $dn(t) = 0$ for these intervals. Passing over these radii $dn(t)$ jumps by an integer equal to the number of zeros on the circle. Therefore,

$$\sum_{j=1}^{N} |z_j|^{-\alpha} = \int_0^T \frac{dn(t)}{t^{\alpha}}, \qquad \text{where } T = |z_N|.$$

By partial integration,

$$\int_0^T \frac{dn(t)}{t^{\alpha}} = \Big/_0^T \frac{n(t)}{t^{\alpha}} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\, dt = \frac{n(T)}{T^{\alpha}} + \alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\, dt.$$

Assume now that $\sum_{j=1}^{\infty} |z_j|^{\alpha}$ converges. Then, for each T,

$$\alpha \int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt \le \int_0^T \frac{dn(t)}{t^{\alpha}} = \sum_{j=1}^{N} |z_j|^{-\alpha} \le \sum_{j=1}^{\infty} |z_j|^{-\alpha} < +\infty.$$

Therefore, $\int_0^{\infty} \frac{n(t)}{t^{\alpha+1}}\,dt$ converges.

Conversely, assume that the integral converges. Then

$$\frac{n(T)}{T^{\alpha}}(1-2^{-\alpha})\frac{1}{\alpha} = n(T)\int_T^{2T} \frac{dt}{t^{\alpha+1}} \le \int_T^{2T} \frac{n(t)}{t^{\alpha+1}}\,dt \le \int_0^{\infty} \frac{n(t)\,dt}{t^{\alpha+1}} =: K < +\infty.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{N} |z_j|^{-\alpha} &= \frac{n(T)}{T^{\alpha}} + \alpha\int_0^T \frac{n(t)}{t^{\alpha+1}}\,dt \\ &\le \frac{K\alpha}{1-2^{-\alpha}} + \alpha\int_0^{\infty} \frac{n(t)}{t^{\alpha+1}}\,dt = \frac{K\alpha}{1-2^{-\alpha}} + \alpha K < +\infty\end{aligned}$$

for each N. Therefore, $\sum_{j=1}^{\infty} |z_j|^{-\alpha}$ converges. □

Corollary 9.5. *Let $f(z)$ be an entire function, $f(0) \neq 0$. Then*

$$\lambda(f) = \inf\Big\{\, \alpha > 0 \;\Big|\; \int_0^{\infty} \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges} \Big\}.$$

Theorem 9.6. $\lambda(f) = \limsup_{r\to\infty} \dfrac{\log n(r)}{\log r}$.

Proof. Denote

$$\sigma := \limsup_{r\to\infty} \frac{\log n(r)}{\log r}.$$

We first prove that $\lambda(f) \le \sigma$ assuming, as we may, that $\sigma < \infty$. Given $\varepsilon > 0$, there exists r_{ε} such that

$$n(r) \le r^{\sigma+\varepsilon}$$

for all $r \ge r_{\varepsilon}$. Then

$$\begin{aligned}\int_0^M \frac{n(t)}{t^{\alpha+1}}\,dt &= \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^{M} \frac{n(t)\,dt}{t^{\alpha+1}} \\ &\le \int_0^{r_\varepsilon} \frac{n(t)\,dt}{t^{\alpha+1}} + \int_{r_\varepsilon}^{M} t^{\sigma-\alpha-1+\varepsilon}\,dt.\end{aligned}$$

As $M \to \infty$, this converges, if $\sigma - \alpha - 1 + \varepsilon < -1$, that is, if $\alpha > \sigma + \varepsilon$. Now, this is true for *all* $\alpha > 0$ such that $\alpha > \sigma + \varepsilon$. Therefore

$$\inf\Big\{\, \alpha > 0 \;\Big|\; \int_0^{\infty} \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges } \Big\} \le \sigma + \varepsilon.$$

By Corollary 9.5, $\lambda(f) \le \sigma + \varepsilon$ and so $\lambda(f) \le \sigma$.

To prove the converse inequality, we may assume that $\sigma > 0$. Take $\varepsilon > 0$ such that $\varepsilon < \sigma$. Then there is a sequence $r_j \to +\infty$ such that

$$\frac{\log n(r_j)}{\log r_j} \ge \sigma - \varepsilon,$$

hence

$$n(r_j) \ge r_j^{\sigma-\varepsilon}.$$

Take now any $\alpha > 0$ such that $0 < \alpha < \sigma - \varepsilon$. For each j, select

$$s_j \ge 2^{1/\alpha} r_j.$$

Since $n(t)$ is increasing, we get

$$\begin{aligned}\int_{r_j}^{s_j} \frac{n(t)\,dt}{t^{\alpha+1}} \ge n(r_j)\int_{r_j}^{s_j} \frac{dt}{t^{\alpha+1}} &\ge r_j^{\sigma-\varepsilon}\frac{1}{\alpha}\left(\frac{1}{r_j^\alpha} - \frac{1}{s_j^\alpha}\right)\\ &\ge \frac{1}{\alpha} r_j^{\sigma-\varepsilon}\frac{1}{r_j^\alpha}(1-\tfrac{1}{2}) = \frac{1}{2\alpha} r_j^{\sigma-\alpha-\varepsilon}.\end{aligned}$$

Since $\alpha < \sigma - \varepsilon$, and so $\sigma - \alpha - \varepsilon > 0$, we see that

$$\int_{r_j}^{s_j} \frac{n(t)}{t^{\alpha+1}}\,dt \to +\infty \qquad \text{as } j \to \infty.$$

Therefore, $\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt$ diverges for all α, $0 < \alpha < \sigma - \varepsilon$. This means that

$$\inf\left\{ \alpha > 0 \;\middle|\; \int_0^\infty \frac{n(t)}{t^{\alpha+1}}\,dt \text{ converges} \right\} \ge \sigma - \varepsilon.$$

Therefore $\lambda(f) \ge \sigma - \varepsilon$, hence of course $\lambda(f) \ge \sigma$. □

Theorem 9.7. (Jensen). *Let $f(z)$ be entire such that $f(0) \ne 0$ and denote*

$$N(r) = N\left(r, \frac{1}{f}\right) = \int_0^r \frac{n(t)}{t}\,dt.$$

Assume that there are no zeros of f on the circle $|z| = r > 0$. Then

$$N(r) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\,d\varphi - \log|f(0)|.$$

Remark. The restriction for zeros on $|z| = r$ is unessential, and may be removed by a rather complicated reasoning.

Proof. Let $a_1, a_2, \dots, a_n$ be the zeros of f in $|z| \le r$. Consider

$$g(z) := f(z) \prod_{j=1}^{n} \frac{r^2 - \overline{a}_j z}{r(z - a_j)}.$$

Then $g(z) \neq 0$ in $|z| \le R$ for an $R > r$. Indeed, for $|z| < \rho \le R$, $\rho \neq r$, this is clear. If $|z| = r$, we see that ($z = re^{i\varphi}$)

$$\left|\frac{r^2 - \overline{a}_j z}{r(z - a_j)}\right| = \left|\frac{r^2 - \overline{a}_j r e^{i\varphi}}{r^2 e^{i\varphi} - a_j r}\right| = \left|\frac{r - \overline{a}_j e^{i\varphi}}{r - a_j e^{-i\varphi}}\right| = \left|\frac{r - \overline{a_j e^{-i\varphi}}}{r - a_j e^{-i\varphi}}\right| = 1$$

and so $|g(z)| = |f(z)| \neq 0$. Since $g \neq 0$ in $|z| < R$, it is an elementary computation (by making use of Cauchy–Riemann equations) that $\log|g(z)|$ is harmonic in $|z| < R$, i.e. that $\Delta\big(\log|g(z)|\big) \equiv 0$. By the mean value property of harmonic functions, CAI, Theorem 10.5, that

$$\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|g(re^{i\varphi})|\, d\varphi.$$

Since

$$|g(0)| = |f(0)| \prod_{j=1}^{n} \frac{r}{|a_j|},$$

we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\varphi})|\, d\varphi &= \frac{1}{2\pi} \int_0^{2\pi} \log|g(re^{i\varphi})|\, d\varphi \\ &= \log|g(0)| = \log\Big(|f(0)| \prod_{j=1}^{n} \frac{r}{|a_j|}\Big) = \log|f(0)| + \sum_{j=1}^{n} \log \frac{r}{|a_j|}. \end{aligned}$$

Comparing this to the assertion, we observe that

$$\int_0^r \frac{n(t)}{t}\, dt = \sum_{j=1}^{n} \log \frac{r}{|a_j|}$$

remains to be proved. Denote $r_j = |a_j|$. Then

$$\begin{aligned} \sum_{j=1}^{n} \log \frac{r}{|a_j|} &= \sum_{j=1}^{n} \log \frac{r}{r_j} = \log\Big(\prod_{j=1}^{n} \frac{r}{r_j}\Big) = \log \frac{r^n}{r_1 \cdots r_n} \\ &= n \log r - \sum_{j=1}^{n} \log r_j = \sum_{j=1}^{n-1} j(\log r_{j+1} - \log r_j) + n(\log r - \log r_n) \\ &= \sum_{j=1}^{n-1} j \int_{r_j}^{r_{j+1}} \frac{dt}{t} + n \int_{r_n}^{r} \frac{dt}{t} = \int_0^r \frac{n(t)}{t}\, dt. \quad \square \end{aligned}$$

Remark. Given $\varphi\colon [r_0, +\infty) \to (0, +\infty)$, the Landau symbols $O\big(\varphi(r)\big)$ and $o\big(\varphi(r)\big)$ are frequently used. They mean any quantity $f(r)$ such that

For $O\big(\varphi(r)\big)$: $\exists K > 0$ such that $|f(r)/\varphi(r)| \le K$ for r sufficiently large,

for $o\big(\varphi(r)\big)$: $\lim_{r\to\infty} \frac{f(r)}{\varphi(r)} = 0$.

Theorem 9.8. *Let $f(z)$ be entire of order ρ. Then for each $\varepsilon > 0$, $n(r) = O(r^{\rho+\varepsilon})$.*

Proof. Recalling that $f(0) = 0$, we may assume that $|f(0)| \ge 1$ by multiplying f by a constant, if needed. By the Jensen formula

$$N(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\, d\varphi \le \frac{1}{2\pi}\int_0^{2\pi} \log M(r,f)\, d\varphi = \log M(r,f).$$

By the definition of the order, $\log M(r,f) \le r^{\rho+\varepsilon}$ for all r sufficiently large. Since $n(t)$ is increasing,

$$\begin{aligned} n(r)\log 2 = n(r)\int_r^{2r} \frac{dt}{t} &\le \int_r^{2r} \frac{n(t)\,dt}{t} \le \int_0^{2r} \frac{n(t)\,dt}{t} \\ &= N(2r) \le \log M(2r,f) \le (2r)^{\rho+\varepsilon} = 2^{\rho+\varepsilon} r^{\rho+\varepsilon} \end{aligned}$$

for r sufficiently large. Therefore

$$n(r) \le \left(\frac{1}{\log 2}\cdot 2^{\rho+\varepsilon}\right) r^{\rho+\varepsilon}. \quad \square$$

Theorem 9.9. *For any entire function $f(z)$, $\lambda(f) \le \rho(f)$.*

Proof. By Theorem 9.8, given $\varepsilon > 0$, there exists $K > 0$ such that

$$n(r) \le K r^{\rho+\varepsilon}, \qquad \rho = \rho(f)$$

for r sufficiently large, say $r \ge r_0$. Then

$$\int_0^M \frac{n(t)}{t^{\alpha+1}}\, dt = \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\, dt + \int_{r_0}^M \frac{n(t)\,dt}{t^{\alpha+1}} \le \int_0^{r_0} \frac{n(t)}{t^{\alpha+1}}\, dt + K\int_{r_0}^M t^{\rho+\varepsilon-\alpha-1}\, dt$$

If now $\alpha > \rho+\varepsilon$, then $\rho+\varepsilon-\alpha-1 < -1$, and therefore the last integral converges as $M \to \infty$, hence

$$\int_0^\infty \frac{n(t)}{t^{\alpha+1}}\, dt \quad \text{converges.}$$

This means that $\lambda(f) \le \rho+\varepsilon$ and so $\lambda(f) \le \rho(f)$. $\square$

10. The Cartan lemma

The Cartan lemma is a purely geometric result addressing the geometry of a finite point set in the complex plane, having a number of applications into the analysis of canonical products.

Lemma 9.1. *Let $z_1, \ldots, z_n$ be given points in $\mathbb{C}$ and $H > 0$ be given. Then there exists closed disks $\Delta_1, \ldots, \Delta_m$, $m \leq n$, such that the sum of the radii of the disks $\Delta_1, \ldots, \Delta_m$ is $\leq 2H$ and that*

$$|z - z_1||z - z_2| \ldots |z - z_n| > (H/e)^n,$$

whenever $z \notin \bigcup_{j=1}^{m} \Delta_j$.

Remark. The points z_j in the assertion above are not necessarily distinct.

Proof. (1) Suppose first that there exists a disk Δ of radius H such that $\{z_1, \ldots, z_n\} \subset H$. Let now Δ_1 denote the disk of radius $2H$, with the same centre as Δ. Consider now any point $z \notin \Delta_1$. Then $|z - z_j| > H$ for each z_j, $j = 1, \ldots, n$. Therefore we obtain

$$|z - z_1||z - z_2| \cdots |z - z_n| > H^n > (H/e)^n.$$

(2) We now define k_1 to be the greatest natural number which satisfies the following condition: There exists a closed disk Δ_1' of radius $k_1 H/n$ such that at least k_1 points z_j are contained in this disk. Obviously, we must have $1 \leq k_1 < n$, the last inequality following as we don't have the case of the first part of the proof. Actually, Δ_1' contains exactly k_1 points z_j. In fact, if not, then Δ_1' contains at least $k_1 + 1$ points z_j. Then the disk of radius $(k_1 + 1)H/n$ with the same centre as Δ_1' results in a contradiction to the definition of k_1.

Renumbering now, if needed, we may assume that $z_1, \ldots, z_{k_1} \in \Delta_1'$ while $z_{k_1+1}, \ldots, z_n \notin \Delta_1'$. We now start repeating the process. So, let k_2 be the greatest natural number such that for a closed disk Δ_2' of radius $k_2 H/n$ at least (actually, exactly) k_2 points of $z_{k_1+1}, \ldots, z_n$ are contained in Δ_2'. Then we have $k_2 \leq k_1$; in fact, otherwise we would have a contradiction to the choice of k_1. We now repeat this process m times, $m \leq n$, so that all points $z_1, \ldots, z_n$ are contained in $\bigcup_{j=1}^{m} \Delta_j'$. Clearly, the disk Δ_j' has radius $k_j H/n$ and $k_1 \geq k_2 \geq \cdots \geq k_m$. Since each Δ_j' contains exactly k_j points of $z_1, \ldots, z_n$, we must have $k_1 + k_2 + \cdots + k_m = n$. Therefore, the sum of their radii is

$$\frac{k_1}{n} H + \cdots + \frac{k_m}{n} H = \frac{k_1 + \cdots + k_m}{n} H = H.$$

Expand now the disks Δ_j', $j = 1, \ldots, n$, concentrically to Δ_j of radius $2\frac{k_j}{n} H$. Hence, the sum of the radii of the disks Δ_j is $= 2H$.

Consider now an arbitrary point $z \notin \bigcup_{j=1}^{m} \Delta_j$. Keep z fixed in what follows. We may assume, by renumbering the points $z_1, \ldots, z_n$ again, if needed, that

$$|z - z_1| \leq |z - z_2| \leq \cdots \leq |z - z_n|.$$

Assuming now that we have been able to prove that

$$|z - z_j| > \frac{j}{n}H, \qquad j = 1, \dots, n, \tag{10.1}$$

we obtain

$$\prod_{j=1}^{n} |z - z_j| > \prod_{j=1}^{n} \frac{j}{n}H = \frac{n!}{n^n}H^n \geq e^{-n}H^n = (H/e)^n.$$

In fact, this is an immediate consequence of

$$e^n = \sum_{j=0}^{\infty} \frac{1}{j!}n^j \geq \frac{1}{n!}n^n.$$

It remains to prove (10.1). We proceed to a contradiction by assuming that there exists at least one j such that $|z - z_j| \leq \frac{j}{n}H$. Let now p be the greatest natural number such that $k_p \geq j$. Such a number p exists. In fact, by monotonicity of the distances $|z - z_j|$, the disk of radius $\frac{j}{n}H$, centred at z, contains at least the points $z_1, \dots, z_j$, and so $k_1 \geq j$. Consider now the pairs of natural numbers (s, q) such that $s \leq j$, $q \leq p$.

We first proceed to prove that $z_s \notin \Delta'_q$. In fact, suppose for a while that we have $z_s \in \Delta'_q$ for some (s, q) such that $s \leq j$, $q \leq p$. By the definition of p, we have $k_q \geq j$. The radius of Δ'_q equals to $\frac{k_q}{n}H$ and Δ'_q contains k_q points of $z_1, \dots, z_n$. Let ζ be the centre of Δ'_q. Then

$$|z - \zeta| \leq |z - z_s| + |\zeta - z_s| \leq |z - z_j| + |\zeta - z_s| \leq \frac{j}{n}H + \frac{k_q}{n}H \leq 2\frac{k_q}{n}H.$$

Therefore, we have $z \in \Delta_q$, contradicting to $z \notin \bigcup_{j=1}^{m} \Delta_j$.

Therefore, we have $z_s \notin \Delta'_q$ for all pairs (s, q) such that $s \leq j$, $q \leq p$. In particular, this means that

$$\{z_1, \dots, z_j\} \subset (\mathbb{C} \setminus \Delta'_p) \cap \cdots \cap (\mathbb{C} \setminus \Delta'_1).$$

Since now

$$|z - z_1| \leq |z - z_2| \leq \cdots \leq |z - z_j| \leq \frac{j}{n}H,$$

the disk of radius $\frac{j}{n}H$, centred at z, contains the points $z_1, \dots, z_j$. By the definition of k_{p+1}, which takes into account points of $z_1, \dots, z_n$, which are outside of $\bigcup_{j=1}^{p} \Delta'_j$, this means that $k_{p+1} \geq j$, a contradiction to the definition of p as the greatest number such that $k_p \geq j$. Therefore, (10.1) holds and we are done.

11. The Hadamard theorem

Recall first the definitions of the Weierstraß factors in Chapter 5:

$$\begin{cases} E_0(z) := 1 - z \\ E_\nu(z) := (1-z)e^{Q_\nu(z)} = (1-z)e^{z+\frac{1}{2}z^2+\cdots+\frac{1}{\nu}z^\nu}, \quad \nu \geq 1, \end{cases}$$

and the notion of the convergence exponent in Chapter 9.

Let $f(z)$ now be an entire function of finite order ρ, and let $(z_n)_{n\in\mathbb{N}}$ be the sequence of its non-zero zeros, arranged according to increasing moduli. Let λ be the convergence exponent of $f(z)$ and define

$$\nu := \begin{cases} [\lambda] = & \text{the integer part of } \lambda \text{, if } \lambda \text{ is not a natural number} \\ \lambda - 1, & \text{if } \lambda \in \mathbb{N} \text{ and } \sum |z_j|^{-\lambda} \text{ converges} \\ \lambda & \text{otherwise.} \end{cases}$$

By Definition 9.2, $\sum |z_j|^{-(\nu+1)}$ converges, and

$$Q(z) = \prod_{j=1}^{\infty} E_\nu\left(\frac{z}{z_j}\right) \tag{11.1}$$

is an entire function with zeros exactly at (z_n). Therefore, $\lambda(Q) = \lambda$. By Theorem 9.9, $\lambda \leq \rho(Q)$.

The infinite product (11.1) is called the *canonical product determined by* (the non-zero zeros) of $f(z)$. Adding a suitable power z^m as an extra factor to $Q(z)$, we may take into account all zeros of $f(z)$.

Theorem 11.1. *For a canonical product, $\lambda(Q) = \lambda = \rho(Q)$.*

Proof. It suffices to prove that $\rho(Q) \leq \lambda$. To this end, we have to find a suitable majorant of $M(r, Q)$. Fix now z, $|z| = r$, and $\varepsilon > 0$. Obviously,

$$\log M(r, Q) = \log \max_{|z|=r} |Q(z)| = \max_{|z|=r} \log |Q(z)|.$$

Clearly,

$$\begin{aligned} \log|Q(z)| = \log \prod_{j=1}^{\infty} \left|E_\nu\left(\frac{z}{z_j}\right)\right| &\leq \sum_{|z/z_j|\geq 1/2} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right| + \sum_{|z/z_j|<1/2} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right| \\ &=: S_1 + S_2. \end{aligned}$$

Observe that S_1 is a finite sum by the standard uniqueness theorem of analytic functions.

To estimate S_2, where $|\frac{z}{z_j}| < 1/2$, recall the property (3) of Weierstraß products from Chapter 5. By this property,

$$\left|E_\nu\left(\frac{z}{z_j}\right) - 1\right| \leq \left|\frac{z}{z_j}\right|^{\nu+1},$$

hence

$$\left|E_\nu\left(\frac{z}{z_j}\right)\right| \le 1 + \left|\frac{z}{z_j}\right|^{\nu+1}.$$

Therefore,

$$\sum_{|z/z_j|<1/2} \log\left|E_\nu\left(\frac{z}{z_j}\right)\right| \le \sum_{|z/z_j|<1/2} \log\left(1 + \left|\frac{z}{z_j}\right|^{\nu+1}\right) \le \sum_{|z/z_j|<1/2} \left|\frac{z}{z_j}\right|^{\nu+1}. \quad (11.2)$$

We now have to analyze all cases in the definition of ν above. In the middle case, the sum (11.2) is majorized by

$$= \sum_{|z/z_j|<1/2} \left|\frac{z}{z_j}\right|^{\lambda} = |z|^\lambda \sum_{|z/z_j|<1/2} |z_j|^{-\lambda} = O(r^{\lambda+\varepsilon}),$$

since $\sum |z_j|^{-\lambda}$ converges. In the remaining two cases, $\nu+1 > \lambda+\varepsilon$ for ε small enough and so

$$\left|\frac{z}{z_j}\right|^{\nu+1} = |z|^{\lambda+\varepsilon}\left|\frac{z}{z_j}\right|^{\nu+1-\lambda-\varepsilon} |z_j|^{-(\lambda+\varepsilon)} \le |z|^{\lambda+\varepsilon}|z_j|^{-(\lambda+\varepsilon)}.$$

Hence, the sum in (11.2) is now

$$\le |z|^{\lambda+\varepsilon} \sum_{|z/z_j|<1/2} |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}),$$

since $\sum |z_j|^{-(\lambda+\varepsilon)}$ converges by the definition of the exponent of convergence.

To estimate S_1, we first consider the case $\nu = 0$; recall that S_1 is a finite sum. Then

$$\begin{aligned} S_1 &= \sum_{|z/z_j|\ge 1/2} \log\left|E_0\left(\frac{z}{z_j}\right)\right| = \sum_{|z/z_j|\ge 1/2} \log\left|1 - \frac{z}{z_j}\right| \\ &\le \sum_{|z/z_j|\ge 1/2} \log\left(1 + \left|\frac{z}{z_j}\right|\right) \le A \sum_{|z/z_j|\ge 1/2} \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|^\varepsilon \sum_{|z/z_j|\ge 1/2} |z_j|^{-\varepsilon}, \end{aligned} \quad (11.3)$$

where A is a suitable constant. If $\lambda = 0$, then $\sum |z_j|^{-\varepsilon}$ converges and by (11.3),

$$S_1 = O(r^\varepsilon) = O(r^{\lambda+\varepsilon}).$$

If $\lambda = 1$ and $\sum |z_j|^{-1}$ converges, we get

$$\begin{aligned} S_1 &= A\sum \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|\sum \left|\frac{z}{z_j}\right|^{\varepsilon-1} |z_j|^{-1} = A|z|\sum \left|\frac{z_j}{z}\right|^{1-\varepsilon} |z_j|^{-1} \\ &\le 2A|z|\sum |z_j|^{-1} = O(r^\lambda) = O(r^{\lambda+\varepsilon}), \end{aligned}$$

provided $\varepsilon < 1$. Since $\nu = 0$, we must have $\lambda \le 1$. Thus, assume now $\lambda \in (0,1)$ and take $\varepsilon < \lambda$. Then

$$S_1 = A \sum \left|\frac{z}{z_j}\right|^{\varepsilon} = A|z|^{\lambda+\varepsilon} \sum \left|\frac{z}{z_j}\right|^{-\lambda} |z_j|^{-(\lambda+\varepsilon)} \le A|z|^{\lambda+\varepsilon} \sum \left|\frac{z_j}{z}\right|^{\lambda} |z_j|^{-(\lambda+\varepsilon)}$$
$$\le 2A|z|^{\lambda+\varepsilon} \sum |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}).$$

Finally, we have to consider the case $\nu > 0$. Then, for each term in S_1,

$$\log \left|E_\nu\left(\frac{z}{z_j}\right)\right| \le \log \left|1 - \frac{z}{z_j}\right| + \left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^{\nu}$$
$$\le 2\left(\left|\frac{z}{z_j}\right| + \cdots + \frac{1}{\nu}\left|\frac{z}{z_j}\right|^{\nu}\right) \le 2\left|\frac{z}{z_j}\right|^{\nu}\left(1 + \left|\frac{z_j}{z}\right| + \cdots + \left|\frac{z_j}{z}\right|^{\nu-1}\right)$$
$$\le 2\left|\frac{z}{z_j}\right|^{\nu}(1 + 2 + \cdots + 2^{\nu-1}) \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\nu}.$$

If now $\nu = \lambda - 1$, then

$$\log \left|E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda-1} = 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda}\left|\frac{z_j}{z}\right| \le 2^{\nu+2}\left|\frac{z}{z_j}\right|^{\lambda}. \tag{11.4}$$

If $\nu \ne \lambda - 1$, and ε is small enough, then $\nu < \lambda + \varepsilon \le \nu + 1$ and $\lambda + \varepsilon + 1 \le \nu + 2$. Therefore,

$$\log \left|E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\nu} = 2^{\nu+1}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon}\left|\frac{z_j}{z}\right|^{\lambda+\varepsilon-\nu}$$
$$\le 2^{\nu+1+\lambda+\varepsilon-\nu}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon} \le 2^{\nu+2}\left|\frac{z}{z_j}\right|^{\lambda+\varepsilon}. \tag{11.5}$$

From (11.4) and (11.5),

$$\sum_{|z/z_j| \ge 1/2} \log \left|E_\nu\left(\frac{z}{z_j}\right)\right| \le 2^{\nu+2} r^{\lambda+\varepsilon} \sum_{|z/z_j| \ge 1/2} |z_j|^{-(\lambda+\varepsilon)}$$
$$\le 2^{\nu+2} r^{\lambda+\varepsilon} \sum_{z_j} |z_j|^{-(\lambda+\varepsilon)} = O(r^{\lambda+\varepsilon}).$$

So, we see that $S_1 = O(r^{\lambda+\varepsilon})$, $S_2 = O(r^{\lambda+\varepsilon})$. This means that

$$\log |Q(z)| = O(r^{\lambda+\varepsilon}),$$

hence

$$\log M(r, Q) = O(r^{\lambda+\varepsilon}),$$

and so

$$\rho(Q) = \limsup_{r\to\infty} \frac{\log\log M(r,Q)}{\log r} \le \lambda + \varepsilon. \quad \square$$

Theorem 11.2. (Hadamard). *Let $f(z)$ be a non-constant entire function of finite order ρ. Then*

$$f(z) = z^m Q(z) e^{P(z)},$$

where (1) *$m \geq 0$ is the multiplicity of the zero of $f(z)$ at $z = 0$,* (2) *$Q(z)$ is the canonical product formed with the non-zero zeros of $f(z)$ and* (3) *$P(z)$ is a polynomial of degree $\leq \rho$.*

Before we can prove the Hadamard theorem, we need the following

Lemma 11.3. *Let $Q(z)$ be a canonical product of order $\lambda = \lambda(Q)$. Given $\varepsilon > 0$, there exists a sequence $(r_n) \to +\infty$ such that for each r_n, the minimum modulus satisfies*

$$\mu(r_n) := \min_{|z|=r_n} |Q(z)| > e^{-r_n^{\lambda+\varepsilon}}. \tag{11.6}$$

Proof. Let (z_j) denote the zeros of $Q(z)$, $0 < |z_1| \leq |z_2| \leq \cdots$. Denote $r_j = |z_j|$. By the definition of the exponent of convergence, $\sum_j r_j^{-(\lambda+\varepsilon)}$ converges. This means that the length of the set

$$E := \bigcup_{j=1}^{\infty} \left[r_j - \frac{1}{r_j^{\lambda+\varepsilon}}, r_j + \frac{1}{r_j^{\lambda+\varepsilon}} \right]$$

is finite. We proceed to prove that (11.4) holds outside of E for all r sufficiently large. From the proof of Theorem 11.1,

$$\log|Q(z)| = S_1 + S_2' = \sum_{|z/z_j|\geq 1/2} \log\left| E_\nu\left(\frac{z}{z_j}\right)\right| + \log \prod_{|z/z_j|<1/2} \left| E_\nu\left(\frac{z}{z_j}\right)\right|.$$

Moreover, from the same proof, making use of the estimate for S_2, $S_2' \leq S_2 = O(r^{\lambda+\varepsilon})$. Recall now again that S_1 is a finite sum. Therefore,

$$S_1 = \sum_{|z/z_j|\geq 1/2} \log\left|1 - \frac{z}{z_j}\right| + \sum_{|z/z_j|\geq 1/2} \log|e^{Q_\nu(z)}| =: S_{11} + S_{12}.$$

Assume now that $r \notin E$ is sufficiently large. Then, as $2r \geq r_j$

$$\left|1 - \frac{z}{z_j}\right| = \frac{|z_j - z|}{|z_j|} \geq \frac{|r - r_j|}{r_j} \geq r_j^{-1-\lambda-\varepsilon} \geq (2r)^{-1-\lambda-\varepsilon}$$

and so

$$S_{11} = \sum_{|z/z_j|\geq 1/2} \log\left|1 - \frac{z}{z_j}\right| \geq -(1+\lambda+\varepsilon)\big(\log(2r)\big)n(2r).$$

By Theorem 9.8, $n(2r) = O(r^{\lambda+\varepsilon})$. Since $r^\varepsilon > (1+\lambda+\varepsilon)\log(2r)$ for r sufficiently large, we get

$$S_{11} \geq -r^{\lambda+2\varepsilon}.$$

For S_{12}, we may apply the proof of Theorem 11.1 to see that

$$S_{12} < S_1 = O(r^{\lambda+\varepsilon}).$$

Writing this as $S_{12} \le Kr^{\lambda+\varepsilon}$ for r large enough, we get

$$\begin{aligned}\log|Q(z)| &\ge |S_{11}| - |S_{12}| - |S_2| \ge -r^{\lambda+2\varepsilon} - Kr^{\lambda+\varepsilon} \\ &= -r^{\lambda+2\varepsilon}(1+Kr^{-\varepsilon}) \ge -2r^{\lambda+2\varepsilon} \ge -r^{\lambda+3\varepsilon}.\end{aligned}$$

By exponentiation, we get

$$|Q(z)| \ge e^{-r^{\lambda+3\varepsilon}},$$

hence (11.6) holds. □

Proof of Theorem 11.2. By the construction of the canonical product, $z^m Q(z)$ has exactly the same zeros as $f(z)$, with the same multiplicities as well. Therefore,

$$f(z)/z^m Q(z)$$

is an entire function with no zeros. By Theorem 4.1, there is an entire function $g(z)$ such that

$$f(z) = z^m Q(z) e^{g(z)}.$$

It remains to prove that $g(z)$ is a polynomial of degree $\le \rho$. Since $f(z)$ is of order ρ,

$$M(r,f) \le e^{r^{\rho+\varepsilon}}$$

for all r sufficiently large. Now the order of $Q(z) = \lambda = \lambda(f) \le \rho$. Take r such that (11.6) is true. Then

$$\max_{|z|=r}|e^{g(z)}| = \max_{|z|=r} e^{\operatorname{Re} g(z)} \le \frac{\max_{|z|=r}|f(z)|}{r^m \min_{|z|=r}|Q(z)|} \le \frac{e^{r^{\rho+\varepsilon}}}{e^{-r^{\lambda+\varepsilon}}} = e^{r^{\rho+\varepsilon}} \cdot e^{r^{\lambda+\varepsilon}} \le e^{2r^{\rho+\varepsilon}}.$$

Recalling Definition 7.4, we observe that

$$A(r,g) \le 2r^{\rho+\varepsilon}.$$

By Theorem 7.6, g is a polynomial of degree $\le \rho+\varepsilon$, hence $\le \rho$. □

Corollary 11.4. *Let $f(z)$ be a nonconstant entire function of finite non-integer order ρ. Then $\lambda(f) = \rho$.*

Proof. If $\rho = 0$, then by Theorem 9.9, we have $0 \le \lambda(f) \le \rho(f) = \rho = 0$. Therefore, we may assume that $\rho > 0$ and that $\lambda(f) < \rho$. By Theorem 11.2, $\deg P(z) = n \le \rho \notin \mathbb{N}$, hence $\deg P(z) < \rho$. By Lemma 7.2,

$$M(r, e^P) \le e^{2|a_n|r^n};$$

here now $P(z) = a_n z^n + \cdots + a_0$. Therefore $\rho(e^P) \le n$. On the other hand,

$$M(r,e^P) = \max_{|z|=r}|e^P| = e^{\max_{|z|=r}\operatorname{Re} P} = e^{A(r,P)} \ge e^{Kr^n}$$

for some $K > 0$ by Theorem 7.5. Hence $\rho(e^P) \ge n$, and so $\rho(e^P) = n < \rho$. By Theorem 7.9,

$$\rho(f) \le \max\big(\rho(z^m), \rho(Q), \rho(e^P)\big) \le \max\big(\lambda(f), n\big) < \rho = \rho(f),$$

a contradiction. □

Corollary 11.5. *If $f(z)$ is transcendental entire and $\rho(f) \notin \mathbb{N}$, then $f(z)$ has infinitely many zeros.*

Proof. If $\rho > 0$, then $\lambda(f) > 0$, and so f must have infinitely many zeros. If then $\rho = 0$, the Hadamard theorem implies that $f(z) = cz^m Q(z)$, $c \in \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$. Since $f(z)$ is not a polynomial, $Q(z)$ cannot be a polynomial and $\rho(Q) = 0$. By the construction of a canonical product, $Q(z)$ is the product of terms of type $E_0(\frac{z}{z_j})$. Since it is not a polynomial, the number of zeros z_j must be infinite. □

To complete what is contained in the two preceding corollaries, we still consider the case of f of an integer order.

Theorem 11.6. *(Borel) If the order of an entire function f is a natural number ρ, then the exponent of convergence $\lambda(a, f)$ of a-points of f equals to ρ, with one possible exceptional value a*

Proof. Suppose there are two exceptional values a, b in the sense that $\lambda(a, f) < \rho$, $\lambda(b, f) < \rho$. By the Hadamard theorem,

$$f(z) - a = z^{m^1} e^{P_1(z)} Q_1(z)$$

and

$$f(z) - b = z^{m_2} e^{P_2(z)} Q_2(z),$$

where $P_1(z), P_2(z)$ are polynomials of degree ρ ($= \rho$ by the assumptions) and $Q_1(z), Q_2(z)$ are canonical products determined by the non-zero a-points, resp. b-points, of f, both being of order $\rho_j < \rho$, $j = 1, 2$. Subtracting we get

$$b - a = z^{m_1} e^{P_1(z)} Q_1(z) - z^{m_2} e^{P_2(z)} Q_2(z),$$

hence

$$z^{m_1} Q_1(z) e^{Q_1(z) - Q_2(z)} = z^{m_2} Q_2(z) + (b - a) e^{-Q_2(z)}.$$

Since $\deg P_2(z) = \rho$, the right-hand side in the preceding identity is of order ρ, hence so is the left-hand side as well. This means that $\deg(P_1(z) - P_2(z)) = \rho$. Differentiating the preceding identity we obtain

$$(z^{m_1} P_1' Q_1 + m_1 z^{m_1 - 1} Q_1 + z^{m_1} Q_1') e^{P_1} = (z^{m_2} P_2' Q_2 + m_2 z^{m_2 - 1} Q_2 + z^{m_2} Q_2') e^{P_2}.$$

Observe that the differentiated identity is correct also, if one of m_1, m_2, or both of them, is zero. Supposing that the differentiation leaves order unchanged, to be proved in the next Proposition 11.7, the order of Q_j' equals to $\rho_j < \rho$, $j = 1, 2$. Therefore, the coefficients, in parenthesis, of e^{P_j} in the differentiated identity above are of order $< \rho$. Since the coefficients are entire functions, we may write the above identity, the the Hadamard theorem again, as

$$z^{m_3} Q_3(z) e^{P_1(z) + P_3(z)} = z^{m_4} Q_4(z) e^{P_2(z) + P_4(z)},$$

where $P_3(z), P_4(z)$ are polynomials of order $< \rho$, hence $\leq \rho - 1$ and $Q_3(z), Q_4(z)$ are the canonical products formed by the non-zero zeros of the respective coefficients. Since the zeros on both sides of the last identity are the same, we must have $m_3 = m_4$ and $Q_3(z) \equiv Q_4(z)$. Therefore, we now see that for some constant integer n,

$$Q_1(z) - Q_2(z) = Q_4(z) - Q_3(z) + 2\pi i n.$$

But this is a contradiction, since $\deg(Q_1(z) - Q_2(z)) = \rho$, while $\deg(Q_4(z) - Q_3(z)) \leq \rho - 1$.

As mentioned in the proof of Theorem 11.6, we still have to establish the following

Proposition 11.7. *Given an entire function g, it is true that $\rho(g') = \rho(g)$.*

Proof. Denote the maximum moduli of g and g' as $M(r, g)$ and $M(r, g')$. Integrating, say along the line segment from the origin to z, we get

$$g(z) = \int_0^z g'(\zeta) d\zeta + g(0).$$

Therefore,

$$|g(z)| \leq M(r, g') + |g(0)|$$

and further

$$M(r, g) \leq M(r, g') + |g(0)|.$$

This immediately results in $\rho(g) \leq \rho(g')$. To prove the reversed inequality, take $|z| = r < R$, and recall that by the Cauchy integral formula

$$g'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = R - r} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta.$$

Taking moduli and estimating upwards we get

$$|g'(z)| \leq \frac{2\pi(R - r)}{2\pi(R - r)^2} M(R, g) = \frac{M(r, g)}{R - r},$$

and so

$$M(r, g') \leq \frac{M(R, g)}{R - r}.$$

Choosing $R = 2r$, and supposing that $r > 1$, as we may, we obtain

$$M(r, g') = \frac{M(2r, g)}{r} \leq M(2r, g).$$

By monotonicity of the logarithm, we obtain

$$\begin{aligned} \frac{\log \log M(r, g')}{\log r} &\leq \frac{\log \log M(2r, g)}{\log r} = \frac{\log \log M(2r, g)}{\log 2r - \log 2} \\ &= \frac{\log 2r}{\log 2r - \log 2} \frac{\log \log M(2r, g)}{\log 2r}, \end{aligned}$$

from which the reversed inequality immediately follows.

Note that Proposition 11.7 may also be proved by use of the Taylor expansions of g and g' and Theorem 7.10.

Remark. By Corollary 11.5 and Proposition 11.7, we have proved the famous Picard theorem: Every transcendental entire function f takes all finite complex values a infinitely often, with one possible exceptional value a.

12. Spherical metrics and normal families

We consider the sphere $\sum$ (Riemann sphere) defined in $\mathbb{R}^3$ by

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4},$$

i.e. a sphere of radius $\frac{1}{2}$, centered in $(0, 0, \frac{1}{2})$. Of course, $\sum$ is tangent to the (x, y)-plane ($= \mathbb{C}$) at the origin $(0, 0, 0)$. We now denote the north pole $(0, 0, 1)$ of the sphere by N, meaning that the origin stands for the south pole S.Clearly, we may set a one-to-one correspondence between the points on $\sum \setminus \{N\}$ and the complex plane $\mathbb{C}$ by defining as the image of $z \in \mathbb{C}$ the point $\neq N$ where the line from N to z intersects the sphere $\sum$. Setting N as the image of ∞, we obtain a one-to-one correspondence between the Riemann sphere $\sum$ and the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Take now an arbitrary point $z = (x, y, 0) \in \mathbb{C}$. To determine the analytic expression of the image $\sum(z)$ of z on $\sum$, we first observe that the line from N to z has the following expression (as a vector from 0 to a point on the line):

$$\overrightarrow{r} = \overrightarrow{k} + t(x\overrightarrow{i} + y\overrightarrow{j} - \overrightarrow{k}) = xt\overrightarrow{i} + yt\overrightarrow{j} + (1-t)\overrightarrow{k}$$

with a real parameter t. Of course, $t = 0$ corresponds to the north pole N, while $t = 1$ stands for the point z in the complex plane. As the image point $\sum(z)$ is on the sphere $\sum$, we obtain

$$(x^2 + y^2)t^2 + (\frac{1}{2} - t)^2 = \frac{1}{4},$$

hence

$$t = \frac{1}{x^2 + y^2 + 1} = \frac{1}{1 + |z|^2}.$$

Therefore, by setting this value of t into the vectorial representation of the line Nz, we obtain that

$$\sum(z) = \left(\frac{x}{1 + |z|^2}, \frac{y}{1 + |z|^2}, \frac{|z|^2}{1 + |z|^2} \right).$$

The length of $\sum(z)$ (as a vector from the origin in $\mathbb{R}^3$ is

$$|\sum(z)| = \frac{|z|}{\sqrt{1 + |z|^2}}.$$

Indeed,

$$|\sum(z)|^2 = \frac{x^2}{(1 + |z|^2)^2} + \frac{y^2}{(1 + |z|^2)^2} + (1 - \frac{1}{1 + |z|^2})^2 = \frac{|z|^2}{1 + |z|^2}.$$

Definition 12.1. The chordal distance $\chi(z_1, z_2)$ of two points $z_1, z_2 \in \widehat{\mathbb{C}}$ is defined as the euclidean distance of $\sum(z_1)$ and $\sum(z_2)$.

Proposition 12.2. *For the chordal distance of two points $z_1, z_2 \in \mathbb{C}$, we have*

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

Moreover, if $z \in \mathbb{C}$, then

$$\chi(z, \infty) = \frac{1}{\sqrt{1+|z|^2}}.$$

Proof. As for the first claim, denote $z_j = (x_j, y_j)$ for $j = 1, 2$. Then a routine computation results in

$$\begin{aligned}
\chi(z_1, z_2)^2 &= \left(\frac{x_2}{1+|z_2|^2} - \frac{x_1}{1+|z_1|^2}\right)^2 + \left(\frac{y_2}{1+|z_2|^2} - \frac{y_1}{1+|z_1|^2}\right)^2 \\
&\quad + \left(\frac{|z_2|^2}{1+|z_2|^2} - \frac{|z_1|^2}{1+|z_1|^2}\right)^2 \\
&= \left(\frac{x_2}{1+|z_2|^2} - \frac{x_1}{1+|z_1|^2}\right)^2 + \left(\frac{y_2}{1+|z_2|^2} - \frac{y_1}{1+|z_1|^2}\right)^2 \\
&\quad + \left(\frac{1}{1+|z_1|^2} - \frac{1}{1+|z_2|^2}\right)^2 \\
&= \frac{x_2^2 + y_2^2 + 1}{(1+|z_2|^2)^2} + \frac{x_1^2 + y_1^2 + 1}{(1+|z_1|^2)^2} - 2\frac{x_1 x_2 + y_1 y_2 + 1}{(1+|z_1|^2)(1+|z_2|^2)} \\
&= \frac{1}{1+|z_1|^2} + \frac{1}{1+|z_2|^2} - 2\frac{x_1 x_2 + y_1 y_2 + 1}{(1+|z_1|^2)(1+|z_2|^2)} \\
&= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1 x_2 - 2y_1 y_2}{(1+|z_1|^2)(1+|z_2|^2)} \\
&= \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(1+|z_1|^2)(1+|z_2|^2)} = \frac{|z_1 - z_2|^2}{(1+|z_1|^2)(1+|z_2|^2)}.
\end{aligned}$$

The second claim is easier:

$$\begin{aligned}
\chi(z, \infty)^2 &= \frac{x^2}{(1+|z|^2)^2} + \frac{y^2}{(1+|z|^2)^2} + \left(1 - \frac{|z|^2}{1+|z|^2}\right)^2 \\
&= \frac{x^2}{(1+|z|^2)^2} + \frac{y^2}{(1+|z|^2)^2} + \frac{1}{(1+|z|^2)^2} = \frac{1}{1+|z|^2}.
\end{aligned}$$

Corollary 12.3. *For all $z_1, z_2 \in \widehat{\mathbb{C}}$, we have*

$$\begin{cases} \chi(z_1, z_2) \le 1, \\ \chi\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = \chi(z_1, z_2), \\ |z_1| \le |z_2| \le \infty \Rightarrow \chi(0, z_1) \le \chi(0, z_2). \end{cases}$$

Proof. The first claim immediately follows from the fact that the diameter of the Riemann sphere equals to 1. The second claim can be seen by the expressions of the chordal distances in the claim. As for the last assertion, if $|z_2| = \infty$, then $\chi(0, \infty) = 1$, and the asserted inequality is trivial by the first assertion. If both of $z_1, z_2 \in \mathbb{C}$, then writing $\chi(0, z_j) = \frac{1}{\sqrt{1+1/|z_j|^2}}$, the assertion becomes trivial.

Proposition 12.4. *χ defines a metric on $\widehat{\mathbb{C}}$. Moreover,*

$$\chi(z_1, z_2) \le |z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{C}$.

Proof. The asserted inequality is trivial by the definition of the chordal metric. That χ in fact defines a metric, only needs to show that the triangle inequality is true. But this is an immediate consequence of the triangle inequality in the three-dimensional euclidean metric.

We next define the spherical arc length element ds on $\sum$ by

$$ds = \frac{|dz|}{1 + |z|^2}$$

and the corresponding spherical area element by

$$dA = \frac{dxdy}{(1 + |z|^2)^2},$$

where $z = x + iy$. Given now a curve γ on the Riemann sphere $\sum$, then its spherical length is naturally defined as

$$L(\gamma) := \int_\gamma \frac{|dz|}{1 + |z|^2}.$$

The spherical length of γ may now be used to define what is called the spherical metric σ on $\sum$ by

$$\sigma(z_1, z_2) := \inf L(\gamma),$$

where the infimum is taken over all differentiable curves on $\sum$ from z_1 to z_2. It is geometrically obvious that $\sigma(z_1, z_2)$ is just the euclidean length of the shorter arc of the great circle on $\sum$ joining z_1 to z_2. Since this shorter great circle arc is by its length $\leq \pi/2$ (as the diameter of the Riemann sphere is one), it is geometrically easy to see that

$$\chi(z_1, z_2) \leq \sigma(z_1, z_2) \leq \frac{\pi}{2}\chi(z_1, z_2).$$

In fact, the left inequality is trivial. To prove the right one, suppose that the angle at the origin in the triangle formed by the center of $\sum$ and the points $\sum(z_1)$ and $\sum(z_2)$ is θ. Then we conclude, by elementary trigonometry, that $\chi(z_1, z_2) = \sin \frac{\theta}{2}$, while $\sigma(z_1, z_2) = \theta/2$. Since $\sin \beta \geq \frac{2}{\pi}\beta$, see p. 9, we obtain that

$$\frac{\sigma(z_1, z_2)}{\chi(z_1, z_2)} = \frac{\theta}{2\sin\frac{\theta}{2}} \leq \frac{\theta}{2\frac{2}{\pi}\frac{\theta}{2}} = \frac{\pi}{2}.$$

The above double inequality means, by elementary topology, that the two metrics χ and σ induce the same topology on $\sum$. Therefore, concerning topological notions such as limits, continuity, openness, compactness etc., we may use either the chordal metric, or the spherical metric equivalently.

Definition 12.5. A sequence (f_n) of functions $f_n : \mathbb{C} \to \widehat{\mathbb{C}}$ converges spherically uniformly to a function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ on a set $E \subset \mathbb{C}$ if for any $\varepsilon > 0$, there exists n_ε such that $n \geq n_\varepsilon$ implies that

$$\chi(f(z), f_n(z)) < \varepsilon$$

for all $z \in E$.

Remark. Since $\chi(z_1, z_2) \leq |z_1 - z_2|$, the usual (euclidean) uniform convergence on E implies the spherical uniform convergence. A partial converse of this observation is contained in the following

Theorem 12.6. *If a sequence (f_n) of functions $f_n : \mathbb{C} \to \widehat{\mathbb{C}}$ converges spherically uniformly to a bounded function $f : \mathbb{C} \to \widehat{C}$ on E, then (f_n) converges uniformly (in the euclidean sense) to f on E.*

Proof. Assume that $|f(z)| \leq M$ on E, Then we get

$$\chi(0, f(z)) \leq \chi(0, M) = \frac{M}{\sqrt{1 + M^2}} < 1.$$

Take now $\varepsilon < 1 - \frac{M}{\sqrt{1+M^2}}$, and fix n_ε so that

$$\chi(f(z), f_n(z)) < \varepsilon$$

for all $n \geq n_\varepsilon$ and all $z \in E$. Then

$$\begin{aligned} \frac{|f_n(z)|}{\sqrt{1 + |f_n(z)|^2}} &= \chi(0, f_n(z)) \leq \chi(0, f(z)) + \chi(f(z), f_n(z)) \\ &< \frac{M}{\sqrt{1 + M^2}} + \varepsilon =: m < 1. \end{aligned}$$

This implies that

$$|f_n(z)| < \frac{m}{\sqrt{1-m^2}} =: M_1$$

for all $n \geq n_\varepsilon$ and all $z \in E$. Therefore,

$$\begin{aligned}|f(z) - f_n(z)| &= \sqrt{1+|f(z)|^2}\sqrt{1+|f_n(z)|^2}\chi(f(z), f_n(z))\\ &< \sqrt{1+M^2}\sqrt{1+M_1^2}\chi(f(z), f_n(z))\end{aligned}$$

for all $n \geq n_\varepsilon$ and all $z \in E$, proving the assertion.

Definition 12.7. A function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is spherically continuous at $z_0 \in \mathbb{C}$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\chi(f(z), f(z_0)) < \varepsilon$$

whenever $|z - z_0| < \delta$.

Proposition 12.8. *If $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic in $\Omega \subset \mathbb{C}$, then f is spherically continuous in Ω.*

Proof. If f is analytic at $z_0 \in \mathbb{C}$, then spherical continuity immediately follows from

$$\chi(f(z), f(z_n)) \leq |f(z) - f(z_0)|.$$

On the other hand, if z_0 is a pole of f. then $1/f$ is analytic, hence continuous at z_0. Therefore,

$$\chi(f(z), f(z_0)) = \chi\left(\frac{1}{f(z)}, \frac{1}{f(z_0)}\right) \leq \left|\frac{1}{f(z)} - \frac{1}{f(z_0)}\right|$$

implies spherical continuity.

We next proceed to define the spherical derivative of a function f meromorphic in a domain $\Omega \subset \mathbb{C}$. Supposing first that $f(z)$ is finite, we set

$$\begin{aligned}f^\sharp(z) &:= \lim_{z' \to z} \frac{\chi(f(z), f(z'))}{|z - z'|}\\ &= \lim_{z' \to z} \frac{|f(z) - f(z')|}{|z - z'|}\frac{1}{\sqrt{1+|f(z)|^2}}\frac{1}{\sqrt{1+|f(z')|^2}}\\ &= \frac{|f'(z)|}{1+|f(z)|^2}.\end{aligned}$$

If then z is pole of f, we then define

$$f^\sharp(z) := \lim_{z' \to z} \frac{|f'(z')|}{1+|f(z')|^2}.$$

By its definition (and the fact that f is continuous, if finite), the spherical derivative $f^\sharp$ is continuous in $\mathbb{C}$. Moreover, it is immediate to see that $f^\sharp(z) = (1/f(z))^\sharp$.

Definition 12.9. A sequence (f_n) of functions $f_n : \Omega \to \mathbb{C}$, resp. $f_n : \Omega \to \widehat{\mathbb{C}}$, on a domain $\Omega \subset \mathbb{C}$ converges uniformly, resp. spherically uniformly, on compact subsets of Ω to a function $f : \Omega \to \mathbb{C}$, resp. $f : \Omega \to \widehat{\mathbb{C}}$, if for any compact set $K \subset \Omega$ and any $\varepsilon > 0$ there exists $N = N(K, \varepsilon)$ such that $n \geq N$ implies that $|f_n(z) - f(z)| < \varepsilon$, resp. $\chi(f_n(z), f(z)) < \varepsilon$ for all $z \in K$.

Definition 12.10. A family $\mathfrak{F}$ of functions $f : \Omega \to \mathbb{C}$ is locally bounded on a domain Ω, if for each $z_0 \in \Omega$ there exists $M = M(z_0)$, $0 \leq M < \infty$, and a disc $D(z_0, r) \subset \Omega$ such that $|f(z)| \leq M$ for all $z \in D(z_0, r)$ and all $f \in \mathfrak{F}$.

Example. Consider the family

$$\mathfrak{F} := \left\{ f_\alpha(z) := \frac{1}{z - e^{i\alpha}} \mid \alpha \in \mathbb{R} \right\}$$

in the unit disc D. The family $\mathfrak{F}$ is not uniformly bounded in D, since given $\alpha \in \mathbb{R}$, $f_\alpha(z) \to \infty$ as $z \to e^{i\alpha}$. But $\mathfrak{F}$ is locally bounded: Given $z_0 \in D$, take $D(z_0, r) \subset D(z_0, \rho) \subset D$, $r < \rho$. Then for any $z \in D(z_0, r)$ and any $\alpha \in \mathbb{R}$, there is $\delta > 0$ so that $|z - e^{i\alpha}| \geq \delta$.

Theorem 12.11. *If $\mathfrak{F}$ is a locally bounded family of analytic functions on a domain Ω, then the family $\mathfrak{F}' := \{ f' \mid f \in \mathfrak{F} \}$ of their derivatives is locally bounded.*

Proof. Let $z_0 \in \Omega$ be arbitrary. Then for some $M < \infty$, $|f(z)| \leq M$ for all $f \in \mathfrak{F}$ and all $z \in \overline{D}(z_0, r)$ in a closed disc $\overline{D}(z_0, r)$ centered at z_0. Given $z \in D(z_0, \frac{r}{2})$ and integrating over the boundary $\partial\overline{D} = \partial\overline{D}(z_0, r)$, the Cauchy formula results in

$$|f'(z)| \leq \frac{1}{2\pi} \int_{\partial\overline{D}} \frac{|f(\zeta)||d\zeta|}{|\zeta - z|^2} < \frac{4M}{r}$$

for all $f' \in \mathfrak{F}'$, so that $\mathfrak{F}'$ is locally bounded.

Remark. The converse of Theorem 12.11 is not true. Indeed, consider the family $\mathfrak{F} := \{ n \mid n \in \mathbb{N} \}$ of constant functions. Clearly, $\mathfrak{F}$ is not locally bounded. However, $\mathfrak{F}' = \{0\}$ is a (locally) bounded family, as it consists of just one bounded function. A partial converse can be given, however, as seen by the next

Theorem 12.12. *Let $\mathfrak{F}$ be a family of analytic functions on a domain $\Omega \subset \mathbb{C}$ such that (1) $\mathfrak{F}' = \{ f' \mid f \in \mathfrak{F} \}$ is locally bounded and (2) there exists a point $z_0 \in \Omega$ such that $\mathfrak{F}$ is bounded at z_0, i.e. that for some finite M, $|f(z_0)| \leq M$ for all $f \in \mathfrak{F}$. Then $\mathfrak{F}$ is locally bounded.*

Proof. Let $z \in \Omega$ be arbitrarily chosen and consider a (small) neighborhood $D(z, r)$ of z. Denote $\rho := |z - z_0|$. If now $f \in \mathfrak{F}$ and $\zeta \in D(z, r)$, then integrating form z_0 to ζ along the path consisting of the line segments $[z_0, z]$ and $[z, \zeta]$ results in

$$|f(\zeta)| \leq |f(z_0)| + \int_{z_0}^{z} |f'(\zeta)||d\zeta| + \int_{z}^{\zeta} |f'(\zeta)||d\zeta| \leq M + M_1(\rho + r),$$

where $M_1 = \max\{ |f'(z)| : z \in [z_0, z] \cup [z, \zeta] \}$. Since $[z_0, z] \cup [z, \zeta]$ is compact and f' is continuous, M_1 is finite, and we are done.

Definition 12.13. A family $\mathfrak{F}$ of analytic functions on a domain $\Omega \subset \mathbb{C}$ is normal in Ω, if every sequence of functions $(f_n) \subset \mathfrak{F}$ contains either a subsequence converging to an analytic limit function f uniformly on each compact subset of Ω, or a subsequence converging uniformly to ∞ on each compact subset.

Normality of a family of functions is a property which holds globally if and only if it is true locally. More precisely, we say that $\mathfrak{F}$ is normal at $z_0 \in \Omega$ if it is normal in some (open) neighborhood of z_0. Then we obtain the following

Theorem 12.14. *A family $\mathfrak{F}$ of analytic functions is normal in a domain Ω if and only if $\mathfrak{F}$ is normal at each point of Ω.*

Proof. Obviously, a normal family is normal at each point locally as well.

To prove the converse assertion, suppose that $\mathfrak{F}$ is normal at each $z \in \Omega$. Choose then a countable dense subset $\{z_n\}$ in Ω. For example, we may take for $z_n = x_n + iy_n$ all points in Ω which rational real and imaginary parts. Denote by $D(z_n, r_n)$ the largest disc about z_n, contained in Ω, in which $\mathfrak{F}$ is normal. Since $\{z_n\}$ is dense in Ω, we clearly have $\bigcup_{n=1}^{\infty} D(z_n, r_n/2) = \Omega$. Take now an arbitrary sequence $(f_n) \subset \mathfrak{F}$. By normality at z_1, we can extract a convergent subsequence $(f_{n_k}^{(1)})$ which converges uniformly in $D(z_1, r_1/2)$ either to an analytic function or to ∞. The subsequence $(f_{n_k}^{(1)})$ in turn has a subsequence $(f_{n_k}^{(2)})$ which converges uniformly in $D(z_2, r_{2/2})$ and in $D(z_1, r_{1/2})$. We now continue in the same way. Picking now the diagonal sequence, let it be $(f_{n_k}^{(k)})$, it is easy to see that it converges uniformly in $D(z_n, r_n/2)$ for $n = 1, 2, 3, \ldots$, in each disc separately either to an analytic function or to ∞. But this divides Ω in two subsets Ω_{anal} and Ω_∞, which are disjoint, open and their union is Ω. Since Ω is a domain, hence connected, one of these two subsets is empty, hence the other one covers the whole Ω. Finally, to see that the convergence is uniform in all compact subsets K of Ω, it is sufficient to observe that K will be covered by finitely many of the discs $D(z_n, r_n/2)$.

Example. The preceding theorem is sometimes useful to verify that a family $\mathfrak{F}$ is normal, resp. non-normal. As an example, take

$$\mathfrak{F} := \{\, f_n(z) = nz \mid n \in \mathbb{N} \,\}.$$

Then we have $f_n(0) \to 0$, while $f_n(z) \to \infty$ for all $z \neq 0$. Therefore, $\mathfrak{F}$ is not normal in any domain containing the origin, while it is normal in any domain not containing the origin.

To continue, we have to recall the notion of equicontinuity:

Definition 12.15. A family $\mathfrak{F}$ of functions defined on a domain $\Omega \subset \mathbb{C}$ is equicontinuous, resp. spherically equicontinuous, at a point $z_0 \in \Omega$ if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, z_0) > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon,$$

resp.

$$\chi(f(z), f(z_0) < \varepsilon$$

for every $f \in \mathfrak{F}$ whenever $|z - z_0| < \delta$. Moreover, $\mathfrak{F}$ is equicontinuous, resp. spherically equicontinuous, on a subset $E \subset \Omega$ if it is equicontinuous, resp. spherically equicontinuous, at each point of E.

Remark. In what follows, it is important to recall that continuity on a compact set is equivalent to being uniformly continuous. The same applies for spherical continuity as well. Also observe that by Proposition 12.4, equicontinuity implies spherical equicontinuity.

Proposition 12.16. *If (f_n) is a sequence of (spherically) continuous functions converging (spherically) uniformly to a function f on a compact subset $E \subset \mathbb{C}$, then f is uniformly (spherically) continuous on E, and the functions $\{f_n\}$ form a (spherically) equicontinuous family of functions on E.*

Proof. We give the proof for the spherical metric only. By the spherically uniform convergence, given $\varepsilon > 0$, we find $n_0 \in \mathbb{N}$ so that

$$\chi(f_n(z), f(z)) < \frac{\varepsilon}{3}$$

for all $z \in E$ whenever $n \geq n_0$. By the uniform spherical continuity of f_{n_0} on E (compact!), there exists $\delta = \delta(\varepsilon, E) > 0$ such that

$$\chi(f_{n_0}(z), f_{n_0}(z')) < \frac{\varepsilon}{3}$$

for all $z, z' \in E$ such that $|z - z'| < \delta$. Then we obtain that

$$\chi(f(z), f(z')) \leq \chi(f(z), f_{n_0}(z)) + \chi(f_{n_0}(z), f_{n_0}(z')) + \chi(f_{n_0}(z'), f(z')) < \varepsilon$$

for $|z-z'| < \delta$. Therefore, f is uniformly spherically continuous on E. The spherical equicontinuity of $\{f_n\}$ now follows from the spherically uniform convergence of (f_n) to f and the spherical continuity of f:

$$\chi(f_n(z), f_n(z')) \leq \chi(f_n(z), f(z)) + \chi(f(z), f(z')) + \chi(f(z'), f_n(z')) < 3\varepsilon$$

for $|z - z'| < \delta$.

Proposition 12.17. *A locally bounded family $\mathfrak{F}$ of analytic functions on a domain $\Omega \subset \mathbb{C}$ is equicontinuous on compact subsets of Ω.*

Proof. By Theorem 12.11, the family $\mathfrak{F}'$ of derivatives is locally bounded, hence uniformly bounded on compact subsets of Ω. Take now a closed disc $K \subset \Omega$ and $M < \infty$ so that $||f'(z) \leq M$ for all $z \in K$ and all $f' \in \mathfrak{F}'$. Given $\varepsilon > 0$ and any two points z, z' in K so that $|z - z'| < \varepsilon/M$, and integrating over the line segment from z to z', we obtain

$$|f(z) - f(z')| \leq \int_z^{z'} |f'(\zeta)||d\zeta| \leq M|z - z'| = \varepsilon,$$

proving equicontinuity on K. Equicontinuity on an arbitrary compact set K follows by a standard compactness argument.

Remark. Note that the converse assertion to Proposition 12.17 is not true. Indeed, $\mathfrak{F} := \{\, z + n \mid n \in \mathbb{N} \,\}$ is equicontinuous, say in the unit disc, but is not locally bounded.

Theorem 12.18. (Montel). *If $\mathfrak{F}$ is a locally bounded family of analytic functions on a domain $\Omega \subset \mathbb{C}$, then $\mathfrak{F}$ is a normal family in Ω.*

Proof. The proof is somewhat similar to the proof of Theorem 12.14. In fact, we again take a countable dense subset $\{z_n\}$ in Ω. Take then any sequence (f_n) from $\mathfrak{F}$ and consider the sequence of complex numbers $\{f_n(z_1)\}$. By hypothesis there is a constant $M < \infty$ so that $|f_n(z_1)| < M$ for all $n \in \mathbb{N}$. As a bounded sequence, $\{f_n(z_1)\}$ has at least one point of accumulation by the Bolzano-Weierstraß principle, hence we can take a convergent subsequence

$$f_{n_1}^{(1)}(z_1), f_{n_2}^{(1)}(z_1), f_{n_3}^{(1)}(z_1), \dots ,$$

converging at z_1. Consider now this subsequence at z_2. Clearly, the sequence $(f_{n_k})^{(1)}(z_2)$ is a bounded sequence of complex numbers, hence we can again extract a convergent subsequence

$$f_{n_1}^{(2)}(z_2), f_{n_2}^{(2)}(z_2), f_{n_3}^{(2)}(z_2), \dots ,$$

converging both at z_1 and z_2. Continuing inductively, we get subsequences $(f_{n_k}^{(p)})$ which converge at $z_1, z_2, \dots , z_p$ for each $p \in \mathbb{N}$. Similarly as in the proof of Theorem 12.14, we take the diagonal sequence $(f_{n_k}^{(k)})$ and this sequence converges at every z_n.

We now proceed to show that the diagonal sequence converges uniformly on compact subsets of Ω. For simplicity of notation, call the diagonal sequence as (g_k), and consider an arbitrary compact set $K \subset \Omega$ and an arbitrary $\varepsilon > 0$. By Theorem 12.17, the original family $\mathfrak{F}$ is equicontinuous on K. Therefore, there exists $\delta = \delta(\varepsilon, K) > 0$ such that

$$|g_n(z) - g_n(z')| < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{N}$, whenever $|z - z'| < \delta$ for $z, z' \in K$. By compactness of K, we find finitely many points z_n, say $z_1, \dots , z_{k_0}$ after having renamed them, if needed, so that $K \subset \cup_{k=1}^{k_0} D(z_k, \delta)$. Since the diagonal sequence converges at every point z_n, we find, by the Cauchy criterium, $n_0 \in \mathbb{N}$ so that

$$|g_n(z_k) - g_m(z_k)| < \frac{\varepsilon}{3}$$

holds good for all $k = 1, \dots , k_0$ whenever $n, m \geq n_0$. The uniform convergence now immediately follows. Indeed, given a compact set $K \subset \Omega$, take $z \in K$ arbitrarily and suppose that $n, m \geq n_0$. Then there is some j, $1 \leq j \leq k_0$ so that $z \in D(z_j, \delta)$. By the preceding inequalities,

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(z_j)| + |g_n(z_j - g_m(z_j)| + |g_m(z_j) - g_m(z)| < \varepsilon.$$

13. Univalent functions in the unit disc

In this final section, we consider analytic functions in the unit disc $D := B(0,1)$ which univalent, i.e. injective mappings $f : D \to \mathbb{C}$. Most of the treatment here is directed to considering normalized univalent functions in

$$\mathfrak{S} := \{ f : D \to \mathbb{C} \mid f(0) = 0, f'(0) = 1 \}.$$

Hence, such a function has the Taylor expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{j=2}^{\infty} a_j z^j,$$

converging in the open unit disc D.

To start with, recall the Green formula (in $\mathbb{R}^2$. Let $G \subset \mathbb{R}^2$ be a smooth domain with positively oriented (i.e. counterclockwise) boundary and let $\overrightarrow{F} = F_1(x,y)\overrightarrow{i} + F_2(x,y)\overrightarrow{j}$ be a differentiable vector field in G. Then the Green formula states that

$$\int_{\partial G} F_1(x,y)dx + F_2(x,y)dy = \int\int_G \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

In addition to the family $\mathfrak{S}$, we also consider the family $\sum$ of functions which are analytic and univalent in $\mathbb{C} \setminus D$ which have a simple pole at infinity. This means that these functions have an expansion of the form

$$g(z) := z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots = z + \sum_{j=0}^{\infty} b_j z^{-j}.$$

It is not difficult to see that whenever $f(z) \in \mathfrak{S}$, then $h(z) := (f(1/z))^{-1} \in \sum$.

Our first result here is the following

Theorem 13.1. (Area theorem). *For each function $g \in \sum$, we have*

$$\sum_{j=1}^{\infty} j|b_j|^2 \leq 1.$$

Proof. Consider the image ∂S_ρ of the circle $|z| = \rho > 1$ under the mapping g. Obviously, ∂S_ρ is a boundary of a domain $S_\rho \subset \mathbb{C}$. Writing $g(z) = u(x,y) + iv(x,y)$, look at the vector field

$$\overrightarrow{F} = -\frac{1}{2} v \overrightarrow{i} + \frac{1}{2} u \overrightarrow{j}$$

in the image plane. Then it is immediate that

$$\frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial v} = 1.$$

Therefore, by the Green formula,

$$\frac{1}{2}\int_{\partial S_\rho} udv - vdu = \int\int_{S_\rho} 1dA = A(S_\rho),$$

the area of the image domain S_ρ bounded by the image curve of the circle $|z| = \rho > 1$. On the other hand,

$$\frac{1}{2}\int_{\partial S_\rho} udv - vdu = \frac{1}{2}\int_{\partial S_\rho} \mathrm{Im}((u-iv)(du+idv)) = \frac{1}{2i}\int_{\partial S_\rho} \overline{g}dg = \frac{1}{2i}\int_{\partial S_\rho} \overline{g}\frac{\partial g}{\partial \theta}d\theta.$$

From the Laurent expansion of g in $z > 1$, we get

$$\overline{g(\rho e^{i\theta})} = \rho e^{-i\theta} + \overline{b_0} + \overline{b_1}\rho^{-1}e^{i\theta} + \overline{b_2}\rho^{-2}e^{2i\theta} + \cdots,$$

and

$$\frac{\partial g(\rho e^{i\theta})}{\partial \theta} = i\rho e^{i\theta} - ib_1\rho^{-1}e^{-i\theta} - 2b_2\rho^{-2}e^{-2i\theta} + \cdots.$$

Therefore,

$$\begin{aligned} A(S_\rho) &= \frac{1}{2i}\int_{\partial S_\rho} \left(\rho e^{-i\theta} + \overline{b_0} + \overline{b_1}\rho^{-1}e^{i\theta} + \cdots\right) i \left(\rho e^{i\theta} - b_1\rho^{-1}e^{i\theta} - 2b_2\rho^{-2}e^{-21\theta} - \cdots\right) d\theta \\ &= \pi\left(\rho^2 - \sum_{j=1}^{\infty} j|b_j|^2\rho^{-2j}\right) \geq 0. \end{aligned}$$

Therefore, we have

$$\sum_{j=1}^{N} j|b_j|^2\rho^{-2j} \leq \sum_{j=1}^{\infty} j|b_j|^2\rho^{-2j} \leq \rho^2$$

for all $\rho > 1$ and all $N \in \mathbb{N}$. Letting now $\rho \to 1+$, we first get

$$\sum_{j=1}^{N} j|b_j|^2 \leq 1$$

and then, by letting $N \to \infty$,

$$\sum_{j=1}^{\infty} j|b_j|^2 \leq 1.$$

Theorem 13.2. (Bieberbach). *If*

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathfrak{S},$$

then $|a_2| \leq 2$.

Proof. First observe that we may choose a square root so that $g(z)\sqrt{f(z^2)} \in \mathfrak{S}$. Trivially, $g(0) = 0$. Since f is univalent, $f(z) = 0$ at the origin only. Therefore, we may fix the branch of the square root so that

$$g(z) = \sqrt{f(z^2)} = z(1 + a_2 z^2 + a_3 z^4 + \cdots)^{1/2} = z + c_3 z^3 + c_5 z^5 + \cdots$$

in $|z| < 1$. Now, g is clearly analytic in D and $g'(0) = 1$. It remains to show that g is univalent in the unit disc. But g is odd, i.e. $g(-z) = -g(z)$. If now $g(z_1) = g(z_2)$, then $f(z_1^2) = f(z_2)^2$ and further $z_1^2 = z_2^2$, hence $z_1 = \pm z_2$. If we have $z_1 = -z_2$, then $g(z_1) = g(z_2) = -g(z_1)$, resulting in $z_1 = 0$ and so $z_1 = z_2$ in this case as well.

Therefore, $h(z) = (g(1/z))^{-1} = (f(1/z^2))^{-1/2} \in \sum$. Starting from the Taylor expansion of f, a routine computation shows that

$$g(z) = z - \frac{a_2}{2} z^{-1} + \cdots .$$

By the area theorem, Theorem 13.1, we conclude that $|a_2| \leq 2$.

Remark. The famous Bieberbach conjecture from 1916 asserts that $|a_n| \leq n$ for all $n \in \mathbb{N}$, provided $f \in \mathfrak{S}$. This was finally proved by de Branges in 1984.

Theorem 13.3. (Koebe). *For each function* $f \in \mathfrak{S}$*, we have* $f(D) \supset \{ w\,;\, |w| < 1/4 \}$.

Proof. Suppose that f omits a value $\omega \in \mathbb{C}$. Then it is immediate to verify that

$$g(z) := \frac{\omega f(z)}{\omega - f(z)} = z + (a_2 + \frac{1}{\omega} z^2 + \cdots$$

is analytic, univalent and $g \in \mathfrak{S}$. By the Bieberbach theorem,

$$|a_2 + \frac{1}{\omega}| \leq 2.$$

If $|1/\omega| > 1/4$, then $1/|\omega| - |a_2| \leq |a_2 + 1/\omega| \leq 2$. Therefore, $1/|\omega| \leq 2 + |a_2| \leq 4$, a contradiction. Hence, $|\omega| \geq 1/4$, and we are done.

Proposition 13.4. *For each $f \in \mathfrak{S}$,*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \le \frac{4r}{1-r^2}$$

for all $|z| = r < 1$.

Proof. Fix $\zeta \in D$ arbitrarily, and consider

$$F(z) := \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + \cdots .$$

It is elementary to see that $F \in \mathfrak{S}$ observing that $\frac{z+\zeta}{1+\bar{\zeta}z}$ maps D onto D. Moreover, a routine computation shows that

$$A_2(\zeta) = \frac{1}{2}\left((1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right).$$

By the Bieberbach theorem, $|A_2(\zeta)| \le 2$. Hence

$$\left|(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right| \le 4.$$

Multiplying by $|\zeta| =: r$ and dividing by $1-r^2$ we obtain

$$\left|\frac{\zeta f''(\zeta)}{f'(\zeta)} - 2\frac{\zeta\bar{\zeta}}{1-r^2}\right| \le \frac{4r}{1-r^2},$$

from which the assertion follows.

Theorem 13.5. (Distortion theorem). *For each $f \in \mathfrak{S}$, we have*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}$$

for all $|z| = r < 1$.

Proof. Since $f'(z) \ne 0$ (by univalence, see remark below) and $f'(0) = 1$, we may determine a unique branch of $\log f'(z)$ which vanishes at the origin. Then we may conclude that

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) = r\frac{\partial}{\partial r}\operatorname{Re}\log f'(z),$$

where $z = re^{i\theta}$. In fact, we first have

$$r\frac{\partial}{\partial r}\log f'(z) = r\frac{\partial}{\partial r}(\log|f'(z)| + i\arg f'(z)).$$

For the left hand side we see that

$$r\frac{\partial}{\partial r}\log f'(z) = r\frac{d(\log f'(z)))}{dz}\frac{\partial z}{\partial r} = re^{i\theta}\frac{f''(z)}{f'(z)} = z\frac{f''(z)}{f'(z)}.$$

Taking the real parts from this and the right hand side of the previous identity we obtain the conclusion.

Recalling the elementary observation that $-c \le \operatorname{Re}\alpha \le c$ when $|\alpha| \le c$, we may use the inequality from Proposition 13.4 to obtain

$$\frac{2r^2-4r}{1-r^2} \le \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \le \frac{2r^2+4r}{1-r^2}.$$

Therefore,

$$\frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log|f'(re^{i\theta})| \le \frac{2r+4}{1-r^2}.$$

Holding now θ fixed, and integrating relative to r from 0 to R, a routine computation results in

$$\log\frac{1-R}{(1+R)^3} \le \log|f(Re^{i\theta}| \le \log\frac{1+R}{(1-R)^3}.$$

The assertion now follows by exponentiation.

Remark. Suppose f analytic and univalent in D, and also that $f'(z_0) = 0$ at some point $z_0 \in D$. For the Taylor expansion of f about z_0 we have

$$f(z) - f(z_0) = \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + \cdots.$$

By the uniqueness theorem of analytic functions, we find a circle $|z-z_0| = r$ such that $f(z)-f(z_0)$ and $f'(z)$ have no zeros in an open disc $0 < |z-z_0| \le r$ contained in D. Define now

$$m := \min_{|z-z_0|=r}|f(z)-f(z_0)| > 0$$

and consider

$$g(z) := f(z) - f(z_0) - a,$$

where we have chosen a to satisfy $0 < |a| < m$. We proceed to show that g has at least two distinct zeros in $|z-z_0| \le r$, hence in D. But then there are at least two distinct points z_j, $j = 1, 2$, so that $f(z_j) = f(z_0) + a$, contradicting the assumption that f is univalent. To this end, observe that $g(z_0) = -a \ne 0$, and that $g'(z) = f'(z) \ne 0$ in $0 < |z-z_0| \le r$, meaning that all zeros of g in $|z-z_0| \le r$ are simple. Now, on the circle $|z-z_0| = r$, we have

$$|f(z) - f(z_0)| \ge m > |a|.$$

Therefore, by the Rouché theorem, g has the same number of zeros in the disc $|z-z_0| \le r$ as does $f(z) - f(z_0)$, that is $k \ge 2$ zeros.

Theorem 13.6. *For each $f \in \mathfrak{S}$, we have*

$$\frac{r}{(1+r)^2}|f(z)| \leq \frac{r}{(1-r)^2}$$

for all $|z| = r < 1$.

Proof. Since $f(0) = 0$, we have by the distortion theorem that

$$|f(z)| = \left| \int_0^r f'(\zeta) d\zeta \right| \leq \int_0^r |f'(\zeta)| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} = \frac{r}{(1-r)^2}.$$

The left inequality is less immediate. If $|f(z)| \geq 1/4$, then it follows from the fact that $F(r) := r/(1+r)^2 < 1/4$ for $0 \leq r \leq 1$. In fact, $F(0) = 0$, $F(1) = 1/4$ and $F'(r) \geq 0$. Hence, we may assume that $|f(z)| < 1/4$. But then, by the Koebe theorem, the line segment from 0 to $f(z)$ lies completely in the image of $f(D)$. Let Γ be the preimage of this line segment. Then Γ is a simple arc from 0 to z, and we obtain, by the construction of Γ, and the distortion theorem,

$$|f(z)| = \left| \int_\Gamma f'(\zeta) d\zeta \right| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3} d\rho = \frac{r}{(1+r)^2}.$$

Remark. By Theorem 13.6, the family $\mathfrak{S}$ is locally bounded. Therefore, by Theorem 12.8, $\mathfrak{S}$ is a normal family. Moreover, since $\mathfrak{S}$ is locally bounded, the same is true for $\mathfrak{S}'$ as well by Theorem 12.11. The, by the Montel theorem, $\mathfrak{S}'$ is a normal family too.

Another distortion type theorem is

Theorem 13.7. *For each $f \in \mathfrak{S}$, we have*

$$\frac{1-r}{1+r} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1+r}{1-r},$$

whenever $|z| = r < 1$.

Proof. Consider the function $F \in \mathfrak{S}$ defined in the proof of Theorem 13.4. By Theorem 13.6, we get for $\zeta \in D$,

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F(-\zeta) \leq \frac{|\zeta|}{(1-|\zeta|)^2}.$$

But $F(-\zeta) = \frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}$, and the assertion follows by combining this with the preceding double inequality.

By the growth theorem, integrating over a circle boundary of radius r, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}| d\theta \leq \frac{r}{(1-r)^2}.$$

However, we can obtain a better result as follows:

Theorem 13.8.. *For each $f \in \mathfrak{S}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \frac{r}{1-r}$$

for all $r < 1$.

Proof. We again consider $g(z) := \sqrt{f(z^2)} \in \mathfrak{S}$, and its Taylor expansion

$$g(z) = z + \sum_{j=2}^{\infty} c_j z^j.$$

By the growth theorem (Theorem 13.6),

$$|g(z)|^2 = |f(z^2)| \leq \frac{r^2}{(1-r^2)^2},$$

$$|h(z)| \leq \frac{r}{1-r^2}.$$

This means that g maps the disc $|z \leq r|$ univalently onto a domain D_r contained in the disc $|w| \leq r/(1-r^2)$. Therefore, the area of D_r satisfies

$$A(D_r) \leq \pi \frac{r^2}{(1-r^2)^2}.$$

On the other hand, we may compute $A(D_r)$ in the same way as in the proof of the area theorem to obtain

$$A(D_r) = \frac{1}{2i}\int_0^{2\pi} \overline{g}\frac{\partial g}{\partial \theta} d\theta = \pi \sum_{j=1}^{\infty} j|c_j|^2 r^{2j}.$$

Therefore,

$$\sum_{j=1}^{\infty} j|c_j|^2 r^{2j-1} \leq \frac{r}{(1-r^2)^2}.$$

Integrating from 0 to r we get

$$\sum_{j=1}^{\infty} |c_j|^2 r^{2j} = \frac{1}{2\pi}\int_{0^{2\pi}} |h(re^{i\theta})|^2 d\theta \leq \frac{r^2}{1-r^2}.$$

Finally, observing that

$$\int_0^{2\pi} |h(re^{i\theta})|^2 d\theta = \int_0^{2\pi} |f(r^2 e^{2i\theta})| d\theta = \frac{1}{2}\int_0^{4\pi} |f(r^2 e^{i\alpha})| d\alpha = \int_0^{2\pi} |f(r^2 e^{i\alpha})| d\alpha.$$

So, we have found

$$\frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{i\theta})| d\theta \leq \frac{r^2}{1-r^2},$$

and the assertion follows by observing that r^2 is an increasing bijection from $[0,1]$ to $[0,1]$.

We are now ready to prove

Theorem 13.9. *If $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ is in $\mathfrak{S}$, then $|a_j| \leq ej$ for all $j \geq 2$.*

Proof. By the Cauchy integral formula, we first obtain

$$a_j = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}} dz$$

for all $j \geq 2$ and all $0 < r < 1$. Therefore,

$$|a_j| \leq \frac{1}{2\pi r^j} \int_0^{2\pi} |f(re^{i\theta})| d\theta.$$

Combining with the preceding theorem, we see that

$$|a_j| \leq \frac{1}{r^{j-1}(1-r)}.$$

Since this inequality is valid for all $0 < r < 1$, and the left hand side is independent of r, we may proceed to minimize the right hand side in $0 < r < 1$, meaning that we have to maximize $r^{n-1}(1-r)$. but this happens at $r = \frac{n-1}{n}$, resulting in

$$|a_j| \leq \frac{j^j}{(j-1)^{j-1}} = j\left(1 + \frac{1}{j-1}\right)^{j-1} < ej.$$