COMPLEX ANALYSIS III

1. Möbius transformations

1.1. **Riemann sphere.** First recall the **Riemann sphere** from previous complex analysis courses. This is a sphere $\sum$ in $\mathbb{R}^3$ defined as

$$\xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}, \tag{1.1}$$

i.e. the sphere of radius $\frac{1}{2}$, centered at $(0, 0, \frac{1}{2})$, and tangent to (x, y)-plane $(= \mathbb{C})$ at the origin. Denoting $N = (0, 0, 1)$, there is an obvious one-to-one correspondence between the points on $\sum \setminus \{N\}$ and the complex numbers $\mathbb{C}$. Indeed, we may define the image of $z \in \mathbb{C}$ to be the point $\neq N$ where the line from N to z intersects $\sum$. Setting then N as the image of ∞, we obtain the required bijective correspondence.

To express the correspondence defined above in an analytic form, let $z = (x, y, 0) \in \mathbb{C}$ be given. To determine its image $\sum(z)$ on the sphere $\sum$, observe that the line from N to z is given by

$$\overline{r} = \overline{k} + t(x\overline{i} + y\overline{j} - \overline{k}) = xt\overline{i} + yt\overline{j} + (1 - t)\overline{k},$$

where t is a real parameter. The image point $\sum(z)$ is on the sphere, hence

$$(x^2 + y^2)t^2 + (\frac{1}{2} - t)^2 = \frac{1}{4},$$

implying that

$$t = \frac{1}{x^2 + y^2 + 1} = \frac{1}{1 + |z|^2}.$$

Therefore, we may express the image point in the form

$$\sum(z) = \left(\frac{x}{1 + |z|^2}, \frac{y}{1 + |z|^2}, \frac{|z|^2}{1 + |z|^2} \right).$$

To obtain corresponding expressions for the inverse mapping $\sum^{-1}$, it appears useful to introduce "geographical coordinates" to determine points on $\sum$, i.e. the longitude λ and the latitude φ. To fix these, let us agree that $\lambda = 0$ in the direction of the positive x-axis, increasing counterclockwise to 2π, while $\varphi = 0$ on the equator, ranging from $-\pi/2$ at the origin S to $\pi/2$ at N. Given now $(\lambda, \varphi) \in \sum$, let $z = re^{i\theta}$ be its image in $\mathbb{C}$ under the inverse mapping $\sum^{-1}$. Looking at the triangle NSz, we immediately observe that

$$\theta = \lambda, \qquad r = \tan \psi,$$

where ψ is the angle of this triangle at N. By elementary geometry, $2\psi = \varphi + \pi/2$, hence $\psi = \frac{\varphi}{2} + \frac{\pi}{4}$. Therefore, we obtain

$$\theta = \lambda \tag{1.2}$$

$$r = \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right). \tag{1.3}$$

If we denote by ρ, resp. ρ', the distance between N and a point z in $\mathbb{C}$, resp. N and $\sum(z)$, then $\rho\rho' = 1$ by an elementary geometric observation.

Exercise. Let $P := (\xi, \eta, \zeta) \in \sum$ be given. Determine $z = \sum^{-1}(P)$ in the form $z = \Re z + i\Im z$.

Before proceeding, we show that the inverse mapping $\sum^{-1}$, also called **stereographic projection**, maps all circles on the Riemann sphere $\sum$ into circles or straight lines on $\mathbb{C}$. In fact, any circle on $\sum$ is the intersection of $\sum$ with a plane in $\mathbb{R}^3$,

$$A\xi + B\eta + C\zeta + D = 0.$$

By (1.1), the image of the circle of the intersection under the stereographic projection is

$$A\frac{x}{1+|z|^2} + B\frac{y}{1+|z|^2} + C\frac{|z|^2}{1+|z|^2} + D = 0,$$

hence

$$(C+D)(x^2+y^2) + Ax + By + D = 0.$$

Obviously, this is a circle, if $C+D \neq 0$ and a straight line, if $C+D=0$. The second alternative appears at $(\xi,\eta,\zeta) = (0,0,1) = N$. Therefore, the image of a circle on $\sum$ under the stereographic projection is a straight line, if and only if the circle is passing through the north pole N. Observe that a straight line may be understood as a circle of radius ∞.

1.2. **Möbius transformations.** By a **Möbius transformation** it will be understood a mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$g(z) := \frac{az+b}{cz+d}, \tag{1.4}$$

where $ad-bc \neq 0$. In what follows, we use the notation $\mathfrak{M}$ for the family of Möbius transformation satisfying the non-singularity condition $ad-bc \neq 0$. Indeed, this condition is needed to ensure that the mapping g is a bijective, continuous map from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. An arbitrary Möbius transformation can be represented as a composition of certain basic transformations. To this end, we observe that $g(z) = z+\alpha$, where $\alpha \in \mathbb{C}$, is a **translation** of the complex plane and, correspondingly, $g(z) = e^{i\theta}z$, where $\theta \in \mathbb{R}$, is a **rotation**. Moreover, by a **dilation** we understand a Möbius transformation of type $g(z) = kz$, where $k > 0$, while $g(z) = \frac{1}{z}$ is an **inversion**. We now obtain

Theorem 1.1. *Each Möbius transformation in $\mathfrak{M}$ may be represented as a composition of certain translations, rotations, dilations and inversions.*

Proof. Suppose first that $c \neq 0$. Using notations $c = |c|e^{i\varphi}$ and $bc-ad = ke^{i\theta}$, we observe that

$$g(z) = \frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+d/c} + \frac{a}{c} = \frac{ke^{i\theta}}{c^2e^{2i\varphi}(z+d/c)} + \frac{a}{c}.$$

Starting from z, we now need to compose a translation, a rotation, a dilation, an inversion, another rotation, another dilation and a final translation to obtain $g(z)$.

If then $c=0$, we may write

$$g(z) = \frac{az+b}{d} = \frac{a}{d}\left(z+\frac{b}{a}\right)$$

to represent g as composed from a translation, a rotation and a dilation. □

Recalling our previous proviso to understand straight lines as circles of radius ∞, we may easily prove

Theorem 1.2. *A Möbius transformation in $\mathfrak{M}$ maps all circles onto circles.*

Proof. By the preceding Theorem 1.1 it is sufficient to prove the claim for arbitrary translations, rotations, dilations and inversions. To this end, recall that a circle (of finite radius r) has a representation as

$$|z-z_0| = r,$$

while a straight line may be represented as

$$\Re(\alpha z) = c, \qquad |\alpha| = 1, \qquad c \in \mathbb{R}.$$

Concerning translations and rotations, the claim is trivial.

Proceeding to a dilation $g(z) = w = kz$, $k > 0$, we observe that $z = w/k$. Therefore, the image of a circle is

$$|w - kz_0| = kr,$$

which is a circle (of radius kr, centered at kz_0), while for a straight line we obtain

$$\Re(\alpha w) = kc,$$

a straight line again.

It remains to see that the claim is valid for the inversion $g(z) = w = \frac{1}{z}$. Now, for a circle $|z - z_0| = r$, we observe that

$$\begin{aligned} 0 = |z - z_0|^2 - r^2 &= (z - z_0)(\overline{z} - \overline{z_0}) - r^2 && (1.5)\\ &= |z|^2 + |z_0|^2 - 2\Re(\overline{z} z_0) - r^2 && (1.6)\\ &= \frac{1}{|w|^2} + (|z_0|^2 - r^2) - 2\Re\left(\frac{1}{\overline{w}}\frac{w}{w} z_0\right) && (1.7)\\ &= \frac{1}{|w|^2} + (|z_0|^2 - r^2) - \frac{2}{|w|^2}\Re(z_0 w). && (1.8) \end{aligned}$$

If now $|z_0| = r$, we immediately see that

$$1 - 2\Re(z_0 w) = 0,$$

and so

$$\Re(z_0 w) = \frac{1}{2},$$

which in the form

$$\Re\left(\frac{z_0}{|z_0|} w\right) = \frac{1}{2|z_0|}$$

represents a straight line.

On the other hand, if $z_0 \neq r$, multiplication by $|w|^2/(|z_0|^2 - r^2)$ results in

$$\begin{aligned} 0 &= \frac{1}{|z_0|^2 - r^2} + |w|^2 - \frac{2}{|z_0|^2 - r^2}\Re(z_0 w) && (1.9)\\ &= \left| w - \frac{z_0}{|z_0|^2 - r^2} \right|^2 - \frac{r^2}{(|z_0|^2 - r^2)^2}, && (1.10) \end{aligned}$$

which clearly represents a circle.

Finally, we have to show that a straight line $\Re(\alpha z) = c$ maps onto a circle (or onto a straight line) under the inversion $g(z) = w = \frac{1}{z}$. From $\Re(\alpha/w) = c$ we obtain (multiplying by $\overline{w}$)

$$\Re(\alpha \overline{w}) = c|w|^2.$$

If now $c = 0$, we conclude that

$$0 = \Re(\alpha \overline{w}) = \Re(\overline{\alpha \overline{w}}) = \Re(\overline{\alpha}) w,$$

hence the image is a straight line. On the other hand, if $c \neq 0$, then

$$|w^2| = \Re(\overline{\alpha} w / c),$$

and this implies that

$$\left|w - \frac{\alpha}{2c}\right|^2 = |w|^2 + \frac{|\alpha|^2}{4c^2} - 2\Re\left(\frac{\overline{\alpha}}{2c}w\right) \tag{1.11}$$

$$= \Re\left(\frac{\overline{\alpha}}{c}w\right) + \frac{|\alpha|^2}{4c^2} - \Re\left(\frac{\overline{\alpha}}{c}w\right) = \frac{1}{4c^2}, \tag{1.12}$$

which means that the image is the circle $|w - \frac{\alpha}{2c}| = \frac{1}{2c}$. □

Remark. Since three distinct points determine a circle uniquely (and this also holds for a straight line in which case the third point is at infinity), we may determine uniquely a Möbius transformation in $\mathfrak{M}$ mapping a given disk onto another given disk. To make use of this fact, we need to give the following

Definition 1.3. Given four distinct extended complex numbers z_1, z_2, z_3, z_4, their **double ratio** will be defined as

$$(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} : \frac{z_2 - z_3}{z_2 - z_4},$$

provided the numbers are finite. If one of the four numbers is infinite, we define

$$(\infty, z_2, z_3, z_4) := \frac{z_2 - z_4}{z_2 - z_3}, \qquad (z_1, \infty, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4},$$

$$(z_1, z_2, \infty, z_4) := \frac{z_2 - z_4}{z_1 - z_4}, \qquad (z_1, z_2, z_3, \infty) := \frac{z_1 - z_3}{z_2 - z_3}.$$

Theorem 1.4. *The double ratio remains invariant under Möbius transformations in $\mathfrak{M}$, i.e. given a Möbius transformation $g(z) = (az+b)/(cz+d)$, $ad - bc \neq 0$, then*

$$(g(z_1), g(z_2), g(z_3), g(z_4)) = (z_1, z_2, z_3, z_4).$$

Proof. A simple proof by "brute force" is to check the claim by mathematical software (Mathematica, Maple). □

Example. Invariance of the double ratio offers a routine tool to determine the Möbius transformation mapping three given extended complex points to another given triple of such points. As an example, we determine a Möbius transformation $g(z)$ such that $g(1) = 2, g(i) = 3, g(-1) = 4$. Then by the invariance we must have

$$(g(z), g(1), g(i), g(-1)) = (w, 2, 3, 4) = (z, 1, i, -1),$$

giving the equality

$$\frac{w-3}{w-4} : \frac{2-3}{2-4} = \frac{z-i}{z+1} : \frac{1-i}{1+1}.$$

After simplification, we obtain

$$g(z) = w = \frac{(2-4i)z + (2+4i)}{(1-i)z + (1+i)}.$$

Theorem 1.5. *The family $\mathfrak{M}$ of Möbius transformations forms a group under mapping composition $\circ$. The composition $\circ$ is not commutative in general.*

Proof. We first observe that the identity map $U(z) \equiv z$ is a unit element for this group. Indeed, given an arbitrary Möbius transformation $g(z) = (az+b)/(cz+d)$, it is immediate to see that $U \circ g = g$ and $g \circ U = g$. Moreover, the Möbius transformation $h(z) := (dz-b)/(-cz+a)$ is the inverse mapping of $g(z)$, since $h \circ g = g \circ h = U$, as one may easily check by mathematical software. Finally, composing two Möbius transformations g_1, g_2, it is again a routine computation (by software) to see that $g_1 \circ g_2$ is a Möbius transformation as well.

Finally, we may see by a simple example that the group of Möbius transformations is not Abelian, i.e. that the composition $\circ$ is not commutative. To this end, consider

$$g_1(z) := \frac{z}{z+1}, \qquad g_2(z) := \frac{z+1}{z-1}.$$

Then it is immediate to see that

$$g_1 \circ g_2(z) = \frac{z+1}{2z}, \qquad g_2 \circ g_1(z) = -2z - 1.$$

□

1.3. **Classification of Möbius transformations.** Given a Möbius transformation

$$g(z) = (az+b)/(cz+d) \in \mathfrak{M},$$

we first determine its fix-points. Of course, fix-points ζ are determined from the equality $\zeta = (a\zeta+b)/(c\zeta+d)$, hence from

$$c\zeta^2 + (d-a)\zeta - b = 0. \tag{1.13}$$

(1) Consider first the case of $c = 0$, assuming that $a \neq d$. Then we obtain $\zeta_1 = b/(d-a)$ from (1.13). But from the original equality $\zeta = (a/d)\zeta + b/d$ we infer that $\zeta_2 = \infty$ is another fix-point in this situation. Observe that we now have

$$g(z) = \frac{a}{d}z + \frac{b}{d}, \qquad \zeta_1 = \frac{a}{d}\zeta_1 + \frac{b}{d}.$$

Therefore,

$$g(z) - \zeta_1 = \frac{a}{d}(z - \zeta_1).$$

Exercise. Describe the geometric meaning of the preceding representation of $g(z)$.

Secondly, if we have $c = 0$ and $a = d$, then the transformation reduces into $g(z) = z + \omega$, where $\omega = b/d$. We now have two possible situations. If $\omega = 0$, then g reduces to an identity mapping, hence all points in the extended complex plane are fix-points. If then $\omega \neq 0$, then g is a translation of the plane, and infinity is the only fix-point.

(2) Suppose next that $c \neq 0$ and that the discriminant $(a-d)^2 + 4bc \neq 0$. Then, obviously, (1.13) has two distinct roots ζ_1, ζ_2 in the complex plane, and so we have two distinct fix-points of g. To see what this means, we shall apply the invariance of the double ratio. Then

$$(g(z), g(z_2), \zeta_1, \zeta_2) = (z, z_2, \zeta_1, \zeta_2),$$

where z_2 is an arbitrary complex number distinct from the fix-points ζ_1, ζ_2. By the double ratio invariance we get

$$\frac{g(z)-\zeta_1}{g(z)-\zeta_2} : \frac{g(z_2)-\zeta_1}{g(z_2)-\zeta_2} = \frac{z-\zeta_1}{z-\zeta_2} : \frac{z_2-\zeta_1}{z_2-\zeta_2}.$$

This may now be written as

$$\frac{g(z)-\zeta_1}{g(z)-\zeta_2} = \lambda \frac{z-\zeta_1}{z-\zeta_2},$$

where λ is a complex number. Observe that λ is independent of the choice of z_2 which may easily be seen by the invariance of the double ratio.

Exercise. Check the independence of λ of the choice of z_2.

(3) It remains to consider the case $c \neq 0$ while $(a-d)^2 + 4bc = 0$. In this case, the fix-point equation (1.13) has exactly one double-root at $\zeta = \frac{a-d}{2c} \in \mathbb{C}$, being therefore the only fix-point of g. To see the behavior of g in this situation, define a Möbius transformation T by

$$T(z) := \zeta + \frac{1}{z},$$

i.e. inversion combined with a translation by ζ. Consider then the transformation $T^{-1} \circ g \circ T$, which is a Möbius transformation by the group property. Next determine the fix-points of this composed transformation. From $T^{-1} \circ g \circ T(\xi) = \xi$ we immediately obtain $g(T(\xi)) = T(\xi)$, and so $T(\xi)$ is a fix-point of g, hence $T(\xi) = \zeta + 1/\xi = \zeta$. Therefore, $\xi = \infty$. By the considerations in part (1) above, we conclude that $T^{-1} \circ g \circ T$ must be a translation, i.e. $T^{-1} \circ g \circ T(z) = z + \omega$ for some complex number $\omega \neq 0$. But then

$$g(T(z)) = T(z+\omega) = \zeta + \frac{1}{z+\omega},$$

hence

$$g(\zeta + \frac{1}{z}) = \zeta + \frac{1}{z+\omega}.$$

Substituting now $z = 1/(y-\zeta)$ we get

$$g(y) = \zeta + \frac{1}{\omega + \frac{1}{y-\zeta}}.$$

Writing again z in place of y, and rearranging we finally obtain

$$\frac{1}{g(z)-\zeta} = \omega + \frac{1}{z-\zeta}.$$

This now easily shows how the transformation g is formed as a composition of translations and inversions.
Any Möbius transformation induces a mapping on the Riemann sphere $\sum$. Under certain conditions, this induced mapping is geometrically a rotation of $\sum$, which means that the diameters of $\sum$ are mapped on diameters of $\sum$. More precisely, it must be that the end-points of a diameter are mapped onto end-points of another diameter (as the mapping is bijective).

Suppose we have this situation, and denote $z^* := g(z)$. Clearly, all five points $N, z, z^*, \sum(z), \sum(z^*)$ are in the same plane in $\mathbb{R}^3$. Looking at this configuration (draw a picture!), we clearly have

$$\arg z^* = \arg z + \pi.$$

Moreover, by elementary geometry,

$$|zz^*| = 1.$$

On the other hand, denoting $z = re^{i\theta}$, we see that

$$z^* = \frac{1}{r}e^{i\theta + i\pi} = -\frac{1}{r}e^{i\theta} = -\frac{1}{re^{-i\theta}} = -\frac{1}{\overline{z}}.$$

Writing now $g(z) = w = (az+b)/(cz+d)$, we then have $g(z^*) = w^* = (az^*+b)/(cz^*+d)$. As the induced mapping on $\sum$ is required to be a rotation, the by the same argument as above, we now get

$$w^* = -\frac{1}{\overline{w}} = \frac{b - a/\overline{z}}{d - c/\overline{z}} = \frac{-a + b\overline{z}}{-c + d\overline{z}}.$$

Therefore,

$$\overline{w} = -\frac{-c + d\overline{z}}{-a + b\overline{z}},$$

and so

$$w = -\frac{-\overline{c} + \overline{d}z}{-\overline{a} + \overline{b}z} = \frac{\overline{d}z - \overline{c}}{-\overline{b}z + \overline{a}}.$$

As this must be the same mapping as our original $g(z)$, the coefficients in these two expressions must be proportional. Therefore, we must have

$$\frac{a}{\overline{d}} = -\frac{b}{\overline{c}} = -\frac{c}{\overline{b}} = \frac{d}{\overline{a}},$$

and so we get

$$|a| = |d|, \qquad |b| = |c|.$$

Hence, for some ε such that $|\varepsilon| = 1$, we see that

$$d = \varepsilon\overline{a}, \qquad c = -\varepsilon\overline{b}.$$

This means that we may write

$$g(z) = \frac{\overline{\varepsilon}az + \overline{\varepsilon}b}{-\overline{b}z + \overline{a}}.$$

But now, it is easy to find a complex number μ such that $\overline{\varepsilon} = \mu/\overline{\mu}$. (**Exercise!**) Making use of this, we immediately see that $g(z)$ may be written in the form

$$g(z) = \frac{\mu az + \mu b}{-\overline{\mu b} + \overline{\mu a}} = \frac{\alpha z + \beta}{-\overline{\beta}z + \overline{\alpha}},$$

where, of course, $\alpha = \mu a$ and $\beta = \mu b$.

Exercise. The representation just obtained is a necessary condition for the situation that a Möbius transformation induces a rotation of the Riemann sphere $\sum$. Show now that this condition is sufficient as well.

To proceed to the anticipated classification of Möbius transformations in $\mathfrak{M}$, we first observe that there is a natural bijective correspondence between the transformations in $\mathfrak{M}$ and non-singular 2×2-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}), \qquad ad - bc \neq 0.$$

A routine computation shows that the composition of Möbius-transformations corresponds to the standard matrix product, and so the group $(\mathfrak{M}, \circ)$ of Möbius transformations is bijectively equivalently with the group of non-singular 2×2-matrices under the matrix product. Although the correspondence described above is not actually used here, it is important to know this fact, which is frequently applied in more advanced developments.

Considering now a Möbius transformation $g(z) = \frac{az+b}{cz+d}$ in $\mathfrak{M}$, we define the **trace** of this transformation as

$$\mathrm{tr}^2 g := \frac{(a+d)^2}{ad - bc}.$$

In what follows, we may assume that $ad - bc = 1$ by multiplying the nominator and denominator of g by a suitable constant. Of course, under this normalization, we have $\mathrm{tr}^2 g = (a + d)^2$.

The notion of trace is one of the two key notions in the classification, the second one being the set F_g of fix-points of g. As we already know from our previous considerations, F_g has three possibilities, namely it is either $\widehat{\mathbb{C}}$, or a singleton (= a set consisting of one point only), or a set of two distinct points in $\widehat{\mathbb{C}}$.

To separate between the three possibilities of F_g, we first recall from our preceding considerations that $g(z) \equiv z$, if we have $F_g = \widehat{\mathbb{C}}$. Therefore, we must have $a - d = c = b = 0$. If then F_g is a singleton, then the determinant $(a-d)^2+4bc = 0$. Therefore, under the normalization $ad-bc = 1$,

$$(a-d)_2 + 4bc = a^2 2ad + d^2 - 4(ad - bc) = (a+d)^2 - 4 = 0,$$

and so we have $\mathrm{tr}^2 g = (a+d)^2 = 4$ in this case. In the remaining case of two distinct fix-points, we must have $\mathrm{tr}^2 g \neq 4$. However, this remaining case contains transformations of quite different behavior. As an example, if $g(z) = kz$ with $|k| \neq 1$, then we have $g \circ g \neq U$, while if $h(z) = -z$, then $h \circ h = U$. Observe that $g(z)$ is not normalized in the sense that $\det g = k + 1 \neq 1$.

A key result for classification is the following

Theorem 1.6. *Given two Möbius transformations g, h in $\mathfrak{M}$, we have*

$$tr^2 g = tr^2(h \circ g \circ h^{-1}).$$

Proof. This is again a routine computation, best being checked by mathematical software. $\square$

Remark. In the situation of this theorem, we say that g and h are **conjugate** to each other. We denote $g \simeq h$. Clearly, the relation of conjugation is an equivalence relation.

By Theorem 1.6, it appears useful to fix certain normal types of Möbius transformations. These will be specified as follows, for each $k \in \mathbb{C} \setminus \{0\}$:

$$m_k(z) = kz, \qquad k \neq 1; \qquad m_1(z) = z + 1.$$

Theorem 1.7. *For all of the normal types of transformations specified above, we have*

$$tr^2 m_k = k + \frac{1}{k} + 2.$$

Proof. First observe for $k \neq 1$ that $m_k(z) = kz \simeq \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. Therefore,

$$\mathrm{tr}^2 m_k = \frac{(k+1)^2}{k} = k + \frac{1}{k} + 2.$$

Secondly, for $k = 1$ we have $m_1(z) = z + 1 \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and so

$$\mathrm{tr}^2 m_1 = \frac{(1+1)^2}{1} = 4 = 1 + \frac{1}{1} + 2.$$

$\square$

To complete the classification task, let now $g \neq U$ be a Möbius transformation, having therefore either two fix-points α, β, or just one fix-point α, in which case we fix another point $\beta \neq \alpha$. In this latter case, of course, β is not a fix-point.

Theorem 1.8. *Each Möbius transformation $g \neq U$ is conjugate to exactly one Möbius transformation m_k of normal type.*

Proof. To begin with, we use the notations given above for the fix-points of g, and we consider first Möbius transformations h which satisfy the following conditions:

(a) $h(\alpha) = \infty$,

(b) $h(\beta) = h(g(\beta)) = 0$, if β is a fix-point, and $h(g(\beta)) = 1$, if β is not a fix-point.

First suppose that g has two fix-points α, β. We then show that one of the transformations h defined above satisfy $h \circ g \circ h^{-1} = m_k$ for some $k \neq 1$. The transformation m_k is independent of the specific choice of h. To this end, we first observe that, independently of the choice of h, the composed mapping $h\circ g\circ h^{-1}$ has two fix-points, namely 0 and ∞. Defining then $k := h(g(h^{-1}(1)))$ we see that $k \neq 1$, as 1 is not a fix-point of $h \circ g \circ h^{-1}$. We now observe that

$$0 = h(g(h^{-1}(1))) = m_k(0),$$

$$\infty = h(g(h^{-1}(0))) = m_k(\infty)$$

and

$$k = h(g(h^{-1}(1))) = m_k(1).$$

Since three points are sufficient to determine a Möbius transformation uniquely, this implies that $h \circ g \circ h^{-1} = m_k$. It remains to show that m_k is independent of the choice of h. To see this, let h_0 be another transformation satisfying the required conditions, and let m_{k_0} be the transformation of normal type determined by h_0. Then we have $h_0 \circ g \circ h_0^{-1} = m_{k_0}$, and therefore $g = h_0^{-1} \circ m_{k_0} \circ h_0$. But then

$$m_k = (h \circ h_0^{-1}) \circ m_{k_0} \circ (h \circ h_0^{-1})^{-1},$$

and so the transformations m_k, m_{k_0} are conjugate. Now, $h \circ h_0^{-1}$ has fix-points 0 and ∞. Defining $k' = h(h_0^{-1}(1))$, we also have $m_{k'}(1) = k'$, and therefore $h \circ h_0^{-1} = m_{k'}$. Hence

$$m_{k'} \circ m_{k_0} \circ m_{k'}^{-1} = m_k.$$

Therefore, since $m_k^{-1}(z) = \frac{1}{k}z$, we obtain, for all z, that

$$m(z) = k'k_0\frac{1}{k'}z = k_0 z,$$

and so we have $m_k = m_{k_0}$.

We still have to consider the case that g has exactly one fix-point α. Then h is being determined uniquely, and we have $h\circ g\circ h^{-1} = m_1$. It is now immediate to see that ∞ is a fix-point of $h\circ g\circ h^{-1}$. On the other hand, if γ is another fix-point, i.e. if $h(g(h^{-1}(\gamma))) = \gamma$, then $g(h^{-1}(\gamma)) = h^{-1}(\gamma)$. Therefore, as α is the only fix-point of g, we have $\gamma = h(\alpha) = \infty$ by the definition of h. Since ∞ is the only fix-point of $h \circ g \circ h^{-1}$, we have $h \circ g \circ h^{-1}(z) = z + b$ for some $b \neq 0$. But $b = h(g(h^{-1}(0))) = 1$, and we are done. □

We now proceed to show that conjugation of Möbius transformations actually depends on their traces only. We first look at transformations in the normal form:

Theorem 1.9. *Möbius transformations m_p, m_q of normal form are conjugate if and only if $tr^2 m_p = tr^2 m_q$ and this happens if and only if $p = q$ or $p = 1/q$.*

Proof. If $m_p \simeq m_q$, then we have $\mathrm{tr}^2 m_p = \mathrm{tr}^2 m_q$ by Theorem 1.6. On the other hand, if $\mathrm{tr}^2 m_p = \mathrm{tr}^2 m_q$, then by Theorem 1.7, we have $p + \frac{1}{p} = q + \frac{1}{q}$. But this implies immediately that either $p = q$ or $p = \frac{1}{q}$. If $p = q$, then we have $m_p = m_q$. Finally, if $p = \frac{1}{q} \neq q$, we define $h(z) = \frac{1}{z}$. Then $h^{-1}(z) = \frac{1}{z}$ as well and so

$$h \circ m_p \circ h^{-1}(z) = h(m_p(\frac{1}{z}) = h(\frac{p}{z}) = 1/(p/z) = \frac{1}{p}z = qz = m_q(z),$$

and so $m_p \simeq m_q$. □

Theorem 1.10. *Two Möbius transformations g, h ($\neq U$) are conjugate if and only if $tr^2 g = tr^2 h$.*

Proof. By Theorem 1.6 it suffices to show that the equality of traces implies conjugation. So, suppose that $\mathrm{tr}^2 g = \mathrm{tr}^2 h$. By Theorem 1.8, there are Möbius transformations m_p, m_q of normal form such that $m_p \simeq g$ and $m_q \simeq h$. Then, by Theorem 1.6,

$$\mathrm{tr}^2 m_p = \mathrm{tr}^2 g = \mathrm{tr}^2 h = \mathrm{tr}^2 m_q.$$

By Theorem 1.9, we infer that $m_p \simeq m_q$, and therefore, $g \simeq h$ by the equivalence property of conjugation. □

We are now prepared to offer the following

Definition 1.11. A Möbius transformation $g \neq U$ is called

(i) **parabolic**, if g admits exactly one fix-point in $\widehat{\mathbb{C}}$ (and then $g \simeq m_1$),

(ii) **loxodromic**, if g has two distinct fix-points in $\widehat{\mathbb{C}}$ and $g \simeq m_k$ with $|k| \neq 1$,

(iii) **elliptic**, if g has two distinct fix-points in $\widehat{\mathbb{C}}$ and $g \simeq m_k$ with $|k| = 1$.

Remark. By Theorem 1.9, the above definition is independent of the choice of m_k, i.e. m_k can be replaced by $m_{1/k}$.

We still have to separate two possibilities in the loxodromic case:

Definition 1.12. A loxodromic Möbius transformation $g \neq U$ is called **hyperbolic**, if there is a disc D (possibly of radius ∞) such that $g(D) = D$. Otherwise, g is said to be **properly loxodromic**.

We are now ready to prove our final result in this section:

Theorem 1.13. *A Möbius transformation $g \neq U$ is*

(i) parabolic if and only if $tr^2 g = 4$,

(ii) elliptic if and only if $0 \leq tr^2 g < 4$,

(iii) hyperbolic if and only if $tr^2 > 4$,

(iv) properly loxodromic if and only if $tr^2 g \neq |tr^2 g|$, meaning that $tr^2 g$ is either non-real or it is real and strictly negative.

Proof. We may assume that $g \simeq m_p \simeq m_{1/p}$, and so we have

$$\mathrm{tr}^2 g = p + \frac{1}{p} + 2.$$

(i) We first observe that g is parabolic if and only $g \simeq m_1$. This happens by the preceding trace formula if and only if $\mathrm{tr}^2 g = 4$.

(ii) In the elliptic case we have $|p| = 1$, hence $p = e^{i\theta}$ for some $0 < \theta \neq \pi$. Therefore,

$$\mathrm{tr}^2 g = e^{i\theta} + e^{-i\theta} + 2 = 2(1 + \cos\theta),$$

from which the claim follows. If then $0 \leq \mathrm{tr}^2 g < 4$, we may fix a unique $\omega \in (0, \pi]$ so that

$$\mathrm{tr}^2 g = 2 + 2\cos\omega.$$

Equation

$$p + \frac{1}{p} + 2 = 2 + 2\cos\omega$$

has two solutions, $p_1 = e^{i\omega}$ and $p_2 = e^{-i\omega}$, both of modulus one. Therefore, g is an elliptic transformation.

(iii) To treat the remaining cases, suppose first that $\mathrm{tr}^2 g > 4$. Now, equation

$$\mathrm{tr}^2 g = p + \frac{1}{p} + 2$$

has two solutions, say $p_1 = k$, $p_2 = \frac{1}{k}$, both of them > 0. But now m_k maps the half-plane $H := \{\Im z > 0\}$ onto itself, since

$$\Im(m_k(z)) = \Im(kz) = k\Im z > 0$$

for all $z \in H$, and reversed. Suppose now that $g = h^{-1} \circ m_k \circ h$ for a Möbius transformation h. Then it is immediate to see that g maps the set $D = h^{-1}(H)$ onto itself. But this set is a disc (possibly of radius ∞) by Theorem 1.2, as $h \in \mathfrak{M}$, and so $h^{-1} \in \mathfrak{M}$ as well.

To complete the proof, suppose now that g is hyperbolic, hence $m_p \simeq g$ as well. As m_p can be replaced by $m_{1/p}$, we may assume that $|p| > 1$. Let then D be a disc (or a half-plane) so that $m_p(D) = D$. If now $z \in D$, then, by assumption,

$$(m_p(z))^n = m_p^n(z) = p^n z \in D$$

for each $n \in \mathbb{Z}$. If $z \neq 0, \infty$, then $|p| > 1$ implies that $p^n z \to \infty$ and $p^{-n}(z) \to 0$ as $n \to \infty$. Therefore, $0, \infty \in \overline{D}$. This means that D must be a half-plane. If now $z \notin D$ is a $\overline{\text{finite,}}$ non-zero point, then again $p^n z \to \infty$ and $p^{-n} z \to 0$ as $n \to \infty$. This means that $0, \infty \in \overline{\widehat{\mathbb{C}} \setminus D}$, and therefore $0, \infty \in \overline{D}$. Hence, D is a half-plane the boundary of which is passing through the origin. Since $m_p(D) = D$, it is geometrically clear that p must be real and strictly positive (actually > 1), hence $\mathrm{tr}^2 g > 4$.

(iv) What is left now, means characterizing the properly loxodromic situation exactly in the asserted form. □

References

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2. Harmonic functions and Poisson formula

Recall that a twice continuously differentiable function $u : D \to \mathbb{R}$ in an open set $D \subset \mathbb{C}$ is called **harmonic** in D, provided its Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

vanishes identically in D. Partial differential equation $\Delta u = 0$ is called **Laplace differential equation**.

Also recall that whenever g is analytic in D, then $u := \Re g$ and $v := \Im g$ both are harmonic functions. This was an immediate consequence of Cauchy–Riemann equations, see previous complex analysis courses.

We start now by proving the classical **Poisson formula**:

Theorem 2.1. *Let g be analytic in $|z| < R$, and let u be its real part. Given r, $0 < r < R$, and any point $a \in \mathbb{C}$ such that $|a| < r$, then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) K(a, r, \theta) d\theta,$$

where the weight function

$$K(a, r, \theta) := \frac{r^2 - |a|^2}{|re^{i\theta} - a|^2}$$

is the ***Poisson kernel.***

Proof. Look first at the Möbius transformation

$$w(z) := r^2 \frac{z + a}{\bar{a}z + r^2}.$$

Clearly $|w(r)| = |w(-r)| = |w(ir)| = r$ and $|w(0)| = |a| < r$. Therefore, by basic properties of Möbius transformations, w maps the disc $|z| < r$ onto $|w| < r$ so that the boundary $|z| = r$ maps onto $|w| = r$.

We next observe that the composed function $g \circ w$ is analytic in some disc $|z| < \rho$ such that $r < \rho < R$. By Cauchy integral formula,

$$g(w(0)) = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(w(z))}{z} dz.$$

The inverse transformation of $w(z)$ may be written as

$$z(w) = -r^2 \frac{w - a}{\bar{a}w - r^2}.$$

By substitution $z = z(w)$ the integration may be carried over the circle $|w| = r$ in the w-plane, resulting in

$$g(a) = g(w(0)) = \frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{-r^2 \frac{w-a}{\bar{a}w - r^2}} z'(w) dw,$$

hence

$$g(a) = \frac{1}{2\pi i} \int_{|w|=r} g(w) \frac{r^2 - |a|^2}{(a - w)(\bar{a}w - r^2)} dw.$$

As now $w = re^{i\theta}$ and so $w\overline{w} = r^2$, $dw = iwd\theta$, we see that

$$g(a) = \frac{1}{2\pi i} \int_0^{2\pi} g(w) \frac{r^2 - |a|^2}{(a - w)(\bar{a}w - w\overline{w})} iwd\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(w)\frac{w}{a-w}\frac{r^2-|a|^2}{w(\overline{a}-\overline{w})}d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(w)\frac{r^2-|a|^2}{(a-w)\overline{(a-w)}}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(w)\frac{r^2-|a|^2}{|a-w|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(w)\frac{r^2-|a|^2}{|re^{i\theta}-a|^2}d\theta.$$

Taking real parts we now obtain

$$u(a) = \Re g(a) = \Re\left(\frac{1}{2\pi}\int_0^{2\pi} g(w)\frac{r^2-|a|^2}{|re^{i\theta}-a|^2}d\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Re g(w)\frac{r^2-|a|^2}{|re^{i\theta}-a|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\frac{r^2-|a|^2}{|re^{i\theta}-a|^2}d\theta.$$

□

Remark. We add here a few simple remarks concerning the Poisson kernel:

(1) Since $|a| < r$, it is trivial to observe that $K(a,r,\theta) = \frac{r^2-|a|^2}{|re^{i\theta}-a|^2} > 0$.

(2) By a simple geometric observation, namely that $|re^{i\theta}-a| \geq r-|a|$, we get

$$K(a,r,\theta) = \frac{(r-|a|)(r+|a|)}{|re^{i\theta}-a|^2} \leq \frac{(r-|a|)(r+|a|)}{(r-|a|)^2} = \frac{r+|a|}{r-|a|}.$$

(3) Considering the constant function $g(z) \equiv 1$, the Poisson formula results in

$$\frac{1}{2\pi}\int_0^{2\pi} K(a,r,\theta)d\theta = 1.$$

If now g is analytic in $|z| < R$ and has no zeros there, we may fix a branch of the logarithm function in the image plane to obtain $\log g(z)$ analytic in $|z| < R$ as well. But then, for some integer k,

$$\Re \log g(z) = \Re(\log|g(z)| + i\arg g(z) + 2\pi ik) = \log|g(z)|.$$

Therefore, since $\Re\log g(z)$ is harmonic, $\log|g(z)|$ is harmonic as well. Theorem 2.1 now applies to result in

$$\log|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} (\log|g(re^{i\theta})|)K(z,r,\theta)d\theta$$

as soon as $0 < r < R$ and $|z| < r$.

Considering now a meromorphic function $w : D \to \mathbb{C}$ instead of an analytic one, the Poisson formula may be extended to the following **Poisson–Jensen–Nevanlinna formula**:

Theorem 2.2. *Let $w(z)$ be meromorphic in a disc $|z| < R \leq \infty$, and let α_j, resp. β_j, denote the zeros, resp. poles, of w in $|z| < R$, each being counted according to its multiplicity. If now $0 < r < R$, $|z| < r$ and if $w(z) \neq 0, \infty$, then*

$$\log|w(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta$$

$$+ \sum_{|\beta_j|<r} \log\left|\frac{r^2-\overline{\beta_j}z}{r(z-\beta_j)}\right| - \sum_{|\alpha_j|<r} \log\left|\frac{r^2-\overline{\alpha_j}z}{r(z-\alpha_j)}\right|.$$

Before proceeding to prove Theorem 2.2, we need to prove an integral convergence lemma:

Lemma 2.3. *Suppose $z \in \mathbb{C}$, $|z| < r$ and $b = re^{i\beta}$ are given. Then the integral*

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} \log|re^{i\theta} - b| K(z, r, \theta) d\theta$$

converges, and $I(r) = \log|z - b|$.

Proof. (1) To prove the convergence, first choose ε_1 small enough to satisfy $|re^{i(\beta+\varepsilon_1)}| < 1$ and $|re^{i(\beta+2\pi-\varepsilon_1)}| < 1$, and consider the following two integrals:

$$I_1(\varepsilon) = \int_{\beta+\varepsilon}^{\beta+\pi} \log|re^{i\theta} - b| K(z, r, \theta) d\theta$$

and

$$I_2(\varepsilon) = \int_{\beta+\pi}^{\beta+2\pi-\varepsilon} \log|re^{i\theta} - b| K(z, r, \theta) d\theta$$

for $0 < \varepsilon < \varepsilon_1$. By the choice of ε_1, we observe that $I_1(\varepsilon)$ decreases as ε decreases, and similarly for $I_2(\varepsilon)$. Therefore, to show their convergence as $\varepsilon \to 0$, hence the convergence of the original integral $I(r)$ as well, it is sufficient to show that the integrals $I_1(\varepsilon), I_2(r)$ remain bounded as $\varepsilon \to 0$. But now

$$|I_1(\varepsilon)| \leq \int_{\beta+\varepsilon}^{\beta+\pi} |\log|re^{i\theta} - re^{i\beta}||K(z, r, \theta\theta) d\theta$$

$$= \int_{\beta+\varepsilon}^{\beta+\pi} |\log|re^{i\beta}||re^{i(\theta-\beta)} - 1||K(z, r, \theta) d\theta = \int_{\beta+\varepsilon}^{\beta+\pi} |\log(r|re^{i(\theta-\beta)} - 1|)|K(z, r, \theta) d\theta$$

$$\leq \int_{\beta+\varepsilon}^{\beta+\pi} |\log(r|re^{i(\theta-\beta}) - 1|)| \frac{r+|z|}{r-|z|} d\theta \leq \frac{r+|z|}{r-|z|} \int_{\beta+\varepsilon}^{\beta+\pi} (|\log r| + |\log|e^{i(\theta-\beta)} - 1||) d\theta.$$

Denoting $\alpha := \theta - \beta$, we observe that the required boundedness follows, if we show that the limit

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi} |\log|e^{i\alpha} - 1|^2| d\alpha$$

exists and is finite. We first observe that

$$\log|e^{i\alpha} - 1|^2 = \log|\cos\alpha - 1 + i\sin\alpha|^2 = \log((\cos\alpha - 1)^2 + \sin^2\alpha)$$

$$= \log(2(1 - \cos\alpha)) = \log\left(4\sin^2\frac{\alpha}{2}\right) = \log 4 + 2\log\sin\frac{\alpha}{2}.$$

Since $0 < \alpha \leq \pi$, then making use of $\sin\frac{\alpha}{2} \geq \frac{\alpha}{\pi}$ **(Exercise!)**, we see that

$$|\log|e^{i\alpha} - 1|^2| \leq \log 4 - 2\log\sin\frac{\alpha}{2} \leq \log 4 - 2\log\frac{\alpha}{\pi} = c - 2\log\alpha,$$

where c is a real constant. Therefore, since

$$\int_{\varepsilon}^{\pi} |\log|e^{i\alpha} - 1|^2| d\alpha \leq \int_{\varepsilon}^{\pi} c\, d\alpha - 2\int_{\varepsilon}^{\pi} \log\alpha\, d\alpha,$$

it is sufficient to show $\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi} \log\alpha\, d\alpha$ exists and is finite. But this follows from

$$\int_{\varepsilon}^{\pi} \log\alpha\, d\alpha = \alpha/_{\varepsilon}^{\pi}(\alpha\log\alpha - \alpha) = \pi\log\pi - \pi - \varepsilon\log\varepsilon + \varepsilon \to \pi\log\pi - \pi$$

as $\varepsilon \to 0$. The case of $I_2(\varepsilon)$ may be treated in a completely similar way, being omitted here.

(2) To prove the assertion $I(r) = \log|z - b|$, we still need to estimate the integral

$$I_\delta(\rho) = \frac{1}{2\pi} \int_{\beta-\delta}^{\beta+\delta} \log|\rho e^{i\theta} - b| K(z, \rho, \theta) d\theta,$$

when $|z| < \rho_0 \leq \rho \leq r$ and δ is small enough. If $\rho = r$, then clearly $I_\delta(r) \to 0$ as $\delta \to 0$ by the first part of the proof. If then $\rho < r$, then the integrand is continuous and bounded, hence convergence of the integral will be trivially ensured. We now proceed to show that the estimate

$$|I_\delta(\rho)| \leq c_1\delta + c_2\delta\log\delta$$

holds in a form where the constants c_1, c_2 are independent of δ, ρ. Denote again $\theta - \beta = \alpha$. Then, similarly as in the first part of the proof, we get

$$2\pi|I_\delta(\rho)| \leq \int_{\beta-\delta}^{\beta+\delta} |\log|\rho e^{i\theta} - b||K(z,\rho,\theta)d\theta \leq \frac{\rho+|z|}{\rho-|z|}\int_{-\delta}^{\delta}|\log|\rho e^{i\alpha} - r||d\alpha$$

$$\leq \frac{1}{2}\frac{r+|z|}{\rho_0-|z|}\int_{-\delta}^{\delta}|\log|\rho e^{i\alpha} - r|^2|d\alpha = K_1\int_{-\delta}^{\delta}|\log|\rho e^{i\alpha} - r|^2|d\alpha.$$

Again, similarly as in the first part, we get the estimate

$$|\rho e^{i\alpha} - r|^2 = |\rho\cos\alpha - r + i\rho\sin\alpha|^2 = (\rho\cos\alpha - r)^2 + \rho^2\sin^2\alpha$$

$$= \rho^2 - 2\rho r\cos\alpha + r^2 = (\rho - r)^2 + 2\rho r(1-\cos\alpha) \geq 4\rho r\sin^2\frac{\alpha}{2} \geq 4\rho_0 r(\alpha/\pi)^2.$$

To estimate the integral $\int_{-\delta}^{\delta}|\log|\rho e^{i\alpha-r}|^2|d\alpha$, let c denote constants which may be different in different occurrences, even within the same expression. Estimating first the integrand, if $|\rho e^{i\alpha} - r| \geq 1$, then

$$|\log|\rho e^{^{i\alpha}} - r|^2| \leq 2\log(2r) \leq c,$$

while if $|\rho e^{i\alpha} - r| < 1$, then

$$|\log|\rho e^{i\alpha} - r|^2| \leq |\log(4\rho_0 r)| + 2|\log(\alpha/\pi)| = c + 2|\log|\alpha||.$$

Now, for δ sufficiently small, we obtain by the preceding estimates that

$$\int_{-\delta}^{\delta}|\log|\rho e^{i\alpha} - r|^2|d\alpha \leq c + c\int_{-\delta}^{\delta}\log t dt$$

and

$$\int_{-\delta}^{\delta}\log t dt = 2\delta - 2\delta\log\delta.$$

Combining the preceding estimates, we obtain the required estimate for $|I_\delta(\rho)|$.

(3) As the final phase to prove the lemma, we now show that $I(r) = \log|z-b|$. Fixing the branch of the logarithm (to ensure that it is analytic) and recalling that $\Re\log(z-b) = \log|z-b|$, we first get by the Poisson formula above that

$$\log|z-b| = \frac{1}{2\pi}\int_0^{2\pi}\log|\rho e^{i\theta} - b|K(z,\rho,\theta)d\theta.$$

Denote now, for notational simplicity,

$$H(\rho,\theta) := \log|\rho e^{i\theta}|K(z,\rho,\theta).$$

Fixing $\varepsilon > 0$ and letting δ be small enough, we see by part (2) of the proof above that

$$\left|\frac{1}{2\pi}\int_{\beta-\delta}^{\beta+\delta}H(\rho,\theta)d\theta\right| \leq \varepsilon$$

for each ρ such that $\rho_0 \le \rho \le r$. If $(\rho,\theta) \in [\rho_0, r]\times[\beta+\delta, \beta+2\pi-\delta]$, we have that $H(\rho,\theta), H(r,\theta)$ are continuous and bounded. Therefore, we may assume that ρ is close enough to r and $\delta > 0$ is small enough to result in

$$\left|\frac{1}{2\pi}\int_{\beta+\delta}^{\beta+2\pi-\delta} H(\rho,\theta)d\theta - \frac{1}{2\pi}\int_{\beta+\delta}^{\beta 2\pi-\delta} H(r,\theta)d\theta\right| \le \varepsilon.$$

Therefore,

$$|I(r) - \log|z-b|| = \left|\frac{1}{2\pi}\int_0^{2\pi} H(r,\theta)d\theta - \log|z-b|\right|$$

$$\le \left|\frac{1}{2\pi}\int_{\beta+\delta}^{\beta+2\pi-\delta} H(r,\theta)d\theta + \frac{1}{2\pi}\int_{\beta-\delta}^{\beta+\delta} H(r,\theta)d\theta - \log|z-b|\right|$$

$$\le \left|\frac{1}{2\pi}\int_{\beta+\delta}^{\beta+2\pi-\delta}(H(r,\theta)-H(\rho,\theta))d\theta\right| + \left|\frac{1}{2\pi}\int_{\beta+\delta}^{\beta+2\pi-\delta} H(\rho,\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} H(\rho,\theta)d\theta\right|$$

$$+\left|\frac{1}{2\pi}\int_0^{2\pi} H(\rho,\theta)d\theta - \log|z-b|\right| + \left|\frac{1}{2\pi}\int_{\beta-\delta}^{\beta+\delta} H(r,\theta)d\theta\right|$$

$$= \left|\frac{1}{2\pi}\int_{\beta+\delta}^{\beta+2\pi-\delta}(H(r,\theta)-H(\rho,\theta))d\theta\right| + \left|\frac{1}{2\pi}\int_{\beta-\delta}^{\beta+\delta} H(\rho,\theta)d\theta\right|$$

$$+\left|\frac{1}{2\pi}\int_{\beta-\delta}^{\beta+\delta} H(r,\theta)d\theta\right| + \left|\frac{1}{2\pi}\int_0^{2\pi} H(\rho,\theta)d\theta - \log|z-b|\right| \le 4\varepsilon.$$

$\square$

Proof. (Proof of Theorem 2.2) For notational convenience, we denote by $\alpha_l,\dots,\alpha_k$, resp. $\beta_1,\dots,\beta_l$, the zeros, resp. poles in $|z| < r$, and by $\alpha_{k+1},\dots,\alpha_p$, resp. $\beta_{l+1},\dots,\beta_s$ the zeros, resp. poles in $|z| = r$. We also denote

$$\phi(z) := \prod_{j=1}^{k} \frac{r^2 - \overline{\alpha_j} z}{r(z-\alpha_j)}, \qquad \chi(z) := \prod_{j=1}^{l} \frac{r(z-\beta_j)}{r^2 - \overline{\beta_j} z},$$

$$\lambda(z) := \left(\prod_{j=l+1}^{s}(z-\beta_j)\right)\left(\prod_{j=k+1}^{p}(z-\alpha_j)^{-1}\right), \qquad g(z) := w(z)\phi(z)\chi(z)\lambda(z).$$

Recall from elementary complex analysis that

$$\left|\frac{r(z-c)}{r^2-\bar{c}z}\right| = \left|\frac{r(z-c)}{z\bar{z}-\bar{c}z}\right| = \frac{r|z-c|}{|z||\overline{z-c}|} = \frac{|z-c|}{\overline{z-c}} = 1,$$

provided $|z| = r$ and $|c| < r$. Therefore, we have

$$|\phi(z)| = 1, \qquad |\chi(z)| = 1$$

for $|z| = r$. By construction of g, this function has neither zeros nor poles inside of the closed disc $|z \le r|$, hence inside of a slightly larger open disc. Therefore, we may fix a branch of $\log g(z)$ to ensure that it will be analytic, and to be able to apply the Poisson formula. Also making use of Lemma 2.3, we obtain

$$\log|g(z)| = \log|w(z)| + \log|\phi(z)| + \log|\chi(z)| + \log|\lambda(z)|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta}|K(z,r,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|\lambda(re^{i\theta})|K(z,r,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{\prod_{j=l+1}^{s}(z-\beta_j)}{\prod_{j=k+1}^{p}(z-\alpha_j)}\right| K(z,r,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta + \sum_{j=l+1}^{s} \frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\theta}-\beta_j|K(z,r,\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\theta}-\alpha_j|K(z,r,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta + \sum_{j=l+1}^{s} \log|z-\beta_j| - \sum_{j=k+1}^{p} \log|z-\alpha_j|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta + \log\lambda(z).$$

Therefore, we obtain

$$\log|w(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta - \log|\phi(z)| - \log|\chi(z)|$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta - \log\prod_{|\alpha_j|<r}\left|\frac{r^2-\overline{\alpha_j}z}{r(z-\alpha_j)}\right| - \log\prod_{|\beta_j|<r}\left|\frac{r(z-\beta_j)}{r^2-\overline{\beta_j}z}\right|$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|K(z,r,\theta)d\theta - \sum_{|\alpha_j|<r}\log\left|\frac{r^2-\overline{\alpha_j}z}{r(z-\alpha_j)}\right| + \sum_{|\beta_j|<r}\log\left|\frac{r^2-\overline{\beta_j}z}{r(z-\beta_j)}\right|.$$

□

To close this section, we add as a corollary a special of the Poisson–Jensen–Nevanlinna formula, known as the **Jensen formula**. This follows from Theorem 2.2 by taking $z = 0$:

Corollary 2.4. *Under the same assumptions as in Theorem 2.2,*

$$\log|w(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|w(re^{i\theta})|d\theta + \sum_{|\beta_j|<r}\log\frac{r}{|\beta_j|} - \sum_{|\alpha_j|<r}\log\frac{r}{|\alpha_j|}.$$

Proof. The claim follows at once from the fact that

$$K(0,r,\theta) = \frac{r^2}{|re^{i\theta}|^2} = 1.$$

□

References

[1] Jank, G. and Volkmann, L., *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel–Boston, 1985.

3. Nevanlinna theory: First Main Theorem

Nevanlinna theory is a theory about the growth and value distribution of meromorphic functions. The three key results are First Main Theorem, Logarithmic Derivative Lemma and Second Main Theorem. First Main Theorem is actually nothing else than a reformulation of the Jensen formula from the preceding section. Despite of its simplicity, Nevanlinna theory, which is one of the greatest mathematical achievements in the last century, would not exist without First Main Theorem. Logarithmic Derivative Lemma on the other hand is a really deep result which has no direct predecessor. Second Main Theorem then follows from Logarithmic Derivative Lemma in a way which is technically somewhat complicated, though basically elementary.

The starting point for the necessary reformulation of the Jensen formula the following decomposition of logarithm in its positive and negative part as follows:

Definition 3.1. For any real number $\alpha \geq 0$, we set

$$\log^+ \alpha := \max(0, \log \alpha).$$

Lemma 3.2. *The operation* $\log^+$ *has the following properties:*

(a) $\log \alpha \leq \log^+ \alpha$,

(b) $\log^+ \alpha \leq \log^+ \beta$, *whenever* $\alpha \leq \beta$,

(c) $\log \alpha = \log^+ \alpha - \log^+ \frac{1}{\alpha}$,

(d) $|\log \alpha| = \log^+ \alpha + \log^+ \frac{1}{\alpha}$,

(e) $\log^+(\alpha\beta) \leq \log^+ \alpha + \log^+ \beta$,

(f) $\log^+ \left(\prod_{j=1}^n \alpha_j\right) \leq \sum_{j=1}^n \log^+ \alpha_j$,

(g) $\log^+(\alpha + \beta) \leq \log^+ \alpha + \log^+ \beta + \log 2$,

(h) $\log^+ \sum_{j=1}^n \alpha_j \leq \log n + \sum_{j=1}^n \log^+ \alpha_j$.

Proof. (a) If $\alpha \geq 1$, then $\log \alpha \geq 0$, hence $\log \alpha = \log^+ \alpha$. On the other hand, if $0 \leq \alpha < 1$, then $\log \alpha < 0 \leq \log^+ \alpha$.

(b) If $1 \leq \alpha \leq \beta$, the claim follows from the monotonicity of the logarithm. If then $0 \leq \alpha, \beta \leq 1$, we have $\log^+ \alpha = \log^+ \beta = 0$. Finally, if $0 \leq \alpha \leq 1 \leq \beta$, then $\log^+ \alpha = 0 = \log^+ 1 \leq \log^+ \beta$.

(c), (d) **Exercise**.

(e) If $\alpha, \beta \geq 1$, then

$$\log^+(\alpha\beta) = \log(\alpha\beta) = \log \alpha + \log \beta = \log^+ \alpha + \log^+ \beta.$$

If next $0 \leq \alpha, \beta < 1$, then $\log^+(\alpha\beta) = 0 = \log^+ \alpha + \log^+ \beta$.

If then $\alpha < 1 \leq \beta$ and $\alpha\beta \geq 1$, we have

$$\log^+(\alpha\beta) = \log(\alpha\beta) = \log \alpha + \log \beta \leq \log^+ \alpha + \log^+ \beta = \log^+ \alpha + \log^+ \beta.$$

Finally, if $\alpha < 1 \leq \beta$ and $\alpha\beta < 1$, then

$$\log^+(\alpha\beta) = 0 \leq \log^+ \alpha + \log^+ \beta.$$

(f) This follows by induction from the preceding case.

(g) If $\alpha + \beta \leq 1$, the assertion is trivial. If then $\alpha + \beta > 1$, we may assume that $\beta \leq \alpha$. Then we have

$$\log^+(\alpha + \beta) = \log(\alpha + \beta) \leq \log(2\alpha) = \log 2 + \log \alpha \leq \log^+ \alpha + \log^+ \beta + \log 2.$$

(h) Induction. □

Definition 3.3. (**Non-integrated counting function.**) Given $a \in \mathbb{C}$, let f be a meromorphic function in $\mathbb{C}$ such that $f - a$ does not vanish identically. Then

$$n(r, a, f) = n\left(r, \frac{1}{f-a}\right)$$

denotes the number of a-points of f in $|z| \leq r$, each such a-point being counted according to its multiplicity. Similarly, we define as

$$n(r, \infty, f) = n(r, f)$$

the non-integrated counting function for poles of f in $|z| \leq r$.

Definition 3.4. (**Counting function.**) For $a \in \mathbb{C}$, we define

$$N(r, a, f) = N\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r.$$

Similarly, for poles of f, we define

$$N(r, \infty, f) = N(r, f) := \int_0^r \frac{n(r, \infty, r) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r.$$

Lemma 3.5. *Let f be meromorphic and let $\alpha_1, \ldots, \alpha_n$ be its a-points, counting multiplcity, in $|z| \leq r$. Moreover, suppose that $f(0) \neq a$, and that $0 < |\alpha_1| \leq \ldots \leq |\alpha_n| \leq r$. Then*

$$\int_0^r \frac{n(t, a, f)}{t} dt = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt = \sum_{0 < |\alpha_j| \leq r} \log \frac{r}{|\alpha_j|}.$$

Proof. Denoting $|\alpha_j| = r_j$ for $j = 1, \ldots, n$, we get by a straightforward computation

$$\begin{aligned}
\sum_{j=1}^n \log \frac{r}{|\alpha_j|} &= O \sum_{j=1}^n \log \frac{r}{r_j} = \log \prod_{j=1}^n \frac{r}{r_j} = \log \frac{r^n}{r_1 \cdots r_n} = n \log r - \sum_{j=1}^n \log r_j \\
&= \sum_{j=1}^{n-1} j(\log r_{j+1} - \log r_j) + n(\log r - \log r_n) \\
&= \sum_{j=1}^n j \int_{r_j}^{r_{j+1}} \frac{dt}{t} + n \int_{r_n}^r \frac{dt}{t} = \int_0^r \frac{n(t, a, r)}{t} dt.
\end{aligned}$$

□

We next apply the Jensen formula to prove the following

Proposition 3.6. *Let f be a meromorphic function with the Laurent expansion $f(z) = \sum_{j=m}^{\infty} c_j z^j$, $c_m \neq 0$. Then*

$$\log |c_m| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta + N(r, f) - N(r, \frac{1}{f}).$$

Proof. Denote by α_j the zeros and β_j the poles of f (outside of the origin). Then apply Jensen formula for the function $h(z) := z^{-m} f(z)$. This results in

$$\log |c_m| = \log |h(0)|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta}| d\theta + \sum_{0 < |\beta_j| < r} \log \frac{r}{|\beta_j|} - \sum_{0 < |\alpha_j| < r} \log \frac{r}{|\alpha_j|}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \log(|f(re^{i\theta})|r^{-m})d\theta\theta + \sum_{0<|\beta_j|<r} \log\frac{r}{|\beta_j|} - \sum_{0<|\alpha_j|<r} \log\frac{r}{|\alpha_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - m\log r + \sum_{0<|\beta_j|<r} \log\frac{r}{||\beta_j} - \sum_{0<|\alpha_j|<r} \log\frac{r}{|\alpha_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta - \left(n(0,\frac{1}{f}) - n(0,f)\right)\log r \\
&\quad + \int_0^r \frac{n(t,f)-n(0,f)}{t}dt - \int_0^r \frac{n(t,\frac{1}{f})-n(0,\frac{1}{f})}{t}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \left(\int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r\right) \\
&\quad - \left(\int_0^r \frac{n(t,\frac{1}{f})-n(0,\frac{1}{f})}{t}dt + n(0,\frac{1}{f})\log r\right) \\
&= \frac{1}{2\pi}\int_0^r \log|f(re^{i\theta})|d\theta + N(r,f) - N(r,\frac{1}{f}),
\end{aligned}
$$

as required. $\square$

Definition 3.7. (**Proximity function**). For the poles of f, we define

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta,$$

and for $a \in \mathbb{C}$,

$$m(r,a,f) = m(r,\frac{1}{f-a}) := \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{1}{f(re^{i\theta}-a}\right|d\theta.$$

Definition 3.8. (**Characteristic function**). For a meromorphic function f, we define its characteristic function as

$$T(r,f) := m(r,f) + N(r,f).$$

Theorem 3.9. *(**First Main Theorem**). Let f be a meromorphic function not being identically equal to a constant. Then, for all $a \in \mathbb{C}$,*

$$T\left(r,\frac{1}{f-a}\right) = T(r,f) + O(1)$$

as $r \longrightarrow\longrightarrow \infty$.

In the preceding theorem, the exact expression of $O(1)$ depends on a, as shown by the following exact form of the First Main Theorem:

Theorem 3.10. *Given $a \in \mathbb{C}$, suppose that a meromorphic function f has the Laurent expansion*

$$f(z) = a + \sum_{j=m}^{\infty} c_j z^j, \qquad c_m \neq 0, \qquad m \in \mathbb{Z}$$

about the origin. Then

$$T\left(r,\frac{1}{f-a}\right) = T(r,f) - \log|c_m| + \varphi(r,a),$$

where $|\varphi(r,a)| \leq \log^+|a| + \log 2$.

Proof. First suppose that $a = 0$. By Proposition 3.6, we see that

$$\log|c_m| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + N(r,f) - N(r,1/f)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+\frac{1}{|f(re^{i\theta})|}d\theta + N(r,f) - N(r,1/f)$$

$$m(r,f) + N(r,f) - (m(r,1/f) + N(r,1/f)) = T(r,f) + T(r,1/f).$$

Therefore,

$$T(r,1/f) = T(r,f) - \log|c_m| + \varphi(r,0),$$

where $\varphi(r,0) \equiv 0$.

As for the general case $a \neq 0$, we define $h := f - a$. Then clearly f has a pole if and if h has a pole and $f(z) = a$ if and only if $h(z) = 0$. Therefore, $N(r,f) = N(r,h)$ and $N(r,1/(f-a)) = N(r,1/h)$. Moreover, by definition, $m(r,1/h) = m(r,1/(f-a))$. Since

$$\log^+|h| = \log^+|f-a| \leq \log^+|f| + \log^+|a| + \log 2$$

and

$$\log^+|f| = \log^+|h+a| \leq \log^+|h| + \log^+|a| + \log 2,$$

we get

$$m(r,h) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|h(re^{i\theta})|d\zeta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} (\log^+|a| + \log 2)d\theta$$

$$= m(r,f) + \log^+|a| + \log 2.$$

A completely parallel reasoning tells that

$$m(r,f) \leq m(r,h) + \log^+|a| + \log 2.$$

Therefore, if we define $\varphi(r,a) := m(r,h) - m(r,f)$, we have established that

$$|\varphi(r,a)| \leq \log^+|a| + \log 2.$$

We may now apply the first part of the proof for the function h, for which $h(0) = 0$. Then we obtain

$$T(t,1/h) = T(r,h) - \log|c_m|,$$

$$m(r,1/h) + N(r,1/h) = m(r,h) + N(r,h) - \log|c_m|,$$

$$m\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-a}\right) = m(r,f) + \varphi(r,a) + N(r,h) - \log|c_m|,$$

$$T\left(r,\frac{1}{f-a}\right) = T(r,f) - \log|c_m| + \varphi(r,a),$$

which proves the assertion. □

The following lemma is a rather trivial fact about polynomials, needed occasionally in what follows:

Lemma 3.11. *Given a polynomial $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, and $\varepsilon > 0$, then there exists $r_\varepsilon > 0$ so that whenever $|z| = r \geq r_\varepsilon$, then*

$$(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n.$$

Proof. The assertion immediately follows from

$$\frac{|P(z)|}{|a_n|r^n} = \left|\frac{P(z)}{a_n z^n}\right| = \left|\frac{a_n z^n + \cdots + a_0}{a_n z^n}\right| = \left|1 + \frac{a_{n-1}}{a_n}\frac{1}{z} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right| \to 1$$

as $r \to \infty$. $\square$

In the subsequent considerations, the **Landau notations** are frequently used. To explain them, let $S : [r_0, \infty) \to [0, \infty)$ be a given nonnegative function on a right-infinite part of the real axis. Then the notation $o(S(r))$ will be used for any real-valued function h on the positive real axis such that

$$\lim_{r\to\infty} \frac{h(r)}{S(r)} = 0.$$

Similarly, $O(S(r))$ means that $h(r)/S(r)$ remains bounded, as $r \to \infty$. In what follows, Landau notations may also be applied for functions defined outside of a (typically small) exceptional set. If so, this state of affairs will be somehow pointed out. In particular, $O(1)$ means a certain quantity remains bounded as $r \to \infty$.

Observe now that First Main Theorem may shortly be expressed in the form

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

Landau notations typically appear in almost all computations related to Nevanlinna Theory. This also applies in the following exercises, as the characteristic function normally cannot be computed exactly, but only modulo a small error term, say of type $O(1)$, or something else.

Exercises. Compute $T(r, f)$, if

(a) $f(z) = z^2$,

(b) $f(z)$ is a polynomial,

(c) $f(z)$ is rational,

(d) $f(z) = e^z$,

(e) $f(z) = \frac{1}{\sin z}$,

(f) $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial.

References

[1] Jank, G. and Volkmann, L., *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel–Boston, 1985.

[2] Hayman, W., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[3] Yang, L., *Value Distribution Theory*, Springer, Berlin, 1993.

4. Nevanlinna theory: Basic results

In this section, we include a collection of basic properties of the Nevanlinna functions which directly follow from their definitions and from previous complex analysis.

Theorem 4.1. *For finitely many meromorphic functions $f_1, \dots, f_n$, the following inequalities hold:*

$$m(r, f_1 + \cdots + f_n) \leq m(r, f_1) + \cdots + m(r, f_n) + \log n,$$

$$N(r, f_1 + \cdots + f_n) \leq N(r, f_1) + \cdots + N(r, f_n),$$

$$T(r, f_1 + \cdots + f_n) \leq T(r, f_1) + \cdots + T(r, f_n) + \log n,$$

$$m(r, f_1 \cdots f_n) \leq m(r, f_1) + \cdots + m(r, f_n),$$

$$N(r, f_1 \cdots f_n) \leq N(r, f_1) + \cdots + N(r, f_n),$$

$$T(r, f_1 \cdots f_n) \leq T(r, f_1) + \cdots + T(r, f_n).$$

Proof. First observe that the assertions concerning the proximity functions immediately follow from Lemma 3.2(f) and Lemma 3.2(h) by integration.

Looking next at the non-integrated counting functions, a pole of a product at a point, say z_0, obviously is of multiplicity at most the sum of pole multiplicities of the components. By logarithmic integration, the assertions concerning the counting functions follow at once.

Finally, the assertions concerning the characteristic functions are an immediate consequence of its definition as the sum of the proximity function and the counting function. □

Remark. Observe that for a power f^k, $k \in \mathbb{N}$, we obtain

$$m(r, f^k) = km(r, f),$$

$$N(r, f^k) = kN(r, f)$$

and

$$T(r, f^k) = kT(r, f).$$

In fact, for non-integrated counting function of poles, we have $n(r, f^k) = kn(r, f)$, from which the claim for $N(r, f^k)$ follows by integration. For the proximity function, it is immediate to see that

$$\log^+ |(f(re^{i\theta}))^k| = k \log^+ |f(re^{i\theta})|.$$

The next result shows that the characteristic function is essentially invariant relative to Möbius transformations:

Theorem 4.2. *Given a Möbius transformation $g \in \mathfrak{M}$ and a non-constant meromorphic function f, then*

$$T(r, g \circ f) = T(r, f) + O(1).$$

Proof. By Theorem 1.1, each Möbius transformation may be represented as a composition of inversions, translations and (complex) dilations. Therefore, it is sufficient to treat each of these special cases separately.

(a) By the Jensen formula, see the proof of Theorem 3.10,

$$T(r, 1/w) = T(r, w) + O(1).$$

(b) We may apply the Jensen formula and First Main Theorem to obtain

$$T(r, w - a) = T\left(r, \frac{1}{w - a}\right) + O(1) = T(r, w) + O(1).$$

(c)Consider now a complex dilation $w \mapsto bw$, where $b \in \mathbb{C} \setminus \{0\}$. First observe that the function bw has exactly the same poles as w, counting multiplicity. Moreover,

$$m(r, w) = m\left(\frac{1}{b}bw\right) \leq m(r, bw) + \log^+ \frac{1}{|b|}$$

and

$$m(r, bw) \leq m(r, w) + \log^+ |b|,$$

meaning that $m(r, bw) = m(r, w) + O(1)$. Therefore,

$$T(r, bw) = m(r, bw) + N(r, bw) = m(r, w) + N(r, w) + O(1) = T(r, w) + O(1).$$

□

Example. To compute $T(r, f)$ for $f(z) = \tan z$, we first represent the tangent function in the form

$$f(z) = \frac{\sin z}{\cos z} = \frac{\frac{1}{2i}(e^{iz} - e^{-iz})}{\frac{1}{2}(e^{iz} + e^{-iz})} = -i\frac{e^{2iz} - 1}{e^{2iz} + 1}.$$

By the preceding theorem, we now obtain

$$T(r, f) = T\left(r, \frac{e^{2iz} - 1}{e^{2iz} + 1}\right) + O(1) = T(r, e^{2iz}) + O(1) = T(r, (e^{iz})^2) + O(1)$$

$$= 2T(r, e^{iz}) + O(1) = 2T(r, e^z) + O(1) = 2\frac{r}{\pi} + O(1).$$

In the special case of entire functions, when $T(r, f) = m(r, f)$, it appears that the characteristic function is qualitatively about the same as the maximum modulus as seen by the following

Theorem 4.3. *Let f be a non-constant entire function. Then*

$$T(r, f) \leq \log M(r, f) \leq \frac{R + r}{R - r}T(R, f),$$

provided $0 < r < R < \infty$ *and r is sufficiently large.*

Proof. By Liouville theorem, f is not bounded. Therefore, the maximum modulus $M(r, f)$ is not bounded as well. By the maximum principle, $M(r, f)$ is an increasing function and so, $M(r, f) \to \infty$ as $r \to \infty$. Therefore, we may take r_0 so that $M(r, f) \geq 1$ for all $r \geq r_0$. For these values of r, we first observe that

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta}|d\theta \leq \log^+ M(r, f) = \log M(r, f).$$

To prove the second inequality, take $r \geq r_0$ arbitrarily and fix it for the rest of the proof. By continuity of f, there exists z_0 so that $|z_0| = r$ and that $|f(z_0)| = M(r, f)$. Clearly, $f(z_0) \neq 0, \infty$ and we may apply the Poisson–Jensen–Nevanlinna formula to obtain

$$\log M(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta}| \frac{R^2-|z_0|^2}{|Re^{i\theta}-z_0|^2}d\theta - \sum_{|a_k|<R} \log\left|\frac{R^2-\overline{a_k z_0}}{R(z_0-a_k)}\right|,$$

where (a_k) means the sequence of zeros of f. Now, it is not difficult to show that for $|a_k| < R$, we have $|R^2 - \overline{a_k} z_0| \geq |R(z_0 - a_k)|$ **(Exercise!)**. Therefore,

$$\sum_{|a_k|<R} \log\left|\frac{R^2-\overline{a_k z_0}}{R(z_0-a_k)}\right| \geq 0.$$

Therefore, recalling Remark (2) on p. 13, we obtain that

$$\log M(r,f) \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta}|K(z_0,R,\theta)d\theta$$

$$\leq \frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta}|d\theta \leq \frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta}|d\theta$$

$$= \frac{R+r}{R-r}m(R,f) = \frac{R+r}{R-r}T(R,f).$$

□

Secondly, it is not difficult to show that the characteristic function behaves like $\log r$ if and only if f is a rational function, while for a non-rational function f, the characteristic functions grows essentially faster:

Theorem 4.4. *A meromorphic function f is rational if and only if $T(r, f) = O(\log r)$.*

Proof. For a rational function f, we have already shown that $T(r, f) = O(\log r)$. Therefore, assume now that $T(r, f) = O(\log r)$ and proceed to show that f then is a rational function. By assumption, we may find $r_0 \geq 1$ and $K > 0$ such that $N(r, f) \leq T(r, f) \leq K \log r$ for all $r \geq r_0$. We may also assume that f is not a constant function. We next show that f has at most finitely many poles. In fact,

$$(n(r,f)-n(0,f))\log r = (n(r,f)-n(0,f))\int_r^{r^2}\frac{dt}{t} = \int_r^{r^2}(n(r,f)-n(0,f))\frac{dt}{t}$$

$$\leq \int_r^{r^2}(n(t,f)-n(0,f))\frac{dt}{t} + n(0,f)\log r^2$$

$$= N(r^2,f) \leq K\log r^2 = 2K\log r,$$

meaning that $n(r, f) \leq n(0, f) + 2K < \infty$ for all $r \geq r_0$.

Let now $b_1, \ldots, b_n$ be the (finitely many) poles of f. next consider the polynomial $P(z) := (z - b_1)\cdots(z - b_n)$ and $g(z) = P(z)f(z)$. Clearly, g is an entire function, as all poles of f are cancelled by the zeros of P. Since deg $P = n$, we get by a preceding exercise that

$$T(r,P) = n\log r + O(1) \leq (n+1)\log r$$

for all r sufficiently large. But then

$$T(r,g) = T(r,Pf) \leq T(r,P) + T(r,f) \leq (K+p+1)\log r.$$

Setting $R = 2r$, and applying Theorem 4.3, we obtain

$$\log M(r,g) \le \frac{2r+r}{2r-r} T(2r,g) = 3T(2r,g) \le 3(K+p+1)\log 2r$$

$$= 3(K+p+1)(\log r + \log 2) \le 6(K+p+1)\log r =: \log r^B.$$

Therefore, $M(r,g) \le r^B$, and we conclude by the Liouville theorem that g is a polynomial. But then, $f = g/P$ must be a rational function. □

The next theorem is the first indication to show that from the two components, the proximity function and the counting function, of the characteristic function, the counting function is usually the dominating one:

Theorem 4.5. ***(Cartan theorem)*** *Let f be a meromorphic function such that $f(0) \neq \infty$. Then*

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right) d\theta + \log^+ |f(0)|.$$

Proof. (1) As the first part of the proof, we show that

$$\log^+ |a| = \frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\psi}| d\psi$$

for all $a \in \mathbb{C}$. To this end, denote $w(z) := a - z$, and suppose first that $|a| \le 1$. By the Jensen formula (Corollary 2.4),

$$\log|a| = \frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\psi}| d\psi - \log\frac{1}{|a|} = \frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\psi}| d\psi + \log|a|.$$

Therefore,

$$\log^+ |a| = 0 = \frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\psi}| d\psi.$$

If next $|a| > 1$, then w has no zeros in the unit disc. Since $\log^+ |a| = \log|a|$, the claim now follows directly from the Jensen formula, applied to w in the unit disc.

(2) We next apply Proposition 3.6 to the function $f(z) - e^{i\theta}$. Writing its Laurent expansion, actually Taylor expansion, about the origin as

$$f(z) - e^{i\theta} = \sum_{j=0}^{\infty} c_j z^j,$$

where $c_0 = f(0) - e^{i\theta}$, we get

$$\log|f(0) - e^{i\theta}| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\psi}) - e^{i\theta}| d\psi + N(r,f) - N\left(r, \frac{1}{f - e^{i\theta}}\right).$$

(3) As the next phase, we proceed to integrate the preceding formula with respect to θ over the unit circle $e^{i\theta}$. Using the notations

$$I_1 := \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\psi}) - e^{i\theta}| d\psi\right) d\theta$$

and

$$I_2 := \frac{1}{2\pi}\int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right) d\theta,$$

the identity from part (2) of this proof results in

$$\log^+ |f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}| d\theta = I_1 + N(r,f) - I_2.$$

The first problem is to show that the integrals I_1, I_2 exist, i.e. are convergent. It is sufficient to show that this is true for I_1. Fix now ψ for a while, and observe that for all $a \in \mathbb{C}$,

$$|\log|a|| = 2\log^+|a| - \log|a|.$$

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} |\log f(re^{i\psi}) - e^{i\theta}||d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \log^+|f(re^{i\psi}) - e^{i\theta}|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\psi}) - e^{i\theta}|d\theta$$

$$\leq 2\log^+|f(re^{i\psi})| + \log 4 - \log^+|f(re^{i\psi})| = \log^+|f(re^{i\psi})| + \log 4.$$

By the theorem of bounded convergence, see [3], Corollary 4.3.6 and the Fubini theorem, [3], Theorem 6.2.1, I_1 is integrable and

$$|I_1| \leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} |\log|f(re^{i\psi}) - e^{i\theta}||d\theta\right)d\psi$$

$$\leq \frac{1}{2\pi}\int_{0^{2\pi}} \log^+|f(re^{i\psi})|d\psi + \log 4 = m(r,f) + \log 4.$$

By the identity obtained in part (2) of the proof, see beginning of this part, the integral I_2 exists as well. Applying the Fubini theorem again, we see that

$$\log^+|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}|d\theta = I_1 + N(r,f) + I_2$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\psi}) - e^{i\theta}|d\theta\right)d\psi + N(r,f) - I_2$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\psi})|d\psi + N(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right)d\theta$$

$$T(r,f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right)d\theta.$$

□

We now close this section by considering the growth of the characteristic function from geometric point of view. More precisely, we show the $T(r,f)$ is increasing with respect to r (which is easy to prove) and convex with respect to $\log r$. Same conclusions hold for the counting function $N(r,f)$ as well, but not for the proximity function $m(r,f)$ in general.

First recall a function $f : \mathbb{R} \to \mathbb{R}$ (or $f : (\alpha,\beta) \to \mathbb{R}$, where $(\alpha,\beta) \subset \mathbb{R}$, is **convex**, for all a, x, b in the domain of definition of f such that $a < x < b$, we have

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}.$$

As one may easily see, this means geometrically, that the graph of f is below of all of its secants. We also apply in what follows the elementary fact that convex functions are continuous. For more details of convex functions, see [4].

Theorem 4.6. *Given a meromorphic function f, the counting function $N(r,f)$ and the characteristic function $T(r,f)$ are increasing with respect to r and convex with respect to $\log r$.*

Proof. To prove the first claim (increasing), observe that the non-integrated counting function $n(r, f)$ is trivially increasing with respect to r, and this property carries over to $N(r, f)$ in integration. Moreover, observe that the first claim also applies to $N(r, 1/(f-a))$ for all $a \in \mathbb{C}$, as $1/f(z, a)$ is a meromorphic function as well.

That $T(r, f)$ is increasing, is an immediate consequence of the Cartan theorem. In fact, given $r_1 \leq r_2$, we conclude that

$$T(r_1, f) = \frac{1}{2\pi}\int_0^{2\pi} N(r_1, e^{i\theta}, f)d\theta + \log^+|f(0)|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} N(r_2, e^{i\theta}, f)d\theta + \log^+|f(0)| = T(r_2, f),$$

provided $f(0) \neq \infty$.
A slight modification may be used to prove that $T(r, f)$ is increasing as well, if f has a pole at the origin.

To prove the convexity, let $0 < r_1 < r < r_2$ and $a \in \mathbb{C}$ be arbitrary. For notational simplicity, denote $n(t) := n(t, a, f) - n(0, a, f)$. By monotonicity of $n(t)$ and elementary estimates we obtain

$$\frac{N(r,a) - N_1(r,a)}{\log r - \log r_1} = \frac{\int_{r_1}^{r} \frac{n(t)}{t}dt + n(0,a)(\log r - \log r_1)}{\log r - \log r_1}$$

$$\leq \frac{(n(r) + n(0,a))(\log r - \log r_1)}{\log r - \log r_1} = \frac{(n(r) + n(0,a))(\log r_2 - \log r)}{\log r_2 - \log r}$$

$$= \frac{n(r)\int_{r}^{r_2} \frac{n(r)}{t}dt + n(0,a)(\log r_2 - \log r)}{\log r_2 - \log r} \leq \frac{\int_{r}^{r_2} \frac{n(t)}{t}dt + n(0,a)(\log r_2 - \log r)}{\log r_2 - \log r}$$

$$= \frac{\int_{0}^{r_2} \frac{n(t)}{t}dt + n(0,a)\log r_2}{\log r_2 - \log r} - \frac{\int_{0}^{r} \frac{n(t)}{t}dt + n(0,a)\log r}{\log r_2 - \log r} = \frac{N(r_2,a) - N(r,a)}{\log r_2 - \log r}.$$

This means that $N(r, a)$ is convex with respect to $\log r$.

Applying the preceding convexity inequality, we obtain by integration

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{N(r, e^{i\theta}) - N(r_1, e^{i\theta})}{\log r - \log r_1}d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{N(r_2, e^{i\theta}) - N(r, e^{i\theta})}{\log r_2 - \log r}d\theta.$$

By a direct application of the Cartan lemma, we find that

$$\frac{T(r,f) - T(r_1,f)}{\log r - \log r_1} = \frac{\frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})d\theta + \log^+|f(0)|}{\log r - \log r_1} - \frac{\frac{1}{2\pi}\int_0^{2\pi} N(r_1, e^{i\theta})d\theta - \log^+|f(0)|}{\log r - \log r_1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{N(r, e^{i\theta}) - N(r_1, e^{i\theta}}{\log r - \log r_1}d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{N(r_2, e^{i\theta}) - N(r, e^{i\theta})}{\log r_2 - \log r}d\theta = \frac{T(r_2,f) - T(r,f)}{\log r_2 - \log r}.$$

□

Since all convex functions are continuous, the following corollary is immediate:

Corollary 4.7. *The counting function and the characteristic function of a meromorphic function f are continuous functions of r. Therefore, the proximity function is continuous as well.*

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5. Second Main Theorem

In this section, we proceed to prove the Second Main Theorem. We first show that a finite sum of proximity functions of type $m(r, 1/(f-c)$, where $c \in \mathbb{C}$, may be expressed, essentially, in terms of $T(r, f)$ and of proximity functions of logarithmic derivatives of $f-c$. This first part towards Second Main Theorem is elementary needing, however, technical computations. Second Main Theorem itself comes out by showing that the proximity functions of logarithmic derivatives of $f-c$ are small in a certain sense to be defined later on. This phase is the deep aspect of the Nevanlinna theory of meromorphic functions.

Before proceeding to the main task of this section, we prove

Lemma 5.1. *Let $P(z) = a_n z^n + \cdots + a_0$ be a polynomial with constant coefficients and f meromorphic function. Then, for the composed function $P \circ f$, we have*

$$T(r, P(f)) = nT(r, f) + O(1).$$

Proof. Clearly, $P(f)$ has a pole at z if and only if f has a pole at this point. If the pole of f there is of multiplicity p, then the pole of $P(f)$ at this point is of multiplicity np. Therefore, summing and integrating, we obtain

$$N(r, P(f)) = nN(r, f).$$

To calculate the proximity function, recall Lemma 3.11. By this lemma,

$$\frac{1}{2}|a_n||z|^n \leq |P(z)| \leq 2|a_n||z^n|,$$

provided $|z|$ is sufficiently large, say $|z| \geq r_0$. Taking logarithms, we get

$$n \log^+ |z| - C \leq \log^+ |P(z)| \leq n \log^+ |z| + C$$

for some constant C. Now, since $\log^+ |P(z)|$ remains bounded in $|z| \leq r_0$, we may increase C so that the preceding inequality remains valid for all $z \in \mathbb{C}$. Therefore, we have

$$n \log +|f(z)| - C \leq \log^+ |P(f(z))| \leq n \log^+ |f(z)| + C.$$

By integrating over the circle $|z| = r$, we obtain

$$nm(r, f) - C \leq m(r, P(f)) \leq nm(r, f) + C,$$

meaning that

$$m(r, P(f)) = nm(r, f) + O(1).$$

Combining this with the previous asymptotic equality for the coounting function, we get the assertion. $\square$

Theorem 5.2. *Let f be a non-constant meromorphic function, and let $c_1, \ldots, c_q$, $q \geq 2$, be distinct complex numbers. Then*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - c_j}\right) \leq 2T(r, f) - N_1(r, f) + S(r, f),$$

where

$$N_1(r, f) = N(r, 1/f') + 2N(r, f) - N(r, f')$$

and

$$S(r, f) = m\left(r, \frac{f'}{f}\right) + \sum_{j=1}^{q} m\left(r, \frac{f'}{f - c_j}\right) + O(1).$$

Proof. Before starting the actual proof, it is important to understand the meaning of $N_1(r,f)$. First, suppose that f has a pole of multiplicity μ at some point. Then f' has a pole of multiplicity $\mu+1$. Therefore, summing $2\mu-(\mu+1)$ over all poles of f, and integrating logarithmically, we get $2N(r,f)-N(r,f')$. On the other hand, if f' has a zero of multiplicity $\nu-1$ at some point, then f has a multiple a-point at this point for some $a\in\mathbb{C}$, of multiplicity ν. Therefore, $N_1(r,f)$ counts multiple points of f, with multiplicity reduced by one, i.e. double points will be counted once, triple points twice, and so on.

To start the proof, we fix a polynomial

$$P(f)=\prod_{j=1}^{q}(f-c_j).$$

Then, by elementary analysis, we may find complex constants A_j so that

$$\frac{1}{P(f)}=\sum_{j=1}^{q}\frac{A_j}{f-c_j}.$$

Multiplying with f', we get

$$\frac{f'}{P(f)}=\sum_{j=1}^{q}A_j\frac{f'}{f-c_j}.$$

Recalling Lemma 3.2, we see that

$$m\left(r,\frac{f'}{P(f)}\right)=m\left(r,\sum_{j=1}^{q}A_j\frac{f'}{f-c_j}\right)\le\sum_{j=1}^{q}m\left(r,\frac{f'}{f-c_j}\right)+\sum_{j=1}^{q}\log^+|A_j|+\log q,$$

and further

$$m\left(r,\frac{1}{P(f)}\right)=m\left(\frac{1}{f'}\frac{f'}{P(f)}\right)\le m\left(r,\frac{1}{f'}\right)+\sum_{j=1}^{q}m\left(r,\frac{f'}{f-c_j}\right)+O(1).$$

Looking at the poles of $P(f)$, we immediately observe that

$$N\left(r,\frac{1}{P(f)}\right)\le\sum_{j=1}^{q}N\left(r,\frac{1}{f-c_j}\right).$$

After these preparations, we proceed to estimate $T(r,f')$. First we obtain

$$T(r,f')=m(r,f')+N(r,f')=m(r,\frac{f'}{f}f)+N(r,f')$$

$$\le m\left(r,\frac{f'}{f}\right)+m(r,f)+N(r,f)+(N(r,f')-N(r,f))$$

$$=T(r,f)+m\left(r,\frac{f'}{f}\right)+(N(r,f')-N(r,f)).$$

We then need to estimate $T(r,f')$ downwards. To this end, we apply First Main Theorem, preceding preparations and Lemma 5.1 to obtain

$$T(r,f')=m\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f'}\right)+O(1)$$

$$\geq m\left(r, \frac{1}{P(f)}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$= T\left(r, \frac{1}{P(f)}\right) - N\left(r, \frac{1}{P(f)}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$= T(r, P(f)) - N\left(r, \frac{1}{P(f)}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$\geq qT(r,f) - \sum_{j=1}^{q} N\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$= \sum_{j=1}^{q} T\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} N\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$= \sum_{j=1}^{q} m\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1).$$

Combining the two inequalities above we obtain

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$\leq T(r,f) + m\left(r, \frac{f'}{f}\right) + (N(r,f') - N(r,f)).$$

Adding now $m(r,f)$ to both sides of the preceding inequality we get

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-c_j}\right) - \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) + N\left(r, \frac{1}{f'}\right) + O(1)$$

$$\leq m(r,f) + T(r,f) + m\left(r, \frac{f'}{f}\right) + (N(r,f') - N(r,f)).$$

Shifting two terms from the left hand side to the right, and adding and subtracting $N(r,f)$ on the right hand side we finally see that

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-c_j}\right) \leq m(r,f) + N(r,f) + T(r,f) + m\left(r, \frac{f'}{f}\right)$$

$$+ \sum_{j=1}^{q} m\left(r, \frac{f'}{f-c_j}\right) - (N(r,1/f') + 2N(r,f) - N(r,f')),$$

completing the proof. □

As $N_1(r,f)$ is the preceding theorem is positive, see begin of the proof, a careful analysis of $m(r,f'/f)$, the proximity function of the logarithmic derivative of f, is needed to find out the real significance of Theorem 5.2. To start this analysis, we first prove

Theorem 5.3. *Let f be a non-constant meromorphic function such that $f(0) \neq 0, \infty$. Then*

$$m\left(r, \frac{f'(z)}{f(z)}\right) \leq 4\log^+ T(R,f) + 3\log^+ \frac{1}{R-r} + 4\log^+ R + 2\log^+ \frac{1}{r} + 4\log^+ \log^+ \frac{1}{|f(0)|} + 10$$

holds whenever $0 < r < R < \infty$.

Proof. The proof of this theorem is relatively long, and so we divide it in several distinct parts.

(1) First observe that the Poisson kernel

$$K(z,R,\theta) = \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2}$$

may be written in the form

$$K(z,R,\theta) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2},$$

where $z = re^{i\phi}$. **Exercise!** On the other hand, we also see that

$$\Re \frac{Re^{i\theta} + z}{Re^{i\theta} - z} = K(z,R,\theta).$$

Exercise!

(2) Take now a point z_0 in $|z| < R$ so that $f(z_0) \neq 0, \infty$. Then $\log f(z)$ is analytic in some neighborhood of z_0. On the other hand, denoting by a_k, resp. b_j the zeros, resp. the poles, of f, we may define

$$g(z) := \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta}| \frac{Re^{i\theta}+z}{Re^{i\theta}-z} d\theta - \sum_{|a_k|<R} \log \frac{R^2 - \overline{a_k}z}{R(z-a_k)} + \sum_{|b_j|<R} \log \frac{R^2 - \overline{b_j}z}{R(z-b_j)}.$$

As $f(z_0) \neq 0, \infty$, the two sum terms in the definition of g are analytic in some neighborhood of z_0. Moreover, the integral term may clearly be differentiated (with respect to z) over the integral, that term is analytic as well in a neighborhood of z_0. Hence, g is analytic in some neighborhood of Z_0.

(3) By Theorem 2.2, the Poisson–Jensen–Nevanlinna formula, and Part (1) of the proof, we may write

$$\log|f(z)| = \Re\log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \Re\frac{Re^{i\theta}+z}{Re^{i\theta}-z} - \sum_{|a_k|<R} \log\left|\frac{R^2 - \overline{a_k}z}{R(z-a_k)}\right| + \sum_{|b_j|<R} \log\left|\frac{R^2 - \overline{b_j}z}{R(z-b_j)}\right|.$$

Now it is immediate to see that $\Re\log f(z) \equiv \Re g(z)$. Therefore, the real parts of two functions $\log f(z)$ and $g(z)$ analytic in a neighborhood of z_0 are identical. By elementary complex analysis, the imaginary parts of these functions differ by a real constant, hence $\log f(z) = g(z) + ic$ for some $c \in \mathbb{R}$ in a neighborhood of z_0. By uniqueness of meromorphic functions, the same identity holds in the whole disc $|z| < R$. Therefore, we have now

$$\log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{Re^{i\theta}+z}{Re^{i\theta}-z} - \sum_{|a_k|<R} \log \frac{R^2 - \overline{a_k}z}{R(z-a_k)} + \sum_{|b_j|<R} \log \frac{R^2 - \overline{b_j}z}{R(z-b_j)} + ic.$$

(4) Differentiating the preceding identity we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2} - \sum_{|a_k|<R} \frac{|a_k|^2 - R^2}{(z-a_k)(R^2 - \overline{a_k}z)} + \sum_{|b_j|<R} \frac{|b_j|^2 - R^2}{(z-b_j)(R^2 - \overline{b_j}z)}.$$

We now proceed to estimate the right hand side of the preceding identity. Recalling that $|z| = r$, a simple geometric observation results in

$$\left|\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\right| \leq \frac{2R}{(R-r)^2}.$$

Similarly, by a geometric argument,

$$\left|\frac{|a_k|^2 - R^2}{(z-a_k)(R^2-\overline{a_k}z)}\right| = \frac{R(R^2-|a_k|^2)}{|R^2-\overline{a_k}z|^2}\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right|$$
$$\leq \frac{R^3}{(R^2-Rr)^2}\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right| = \frac{R}{(R-r)^2}\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right|.$$

Exactly same reasoning yields for the poles

$$\left|\frac{|b_j|^2 - R^2}{(z-b_j)(R^2-\overline{b_j}z)}\right| \leq \frac{R}{(R-r)^2}\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right|.$$

Taking modulus of $|f'(z)/f(z)|$, applying triangle inequality in the above identity, and using the preceding estimates we get

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2R}{(R-r)^2)}\left(\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\theta}||d\theta + \sum_{|a_k|<R}\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right| + \sum_{|b_j|<R}\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right|\right).$$

By Lemma 3.2(d) and the First Main Theorem (or actually the proof of Theorem 3.10),

$$\frac{1}{2\pi}\int_0^{2\pi}|\log|f(Re^{i\theta})||d\theta = m(R,f) + m(R,1/f) \leq 2T(R,f) + \log\frac{1}{|f(0)|}.$$

Therefore,

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2R}{(R-r)^2}\left(2T(R,f) + \log\frac{1}{|f(0)|} + \sum_{|a_k|<R}\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right| + \sum_{|b_j|<R}\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right|\right).$$

Taking now plus-logarithms on both sides, observe that the zero-sum, resp. the pole-sum, has at most $n(R, 1/f)$, resp. $n(R, f)$ terms, and so the right hand side has at most $n(R, f)+n(R, 1/f)+2$ sum terms altogether. Therefore, we get

$$\log^+\left|\frac{f'(z)}{f(z)}\right| \leq \log^+\frac{2R}{(R-r)^2} + \log^+ 2T(R,f) + \log^+\log^+\frac{1}{|f(0)|}$$
$$+\sum_{|a_k|<R}\log^+\left|\frac{R^2-\overline{a_k}z}{R(z-a_k)}\right| + \sum_{|b_j|<R}\log^+\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right| + \log(n(R,f)+n(R,1/f)+2).$$

(5) The next phase is to apply the Jensen formula, see the proof of Theorem 3.10,

$$T(r,1/h) = T(r,h) - \log|h(0)|$$

to the meromorphic function $h(z) := \frac{R^2-\overline{a_k}z}{R(z-a_k)}$, where $|a_k| < R$. First observe that $1/h$ is analytic in $|z| < R$. Since for each z such that $|z| = R$, we have $|h(z)| = 1$, we have $|1/h(z)| \leq 1$ for all z such that $|z| < R$ by the maximum principle. Therefore, $m(r, 1/h) = 0$ and $N(r, 1/h) = 0$. Moreover, the only pole of h in $|z| \leq r$ is at $z = a_k$, and we may apply Lemma 3.5 to conclude that $N(r, h) = \log^+ \frac{r}{a_k}$. Finally, $h(0) = R/a_k$. Hence, the Jensen formula now implies that

$$0 = m(r,h) + \log^+\frac{r}{|a_k|} - \log\frac{R}{|a_k|}.$$

Summing over all zeros a_k of f in $|z| < R$, we see that

$$\sum_{|a_k|<R} m\left(r, \frac{R^2 - \overline{a_k} z}{R(z-a_k)}\right) = \sum_{|a_k|<R} \log \frac{R}{|a_k|} - \sum_{|a_k|<R} \log^+ \frac{r}{|a_k|} = N(R, 1/f) - N(r, 1/f).$$

Of course, the same reasoning for poles as well, and we get

$$\sum_{|b_j|<R} m\left(r, \frac{R^2 - \overline{b_j} z}{R(z-b_j)}\right) = N(R, f) - N(r, f).$$

Now, integrating the expression for $\log^+ |f'(z)/f(z)|$ at the end of Part (4), and substituting the above proximity function sums into the result of integration, we get

$$m(r, f'/f) \leq 2\log 2 + \log^+ R + 2\log\frac{1}{R-r} + \log^+ T(R, f) + \log^+ \log^+ \frac{1}{|f(0)|}$$

$$+N(R, f) - N(r, f) + N(R, 1/f) - N(r, 1/f) + \log(n(R, f) + n(R, 1/f) + 2).$$

Finally, as this inequality holds for all $R > r$, we take ρ to satisfy $r < \rho < R$ and replace R by ρ in the inequality above. Therefore, we have

$$m(r, f'/f) \leq 2\log 2 + \log^+ \rho + 2\log\frac{1}{\rho - r} + \log^+ T(\rho, f) + \log^+ \log^+ \frac{1}{|f(0)|}$$

$$N(\rho, f) - N(r, f) + N(\rho, 1/f) - N(r, 1/f) + \log(n(\rho, f) - n(\rho, 1/f) + 2).$$

(6) We next proceed to estimate the quantity $n(t) := n(t, f) + n(t, 1/f)$. Denote by $N(t)$ the integrated counting function determined by $n(t)$. Then

$$N(R) \geq \int_\rho^R \frac{n(t)}{t} dt \geq n(\rho)\frac{R-\rho}{R}.$$

Therefore,

$$n(\rho) \leq \frac{R}{R-\rho} N(R) \leq \frac{R}{R-\rho}(2T(R, f) + \log^+ \frac{1}{|f(0)|}).$$

Applying now plus-logarithm in this inequality we obtain

$$\log^+(n(\rho) + 2) \leq \log^+ R + \log^+ \frac{1}{R-\rho} + \log^+ \log^+ \frac{1}{|f(0)|} + \log^+ T(R, f) + 4\log 2.$$

To obtain estimate for the integrated counting function $N(t)$, we first observe that $N(t)$ is convex with respect to $\log t$ as the sum of two convex functions, namely of $N(t, f)$ and $N(t, 1/f)$. By convexity,

$$\frac{N(\rho) - N(r)}{\log \rho - \log r} \leq \frac{N(R) - N(r)}{\log R - \log r},$$

and therefore,

$$N(\rho) - N(r) \leq \frac{\log(\rho/r)}{\log R/r} N(R).$$

Since

$$\log\frac{\rho}{r} = \int_r^\rho \frac{dt}{t} \leq \frac{\rho - r}{r}, \qquad \log\frac{R}{r} = \int_r^R \frac{dt}{t} \geq \frac{R-r}{R},$$

we finally obtain

$$N(\rho) - N(r) \leq \frac{R}{r}\frac{\rho - r}{R - r}\left(2T(R, f) + \log^+ \frac{1}{|f(0)|}\right).$$

(7) The final trick to complete the proof is now to select ρ in a suitable way, namely by taking

$$\rho = r + \frac{r(R-r)}{2R(T(R,f) + \log^+ \frac{1}{|f(0)|} + 1)}.$$

This choice implies that

$$0 < r < \rho < R,$$

$$N(\rho) - N(r) < 1,$$

$$\log^+ \frac{1}{\rho - r} \leq \log^+ \frac{1}{r} + \log^+ \frac{1}{R-r} + \log^+ R + \log^+ T(R,f) + \log^+ \log^+ \frac{1}{|f(0)|} + \log 6.$$

Moreover, $\rho - r < \frac{R-r}{2}$, hence $R - \rho = (R-r) - (\rho - r) > \frac{R-r}{2}$, and therefore,

$$\log^+ \frac{1}{R-\rho} < \log 2 + \log^+ \frac{1}{R-r}.$$

Substituting now the expressions we have obtained above into the expression of $m(r, f'/f)$ at the end of Part (5), we finally obtain

$$m(r,f'/f) \leq 4\log^+ T(R,f) + 3\log^+ \frac{1}{R-r} + 4\log^+ R + 2\log^+ \frac{1}{r} + 4\log^+ \log^+ \frac{1}{|f(0)|} + 9\log 2 + 2\log 3 + 1.$$

This completes the proof as the numerical constant on the right hand side is $< 9.44 < 10$. $\square$

Remark. Theorem 5.3 essentially remains unchanged, if $f(0) = 0$ or $f(0) = \infty$. Indeed, in both cases, we look at the Laurent expansion

$$f(z) = \sum_{j=m}^{\infty} c_j z^j, \qquad c_m \neq 0,$$

and define $g(z) = z^{-m} f(z)$. Then $g(0) \neq 0, \infty$, and so Theorem 5.3 applies to g. Since

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \frac{g'(z)}{g(z)},$$

we have

$$m(r, f'/f) \leq m(r, g'/g) + \log r + O(1).$$

Therefore, as the final outcome is a slight change in the coefficients on the right hand side of the inequality in Theorem 5.3. As we see immediately, this change is of no importance for non-rational functions f.

The essential problem to understand the real contents of Theorem 5.3 is the term $\log^+ T(R,f)$ as we don't know enough about the growth of $T(R,f)$ in terms of R. To obtain control over this, we need to apply the following important lemma due to E. Borel in 1987:

Lemma 5.4. *Let $T(r)$ be a continuous, non-decreasing function defined in the positive real axis. If for some r_0, $T(r_0) \geq 1$, then*

$$T\left(r + \frac{1}{T(r)}\right) < 2T(r) \tag{5.1}$$

outside of an exceptional set E of finite linear measure.

Proof. We may restrict ourselves to considering the interval $[r_0, \infty)$ only. If (5.1) holds for all values of r in this restricted interval, then there is nothing to be proved. So, let us denote by $E \subset [r_0, \infty)$ the set where (5.1) does not hold, i.e.

$$T\left(r + \frac{1}{T(r)}\right) \geq 2T(r) \tag{5.2}$$

for each $r \in E$. Since $T(r)$ is continuous, the set E is closed. We now define

$$r_1 := \min E, \qquad r_1' := r_1 + \frac{1}{T(r_1)},$$

and then, inductively,

$$r_n := \min E \cap [r_{n-1}', \infty), \qquad r_n' = r_n + \frac{1}{T(r_n)}$$

for each $n > 1$. It is now easy to see that each interval $[r_{n-1}', r_n) \subset [r_0, \infty) \setminus E$, i.e. the inequality (5.1) holds in each of the intervals $[r_{n-1}', r_n)$. Therefore, E is contained in the union of the intervals $[r_n, r_n']$. If the inductive process to determine the sequence (r_n) terminates after finitely many steps, then clearly E is of finite linear measure $|E|$. If the process is not finite, then the sequence (r_n) has a limit point as an increasing sequence. If the limit point is finite, then (r_n') has the same limit point, since $r_n < r_n' \leq r_{n+1}$. But then

$$r_n' - r_n = \frac{1}{T(r_n)} \geq \frac{1}{T(r_0)} \geq 1 > 0,$$

a contradiction. Therefore, $\lim_{n\to\infty} r_n = \infty$, and so $E \subset \bigcup_{n=1}^{\infty} [r_n, r_n']$. Observing first that

$$T(r_n) \geq T(r_{n-1}') = T(r_{n-1} + \frac{1}{r_{n-1}}) \geq 2T(r_{n-1}) \geq \cdots \geq 2^{n-1} T(r_1) \geq 2^{n-1},$$

we conclude that

$$|E| \leq \sum_{n=1}^{\infty} (r_n' - r_n) = \sum_{n=1}^{\infty} \frac{1}{T(r_n)} \leq \sum_{n=1}^{\infty} 2^{n-1} = 2,$$

and we are done. □

To offer the final formulation of the Second Main Theorem, recall Theorem 5.2, we need to define the order of a meromorphic function f:

Definition 5.5. The **order** of a meromorphic function f will be defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Exercise. Recall from Complex Analysis II that the order of an entire function has been defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

by using the maximum modulus of f. Show that these two definitions of order are equal for an entire function.

Exercise. Determine the order of the meromorphic function $f(z) = \sin\sqrt{z}/\sqrt{z}$.

Looking at the expression of $S(r, f)$ in Theorem 5.2, it is clearly sufficient to consider the proximity function of f'/f as the terms of type $m(r, f'/(f - c_j))$ are proximity functions of the logarithmic derivatives $(f - c_j)'/(f - c_j)$. Therefore, the key for the final formulation of the Second Main Theorem is

Lemma 5.6. *Given a non-constant meromorphic function f of finite order, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

If f is of infinite order, then

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r,f))) = O(\log T(r,f)) + O(\log r)$$

outside of a possible exceptional set of finite linear measure.

Proof. In the finite order case, we have $T(r,f) \leq r^s$ for some $s > 0$, at least for all sufficiently large r. Taking now $R = 2r$ in Theorem 5.3, we immediately obtain the claim.

Suppose now that $\rho(f) = \infty$. Let E be the exceptional set determined by the Borel lemma, lemma 5.4, and take r outside of this exceptional set. Fixing $R = r + 1/T(r,f)$, we may again apply Theorem 5.3 to obtain the claim. $\square$

This enables us now to write

Theorem 5.7. *Let f be a non-constant meromorphic function, and let $c_1, \ldots, c_q$, $q \geq 2$, be distinct complex numbers. Then*

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - c_j}\right) \leq 2T(r,f) - N_1(r,f) + S(r,f),$$

where

$$N_1(r,f) = N(r, 1/f') + 2N(r,f) - N(r,f') \geq 0$$

and

$$S(r,f) = O(\log r),$$

if f is of finite order, while

$$S(r,f) = O(\log(rT(r,f))) = O(\log T(r,f)) + O(\log r)$$

outside of a possible exceptional set of finite linear measure, if f is of infinite order.

In what follows, let $S(r,f)$ be the same quantity as in Theorem 5.7 above. In particular, this means that the estimate included in the notation $S(r,f)$ holds outside of a possible exceptional set of finite linear measure. We then easily get the following simple modification of Second Main Theorem:

Theorem 5.8. *Let f be a non-constant meromorphic function, and let $c_1, \ldots, c_q$, $q \geq 3$, be distinct number in the extended complex plane $\widehat{\mathbb{C}}$. Then we have*

$$\sum_{j=1}^{q} m(r, c_j, f) \leq 2T(r,f) - N_1(r,f) + S(r,f)$$

and

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} N(r, c_j, f) - N_1(r,f) + S(r,f).$$

Proof. Concerning the first assertion, if $c_j = \infty$ for some j, then the assertion is exactly as in Theorem 5.7, and there is nothing to prove. If $c_j \neq \infty$ for all j, then we have

$$m(r,f) + \sum_{j=1}^{q} m(r, c_j, f) \leq 2T(r,f) - N_1(r,f) + S(r,f)$$

and so

$$\sum_{j=1}^{q} m(r, c_j, f) \leq 2T(r, f) - m(r, f) - N_1(r, f) + S(r, f) \leq 2T(r, f) - N_1(r, f) + S(r, f),$$

since $m(r, f) \geq 0$. The second assertion now follows from the first one by adding $\sum_{j=1}^{q} N(r, c_j, f)$ on both sides of the inequality, and then applying the First Main Theorem. □

The following theorem, actually a corollary of Theorem 5.8, is a slight improvement of the classical Picard theorem:

Theorem 5.9. *A transcendental meromorphic function f has at most two Picard exceptional values c in $\widehat{\mathbb{C}}$, i.e., the equation $f(z) = c$ has at most finitely many solutions in the complex plane.*

Proof. Suppose, contrary to the assertion, that f has three Picard values, say a, b, c. Then f has at most finitely a-points in the complex plane, and so $N(r, a, f) = O(\log r)$. Similarly, $N(r, b, f) = O(\log r)$ and $N(r, c, f) = O(\log r)$. By Theorem 5.8 we conclude that

$$T(r, f) \leq N(r, a, f) + N(r, b, f) + N(r, c, f) - N_1(r, f) + S(r, f)$$

$$\leq O(\log r) + S(r, f) = O(\log r) + O(\log T(r, f)),$$

outside of a possible exceptional set E of finite linear measure. Dividing now this by $T(r, f)$ and letting $r \to \infty$ outside of E, we obtain for some $K > 0$,

$$1 \leq K \left(\limsup_{r\to\infty, r\notin E} \frac{\log r}{T(r, f)} + \limsup_{r\to\infty, r\notin E} \frac{\log T(r, f)}{T(r, f)} \right) = 0,$$

a contradiction. □

Remark. Since the exponential function $f(z) = e^z$ completely omits the values 0 and ∞, two Picard values may appear.

We next proceed to our final version of the Second Main Theorem by considering distinct a-points of f. In other words, each a-point will be counted only once, independently of its multiplicity. Denoting by $\overline{n}(r, a, f)$ the number of distinct a-points of f in $|z| \leq r$, and defining $\overline{n}(0, a, f) = 1$, if $f(0) = a$, and $\overline{n}(0, a, f) = 0$, $f(0) \neq a$, we may define the **integrated counting function for distinct a-points** similarly as we defined the usual integrated counting function:

$$\overline{N}(r, a, f) := \int_0^r \frac{\overline{n}(t, a, f) - \overline{n}(0, a, f)}{t} dt + \overline{n}(0, a, f) \log r.$$

Theorem 5.10. *Let f be a non-constant meromorphic function, and let $c_1, \ldots, c_q$, $q \geq 3$, be distinct numbers in the extended complex plane. Then*

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, c_j, f) + S(r, f).$$

Proof. Recall our analysis of $N_1(r, f)$ in the proof of Theorem 5.2:

$$N_1(r, f) = 2N(r, f) - N(r, f') + N(r, 1/f')$$

counts a μ-fold pole of f $\mu - 1$ times, and $N(r, 1/f')$ counts all finite multiple a-points so that a μ-fold a-point will be counted $\mu - 1$ times in N(r,1/f'). Therefore,

$$\sum_{j=1}^{q} N(r, c_j, f) - N_1(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, c_j, f).$$

Substituting this into the right hand side of the second assertion in Theorem 5.8, we obtain the inequality asserted in this theorem. □

The final notion in this section is the **deficiency**:

Definition 5.11. Let f be a transcendental meromorphic function and $a \in \widehat{\mathbb{C}}$. Then the deficiency of f with respect to a is defined by

$$\delta(a,f) := \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}.$$

Clearly, $0 \le \delta(a,f) \le 1$. If a is a Picard value of f, then $N(r,a,f) = O(\log r)$, and since f is transcendental, we have $\delta(a,f) = 1$ for any Picard value a. If $\delta(a,f) > 0$, a is said to be a deficient value of f.

Theorem 5.12. *Let f be a transcendental meromorphic function in the complex plane. Then f has at most countably many deficient values in the extended complex plane and*

$$\sum_{a\in\widehat{\mathbb{B}}} \delta(a,f) \le 2.$$

Proof. Let first $c_1, \ldots, c_q$, $q \ge 3$ be any collection of extended complex numbers. By Theorem 5.8, we have

$$\sum_{j=1}^{q} m(r,c_j,f) \le 2T(r,f) + S(r,f).$$

Dividing by $T(r,f)$ and letting $r \to \infty$ outside of the possible exceptional set E determined by $S(r,f)$, we observe that

$$\sum_{j=1}^{q} \delta(c_j,f) = \sum_{j=1}^{q} \liminf_{r\to\infty} \frac{m(r,c_j,f}{T(r,f)} \le \liminf_{r\to\infty} \sum_{j=1}^{q} \frac{m(r,c_j,f}{T(r,f)}$$

$$\le \liminf_{r\to\infty, r\notin E} \sum_{j=1}^{q} \frac{m(r,c_j,f)}{T(r,f)} \le 2 + \liminf_{r\to\infty, r\notin E} \frac{S(r,f)}{T(r,f)} = 2.$$

Therefore, for any collection of finitely many extended complex numbers $c_1, \ldots, c_q$, $\sum_{j=1}^{q} \delta(c_j,f) \le 2$. By this fact, there can be at most $2n$ extended complex numbers c_j so that $\delta(c_j,f) \ge 1/n$. Therefore, the set

$$A_n := \{a \in \widehat{\mathbb{C}}; \frac{1}{n} \le \delta(a,f) < \frac{1}{n-1}\}$$

is a finite set, having at most $2n$ elements. Therefore, the set

$$A := \{a \in \widehat{\mathbb{C}}; \delta(a,f) > 0\} = \bigcup_{n=1}^{\infty} A_n$$

is a countable set. Let us now denote $A = \{c_j; j \in \mathbb{N}\}$. Then

$$\sum_{a\in\widehat{\mathbb{C}}} \delta(a,f) = \sum_{j=1}^{\infty} \delta(c_j,f) = \lim_{q\to\infty} \sum_{j=1}^{q} \delta(c_j,f) \le 2,$$

and we are done. □

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6. Uniqueness theorems

In this section, we show a few examples of the extensive uniqueness theory of meromorphic functions, base on the use of the Nevanlinna theory. First recall that Theorem 5.2 had been proves for all non-constant meromorphic functions, hence for rational functions as well. Let now $R(z)$ be a non-constant rational function. Then it is well known that $R(z)$ may be written in form

$$R(z) = c\frac{\prod_{j=1}^{p}(z-a_j)}{\prod_{j=1}^{q}(z-b_j},$$

where c is a complex constant, and a_j, resp. b_j, are the zeros, resp. poles, of $R(z)$. But then

$$\frac{R'(z)}{R(z)} = \sum_{j=1}^{p}\frac{1}{z-a_j} - \sum_{j=1}^{q}\frac{1}{z-b_j}.$$

Therefore, $|R'(z)/R(z)| < 1$ for all z large enough. This means that $m(r, R'/R) = 0$ for all z large enough. We therefore conclude that in Theorem 5.2 is $S(r, f) = O(1)$ for all non-constant rational functions.

We now say that two meromorphic functions f_1, f_2 share an extended complex value a CM, if $f_1(z) = a$ if and only if $f_2(z) = a$ and the multiplicity of an a-point of f_1 at z is the same as the multiplicity of the a-point of f_2 at z. Moreover, we say that f_1, f_2 share an extended complex value a IM, if $f_1 = a$ if and only if $f_2 = a$, but the multiplicities can be different. Clearly, sharing CM is a subcase of sharing IM. We can now state the classical five-value theorem due to R. Nevanlinna in 1926, see [6]:

Theorem 6.1. *If two meromorphic functions f_1, f_2 share five distinct extended complex values $a_1, \ldots, a_5$ IM, then either $f_1 \equiv f_2$, or both functions are constant.*

Proof. Suppose first that one of these two functions, say f_1, is constant. Then f_2 completely omits at least four of the five values $a_1, \ldots, a_5$. This is a contradiction to the Picard theorem, Theorem 5.9, unless f_2 is a constant as well.

Therefore, we now assume that both of f_1, f_2 are non-constant meromorphic functions. Assume also, contrary to the assertion that $f_1 \neq f_2$ and that all values a_k are finite. By assumption, we have

$$\overline{N}(r, a_k, f_1) = \overline{N}(r, a_k, f_2) =: N_k(r)$$

for $k = 1, \ldots, 5$. Applying now Theorem 5.10 for f_1 and f_2, with $q = 5$ and taking $a_1, \ldots, a_5$ in place of $c_1, \ldots, c_q$, we obtain

$$3T(r, f_j) \leq \sum_{k=1}^{5} N_k(r) + S(r, f_j)$$

for $j = 1, 2$. Recalling Lemma 5.6 and what we observed above concerning rational functions, we get for $j = 1, 2$ that

$$T(r, f_j) \leq \frac{1}{3}\sum_{k=1}^{5} N_k(r) + o(T(r, f_j)),$$

where $o(T(r, f_j))$ is a quantity that when divided by $T(r, f_j)$ goes to zero as $r \to \infty$, at least outside of a possible exceptional set of finite linear measure. By the preceding inequality, we have

$$(1 + o(1))T(r, f_j) \leq \frac{1}{3}\sum_{k=1}^{5} N_k(r)$$

for $j = 1, 2$. Adding results in

$$(1 + o(1))(T(r, f_1) + T(r, f_2)) \leq \frac{2}{3} \sum_{k=1}^{5} N_k(r),$$

again outside of a possible exceptional set. Now, all a_k-points of f_1 (and of f_2 at the same time) are zero-points of $f_1 - f_2$. By the First Main Theorem,

$$\sum_{k=1}^{5} N_k(r) \leq N(r, 0, f_1 - f_2) \leq T(r, f_1 - f_2) + O(1) \leq T(r, f_1) + T(r, f_2) + O(1).$$

Combining this and the preceding inequality we observe that

$$(1 + o(1))(T(r, f_1) + T(r, f_2)) \leq \frac{2}{3}(T(r, f_1) + T(r, f_2) + O(1)).$$

This brings an immediate contradiction by letting $r \to \infty$ outside of the possible exceptional set.

Finally, if one of the values $a_k = \infty$, we may take $a \in \mathbb{C}$ to be distinct of the values a_k, and consider the functions $1/(f_1 - a)$ and $1/(f_2 - a)$ instead of f_1, f_2. Then the situation returns back to the preceding case of five finite values, and we are done. □

Remark. The number five in the previous theorem is best possible. It is sufficient to look at $f_1(z) = e^z$ and $f_2(z) = e^{-z}$. These those functions share the values $0, 1, -1, \infty$ IM, but are neither constants nor identically equal.

In what follows, we denote by $S(r, f)$ any quantity such that

$$\limsup_{r \to \infty, r \notin E} \frac{S(r, f)}{T(r, f)} = 0,$$

where E is an exceptional set of finite linear measure. In particular, observe that the $S(r, f)$ in Theorem 5.2 and later satisfies this condition, even for non-constant rational functions. Therefore, this new agreement does not bring a contradiction.
We next prove the four-value theorem, also due to Nevanlinna:

Theorem 6.2. *Suppose that two meromorphic functions f, g share four distinct extended complex values $a_1, \ldots, a_4$ CM. Then either $f \equiv g$, or there exists a Möbius transformation T so that $f = T \circ g$ and that*

$$T(a_1) = a_1, \quad T(a_2) = a_2, \quad T(a_3) = a_4, \quad T(a_4) = a_3.$$

Moreover, a_3, a_4 are values completely omitted by f, g and the double ratio $(a_1, a_2, a_3, a_4) = -1$.

Proof. Suppose that $f \not\equiv g$. By Theorem 5.10,

$$2T(r, f) - S(r, f) \leq \sum_{k=1}^{4} \overline{N}(r, a_k, f) \leq N(r, 0, f - g) \leq T(r, f - g) + O(1) \leq T(r, f) + T(r, g) + O(1).$$

Therefore, $T(r, f) \leq T(r, g) + S(r, f)$. Changing f and g, we obtain $T(r, g) \leq T(r, f) + S(r, g)$. Denote further $S(r) := \max(S(r, f), S(r, g))$, hence $S(r, f) \leq S(r)$ and $S(r, g) \leq S(r)$. Then we see from the preceding inequalities that $2T(r, f) \leq \sum_{k=1}^{4} \overline{N}(r, a_k, f) = S(r)$ and $\sum_{k=1}^{4} \overline{N}(r, a_k, f) \leq 2T(r, f) + S(r)$. Therefore, we may write

$$2T(r, f) = \sum_{k=1}^{4} \overline{N}(r, a_k, f) + S(r),$$

and similarly for g.

Applying now the double ratio theorem, Theorem 1.4, it is not difficult to see that we may find $c \neq 0, 1, \infty$ and a Möbius transformation S so that $S(a_1) = 0$, $S(a_2) = 1$, $S(a_3) = \infty$ and $S(a_4) = c$ **(Exercise)!**. Considering $S \circ f$ and $S \circ g$ instead of f and g, we see that $S \circ f$ and $S \circ g$ share the values $0, 1, \infty, c$ CM **(Exercise)!**. Therefore, we may now proceed by the original notations f, g, but assuming that the CM-shared values are $0, 1, \infty, c$. For shortness, we may use the notations $a_1, \ldots, a_4$ as well.

We now consider the following point sets in the complex plane: Let A be the set of all points z in the complex plane where either f or g takes a multiple value $d \neq 0, 1, \infty, c$, and let B be the set of all points s in the complex plane where f (and hence g as well by the CM-sharing assumption) takes a multiple value $d \in \{0, 1, \infty, c\}$. Let next $N^*(r)$ be the integrated counting function, defined in the usual way for the non-integrated counting function of the set $A \cup B$ as follows: (1) If $z^* \in A$ is a multiple value of f, resp. of g, of multiplicity k, resp. of k then z will be counted $k-1$ times; if f and g both have a multiple point at $z^* \in A$ of multiplicity k_f, resp. of k_g simultaneously, then z^* will be counted $k_f + k_g - 2$ times. (2) If $z^* \in B$ and is of multiplicity p, then, z^* will be counted $2(p-1)$ times.

We show now that $N^*(r)$ is small in the sense that $N^*(r) = S(r)$. To this end, we consider the following auxiliary function

$$\psi := \frac{f'g'(f-g)^2}{f(f-1)(f-c)g(g-1)(g-c)}.$$

The function ψ is an entire function. To show this, the possible poles of ψ must be at the a_k-points of f (and of g at the same time). As zeros, one-points and c-points have the same behavior with respect to ψ, it is sufficient to see what happens at the zeros and poles of f (and of g). Suppose first that z^* is a p-fold zero of f, hence a p-fold zero of g as well. But then $(f-g)^2$ has a $2q$-fold zero at z^*, where $q \geq p$. Then ψ has a $2q - 2 \geq 2p - 2 \geq 0$-fold zero at z^*, hence either a zero or a point of analyticity. Let then z^* be a p-fold pole of f, hence of g as well. Then $f(f-1)(f-c)g(g-1)(g-c)$ has a $6p$-fold pole at z^*. On the other hand, $f - g$ has a q-fold pole at z^* with $q \leq p$. But then ψ has a $4q + 2 - 6p \leq 2(1-p)$-fold pole at z^*. But $1 - p \leq 0$, hence z^* is a point of analyticity of ψ. Therefore, ψ is entire.

Writing now

$$\psi = \frac{ff'}{(f-1)(f-c)}\frac{g'}{g(g-1)(g-c)} - 2\frac{f'}{(f-1)(f-c)}\frac{g'}{(g-1)(g-c)} + \frac{f'}{f(f-1)(f-c)}\frac{gg'}{(g-1)(g-c)},$$

it is easy to see that ψ is a sum of products of logarithmic derivatives. Therefore, by the logarithmic derivative lemma, Lemma 5.6, we conclude that ψ is a small function in the sense that

$$T(r, \psi) = m(r, \psi) = S(r).$$

To compute $N^*(r)$, we first observe that a point $z^* \in A$ will be counted equally many times as is the multiplicity of a zero of ψ at z^*, unless f and g both have a multiple point at z^*. In this case the contribution of z^* to $N^*(r)$ is at most the same as the contribution to $N(r, 1/\psi)$ (inequality appears if $f(z^*) - g(z^*) = 0$). Concerning a point $z^* \in B$, it suffices to consider zeros and poles of f, as the one-points and c-points of f behave exactly like the zeros. But looking back at the reasoning we used to prove that ψ is entire, we observe that again z^* contributes to $N^*(r)$ less or equal as to $N(r, 1/\psi)$. Therefore, altogether, we have

$$N^*(r) \leq N(r, 1/\psi) \leq T(r, \psi) + O(1) = S(r).$$

Next observe that if $\overline{N}(r, a, f) = S(r, f)$ for three of $a = 0, 1, \infty, c$, then by Theorem 5.10, $T(r, f) = S(r, f)$, which is a contradiction. Therefore, we may assume that $\overline{N}(r, a, f) \neq S(r, f)$

for at least two of $a = 0, 1, \infty, c$. By an additional Möbius transformation, we may assume that $\overline{N}(r, f) \neq S(r, f)$ and $\overline{N}(r, 1/f) \neq S(r, f)$.

We proceed to consider the following function

$$H := \frac{f''}{f'} - \frac{g''}{g'}.$$

At simple poles of f (and of g), H has zeros by the assumption of CM-sharing. At multiple poles of f (and simultaneously of g), H is analytic. Therefore, poles of H, all simple!), appear on zeros of f' and g' which are multiple points of f or g. As these will be counted in $N^*(r)$, we have $N(r, H) = S(r)$. Moreover,

$$N(r, f) = \overline{N}(r, f) + S(r).$$

If now $H \neq 0$, then

$$\overline{N}(r, f) + S(r) \leq N(r, \frac{1}{H}) \leq T(r, H) + O(1) = m(r, H) + S(r) = S(r),$$

since H is the difference of two logarithmic derivatives. This now contradicts the assumption that $\overline{N}(r, f) \neq S(r)$, hence we have $H \equiv 0$. Therefore, by integrating twice we get $f = Ag + B$ for some complex constants A, B. Since $\overline{N}(r, 1/f) \neq S(r)$, f has zero-points (which are zero-points of g as well), we must have $B = 0$, hence $f = Ag$ for some constant A. Since $f \neq g$ by assumption, we have $A \neq 1$. But since 1 and c are shared values counting multiplicity, these values must be completely omitted by f and g. If now $f(z^*) = A$, then $g(z^*) = 1$. Since g omits 1, f must omit the value A. By continuity, $A = c$. Similarly, if $g(z^*) = 1/A$, then $f(z^*) = 1$. Since f omits 1, g must omit the value $1/A$. Therefore, $1/A = c$, and so $A = 1/A$. Since $A \neq 1$, we must have $A = -1$, hence $f = -g$, and we are done. □

Remark. A lot of work has been to understand in what sense the assumptions in Theorem 6.6 could be relaxed. The following curious example shows that CM can not be replaced by IM: Take

$$f(z) := \frac{e^z + 1}{(e^z - 1)^2}, \qquad g(z) := \frac{(e^z + 1)^2}{8(e^z - 1)}.$$

Then f and g share the values $0, 1, -1/8, \infty$, but IM. In fact, all zeros and 1-points of f are simple, while the zeros and 1-points of g are double. On the other hand, poles and $-1/8$-points of f are double, while for g they are simple. Clearly, f and g are not Möbius transformations of each other. **Exercise:** Show the claims of this remark.

We remark that the assumption 4CM of Theorem 6.6 can be relaxed to 3CM+IM, see [1] or [3], Satz 9.19, and even to 2CM+2CM, see [2]. As the case 4IM is not possible by the example above, it remains the case CM+3IM. This is an open problem.

Another typical collection of uniqueness theorems comes out by assuming certain value sharing properties between a meromorphic function f and its derivative f'. Such results may also be understood as characterizations of the exponential function. Before proceeding to this final theorem in this section, we need a couple of additional lemmas:

Lemma 6.3. *Suppose f is a non-constant meromorphic function and let $a_1, \dots, a_q$ be distinct complex numbers, $q \geq 1$. Then*

$$\sum_{j=1}^{q} m(r, \frac{1}{f - a_j}) \leq m(r, \sum_{j=1}^{q} \frac{1}{f - a_j}) + O(1).$$

Proof. If $q = 1$, the assertion is trivial. Therefore, suppose that $q \geq 2$, and let $\delta > 0$ be such that $|a_\mu - a_\nu| \geq \delta$ for all $\mu \neq \nu$. Fix $z \in \mathbb{C}$. Suppose first that $|f(z) - a_\nu| \leq \delta/(3q)$ for some ν. Then, for all $\mu \neq \nu$, by triangle inequality,

$$|f(z) - a_\mu| \geq |a_\mu - a_\nu| - |f(z) - a_\nu| \geq \delta - \delta/(3q) \geq 2\delta/3 > \delta/2.$$

Therefore, if $\mu \neq \nu$, we have

$$\frac{1}{|f(z) - a_\mu|} \leq \frac{3}{2\delta} \leq \frac{1}{2q}\frac{1}{|f(z) - a_\nu|}.$$

Therefore, by triangle inequality again,

$$\left|\sum_{\mu=1}^{q} \frac{1}{f(z) - a_\mu}\right| \geq \frac{1}{|f(z) - a_\nu|} - \sum_{\mu \neq \nu} \frac{1}{|f(z) - a_\mu|} \geq \left(1 - \frac{q-1}{2q}\right)\frac{1}{|f(z) - a_\nu|} \geq \frac{1}{2|f(z) - a_\nu|}.$$

Taking plus-logarithms on both sides, we get

$$\log^+ \left|\sum_{\mu=1}^{q} \frac{1}{f(z) - a_\mu}\right| \geq \log^+ \frac{1}{|f(z) - a_\nu|} - \log 2.$$

But then

$$\begin{aligned}\log^+ \left|\sum_{\mu=1}^{q} \frac{1}{f(z) - a_\mu}\right| &\geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z)| - a_\mu} - q\log^+ \frac{2}{\delta} - \log 2 \\ &\geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z) - a_\mu|} - q\log^+ \frac{3q}{\delta} - \log 2,\end{aligned}$$

since all terms with $\mu \neq \nu$ in the sum on the right hand side are $\leq \log^+(2/\delta)$. On the other hand, if $|f(z) - a_\mu| \geq \delta/(3q)$ for all μ, then the same inequality

$$\log^+ \left|\sum_{\mu=1}^{q} \frac{1}{f(z) - a_\mu}\right| \geq \sum_{\mu=1}^{q} \frac{1}{|f(z) - a_\mu|} - q\log^+ \frac{3q}{\delta} - \log 2$$

is true again, since the left hand side and $\log 2$ are positive, while the sum on the left hand side plus the middle term sum up to at most 0. Therefore, the inequality is always true. By integration over the circle boundary of radius r, centered at the origin, we obtain the assertion. $\square$

Lemma 6.4. *Suppose f is a non-constant meromorphic function, and let $a_1, \dots, a_q$ be distinct complex numbers, $q \geq 1$. Then*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f(z) - a_j}\right) \leq m\left(r, \frac{1}{f'}\right) + S(r, f).$$

Proof. By the preceding lemma, and the logarithmic derivative lemma, Lemma 5.6,

$$\begin{aligned}\sum_{j=1}^{q} m\left(r, \frac{1}{f(z) - a_j}\right) &\leq m\left(r, \sum_{j=1}^{q} \frac{1}{f(z) - a_j}\right) + O(1) \\ &\leq m\left(r, \frac{1}{f'}\right) + m\left(f' \sum_{j=1}^{q} \frac{1}{f(z) - a_j}\right) + O(1) \leq m\left(r, \frac{1}{f'}\right) + S(r, f).\end{aligned}$$

$\square$

Theorem 6.5. *Suppose f is a non-constant meromorphic function so that f and f' share three finite distinct complex values a, b, c IM. Then $f = f'$ and so $f(z) \equiv Ce^z$ for a constant $C \neq 0$.*

Proof. Suppose first that $f - f'$ is not constant. Using the First Main Theorem and the logarithmic derivative lemma we conclude that

$$\begin{aligned} N\left(r, \frac{1}{f-f'}\right) &\leq T(r, f-f') + O(1) = N(r, f-f') + m(r, f-f') + O(1) \\ &= N(r, f') + m\left(r, f\left(1 - \frac{f'}{f}\right)\right) + O(1) \leq N(r, f) + \overline{N}(r, f) + m(r, f) + S(r, f) \\ &= T(r, f) + \overline{N}(r, f) + S(r, f). \end{aligned}$$

Dividing the reasoning in two cases, suppose that $abc \neq 0$. Then we obtain, by the preceding inequality,

$$\overline{N}(r, a, f') + \overline{N}(r, b, f') + \overline{N}(r, c, f') \leq N\left(r, \frac{1}{f-f'}\right) \leq T(r, f) + \overline{N}(r, f) + S(r, f). \tag{6.1}$$

Moreover,

$$N(r, a, f') - \overline{N}(r, a, f') + N(r, b, f') - \overline{N}(r, b, f') + N(r, c, f') - \overline{N}(r, c, f') \leq N(r, 1/f''). \tag{6.2}$$

Applying next Lemma 6.4 to f' (and keeping in mind that all of a, b, c are non-zero), we get

$$m\left(r, \frac{1}{f'}\right) + m(r, a, f') + m(r, b, f') + m(r, c, f') \leq m\left(r, \frac{1}{f''}\right) + S(r, f).$$

adding now the preceding three inequalities (and using the First Main Theorem), we get

$$m\left(r, \frac{1}{f'}\right) + 3T(r, f') \leq T(r, f'') + T(r, f) + \overline{N}(r, f) + S(r, f). \tag{6.3}$$

On the other hand, by assumption,

$$N(r, a, f) + N(r, b, f) + N(r, c, f) \leq N\left(r, \frac{1}{f-f'}\right) \leq T(r, f) + \overline{N}(r, f) + S(r, f)$$

and, by Lemma 6.4,

$$m(r, a, f) + m(r, b, f) + m(r, c, f) \leq m\left(r, \frac{1}{f'}\right) + S(r, f).$$

Adding, and applying the First Main Theorem again, we obtain

$$3T(r, f) \leq T(r, f) + \overline{N}(r, f) + m\left(r, \frac{1}{f'}\right) + S(r, f). \tag{6.4}$$

Adding now (6.3) and (6.4), we conclude that

$$T(r, f) + 3T(r, f') \leq T(r, f'') + 2\overline{N}(r, f) + S(r, f). \tag{6.5}$$

By elementary considerations, for any meromorphic function f,

$$T(r, f') = m(r, f') + N(r, f') \leq m(r, f) + N(r, f) + \overline{N}(r, f) + S(r, f) = T(r, f) + \overline{N}(r, f) + S(r, f).$$

Since $S(r, f') = S(r, f)$, see Section 7, we obtain

$$T(r, f'') \leq T(r, f') + \overline{N}(r, f) + S(r, f).$$

Combining this with (6.5), we conclude that

$$T(r, f) + 2T(r, f') \leq 3\overline{N}(r, f) + S(r, f). \tag{6.6}$$

On the other,
$$2T(r,f') \geq 2N(r,f') = 2N(r,f) + 2\overline{N}(r,f) \geq 3\overline{N}(r,f).$$
Finally, combining this with (6.6), we get $T(r,f) = S(r,f)$, a contradiction.

Therefore, we may now assume that one of a, b, c, say $c = 0$. First observe that the inequalities (6.1) and (6.2) remain valid in this case with $c = 0$. As in the preceding case, Lemma 6.4 now implies that
$$m(r,a,f') + m(r,b,f') + m(r,1/f') \leq m(r,1/f'') + S(r,f). \tag{6.7}$$
Adding now (6.7), (6.1) and (6.2), we get
$$3T(r,f') \leq T(r,f'') + T(r,f) + \overline{N}(r,f) + S(r,f). \tag{6.8}$$
Since $ab \neq 0$, we know, as before, that a- and b-points of f are simple. Moreover, if f has a zero of multiplicity p, then the multiplicity of f' at this point is $p-1$, and $f - f'$ has a zero at this same point, again of multiplicity $p-1$. Therefore, we have
$$N(r,a,f) + N(r,b,f) + N(r,1/f') \leq N(r,1/(f-f')) \leq T(r,f) + \overline{N}(r,f) + S(r,f). \tag{6.9}$$
Since f, f' share the value 0, and the multiplicity goes down by one when considering f' instead of f, we see that
$$N(r,1/f) - N(r,1/f') = \overline{N}(r,1/f) = \overline{N}(r,1/f') \leq N(r,1/f').$$
Combining this with (6.9), we obtain
$$N(r,a,f) + N(r,b,f) + N(r,1/f) \leq T(r,f) + \overline{N}(r,f) + N(r,1/f') + S(r,f). \tag{6.10}$$
By Lemma 6.4 again, we get
$$m(r,a,f) + m(r,b,f) + m(r,1/f) \leq m(r,1/f') + S(r,f). \tag{6.11}$$
Adding now (6.8), (6.10) and (6.11), we see, after a slight simplification, that
$$T(r,f) + T(r,f') \leq 3\overline{N}(r,f) + S(r,f). \tag{6.12}$$
Therefore,
$$m(r,f) + N(r,f) + m(r,f') + N(r,f) + \overline{N}(r,f) \leq 3\overline{N}(r,f) + S(r,f),$$
hence
$$m(r,f) + m(r,f') + 2(N(r,f) - \overline{N}(r,f)) = S(r,f).$$
Writing this in the form
$$(m(r,f) + N(r,f) - \overline{N}(r,f)) + m(r,f') + (N(r,f) - \overline{N}(r,f)) = S(r,f),$$
and observing that all three terms on the left-hand side are non-negative, we conclude that
$$T(r,f) = \overline{N}(r,f) + S(r,f).$$
Moreover,
$$0 \leq m(r,f') + N(r,f) - \overline{N}(r,f) = T(r,f') - 2\overline{N}(r,f) = S(r,f),$$
hence
$$T(r,f') = 2\overline{N}(r,f) + S(r,f) = 2T(r,f) + S(r,f).$$
Finally,
$$\begin{aligned} T(r,f'') &= m(r,f') + N(r,f') + \overline{N}(r,f') = T(r,f') + \overline{N}(r,f) + S(r,f) \\ &= 3\overline{N}(r,f) + S(r,f) = 3T(r,f) + S(r,f). \end{aligned}$$

Substituting now the expressions we got for $T(r,f), T(r,f')$ and $T(r,f'')$ into (6.8), we obtain

$$6T(r,f) \leq 5T(r,f) + S(r,f),$$

a contradiction.

By the contradicting we have obtained, we conclude that $f - f'$ must be a constant. By the Picard theorem, f takes at least one of the values a, b, c, say a at a point z^*. By assumption,

$$f(z^*) - f'(z^*) = a - a = 0,$$

hence $f = f'$, and the claim has been proved. □

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7. Nevanlinna theory: more advanced results

In this section, we present three more advanced results of meromorphic functions, frequently applied in the theory of complex differential equations, namely the theorems due to Clunie, Mohon'ko and Valiron–Mohon'ko. Although the results itself are more advanced, their proofs needs nothing but elementary properties of the Nevanlinna theory. The Valiron–Mohon'ko theorem is an exception, as the proof is rather long, needing the Euclidean algorithm in addition to the Nevanlinna theory.

Before proceeding to the actual goal of this section, recall that the logarithmic derivative lemma, Lemma 5.6, may shortly be written as

$$m(r, f'/f) = S(r, f). \tag{7.1}$$

We now prove the generalized logarithmic derivative lemma as follows:

Lemma 7.1. *Given a natural number n, we have*

$$m\left(r, \frac{f^{(n)}}{f}\right) = S(r, f). \tag{7.2}$$

Proof. By Lemma 5.6, the claim is true for $n = 1$. We now proceed by the usual induction. So, suppose that the assertion has been proved for $n = k$, i.e. we have $m(r, f^{(k)}/f) = S(r, f)$. Then we get, by elementary properties of the proximity function that

$$m(r, f^{(k)}) \le m\left(r, \frac{f^{(k)}}{f}\right) + m(r, f) = m(r, f) + S(r, f).$$

If now f has a pole of multiplicity p at z_0, then $f^{(k)}$ has a pole of multiplicity $p + k \le (k+1)p$ at the same point z_0. Therefore,

$$N(r, f^{(k)}) \le (k + 1)N(r, f).$$

Adding we obtain

$$T(r, f^{(k)}) = m(r, f^{(k)}) + N(r, f^{(k)}) \le m(r, f) + (k+1)N(r, f) + S(r, f) \le (k+1)T(r, f) + S(r, f).$$

This means that

$$m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) = S(r, f^{(k)}) = S(r, f).$$

In fact, for r large enough, and outside of the possible exceptional set, we have $T(r, f^{(k)}) \le 2(k + 1)T(r, f)$, and therefore

$$\frac{S(r, f^{(k)})}{T(r, f)} \le 2(k+1)\frac{S(r, f^{(k)})}{T(r, f^{(k)})} \to 0 \tag{7.3}$$

as $r \to \infty$ outside of the possible exceptional set. But then

$$m\left(r, \frac{f^{(k+1)}}{f}\right) \le m\left(\frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$$

completing the induction. □

Remark. (1) Observe the essential meaning of (7.3):

$$S(r, f^{(k)}) = S(r, f)$$

for all natural numbers k.

(2) Observe the important inequality

$$T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f) \tag{7.4}$$

in the preceding proof. In particular, $T(r, f') \leq 2T(r, f) + S(r, f)$.

We now proceed to the Clunie lemma, which is perhaps the most cited result in the field of complex differential equations. After having proved the lemma, we give a couple of simple examples to point out the importance of this lemma.

Theorem 7.2. *Let f be a transcendental meromorphic function that satisfies an identity of the form*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f), Q(z, f)$ are polynomials in f and finitely may of its derivatives with meromorphic coefficients. Suppose that all coefficients of $P(z, f), Q(z, f)$ are small in the sense that whenever $a(z)$ is any such coefficient, then $m(r, a) = S(r, f)$. If now the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is $\leq n$, then

$$m(r, P(z, f)) = S(r, f).$$

Proof. Denoting

$$E_1(r) := \{\theta \in [0, 2\pi); |f(re^{i\theta})| < 1\}, \qquad E_2(r) := [0, 2\pi) \setminus E_1(r),$$

we may compute $m(r, P(z, f))$ in two parts as follows:

$$2\pi m(r, P(z, f)) = \int_{E_1(r)} \log^+ |P| d\theta + \int_{E_2(r)} \log^+ |P| d\theta.$$

Denoting by $\lambda = (l_0, \ldots, l_\nu)$ a multi-index, and recalling that $P(z, f)$ is a polynomial (in several variables), we may write $P(z, f)$ in the form

$$P(z, f) = \sum_{\lambda \in I} P_\lambda(z, f) = \sum_{\lambda \in I} a_\lambda(z) f^{l_0} (f')^{l_1} \cdots (f^{(\nu)})^{l_\nu}.$$

Since $|f(re^{i\theta})| < 1$ in $E_1(r)$, we obtain for all $z \in E_1(r)$ that

$$|P_\lambda(z, f)| \leq |a_\lambda(z)| \left| \frac{f'}{f} \right|^{l_1} \cdots \left| \frac{f^{(\nu)}}{f} \right|^{l_\nu}.$$

Therefore, by the generalized logarithmic derivative lemma, Lemma 7.1, and the fact that all coefficients $a_\lambda(z)$ are small in the $S(r, f)$ sense, we obtain

$$\int_{E_1(r)} \log^+ |P_\lambda(r, f)| d\theta \leq 2\pi m(r, a_\lambda + \sum_{j=1}^{\nu} 2\pi l_j m \left(r, \frac{f^{(j)}}{f} \right) = S(r, f).$$

Therefore, adding over all finitely many terms of $P(z, f)$, we get

$$\int_{E_1(r)} \log^+ |P(z, f)| d\theta \leq \sum_{\lambda \in I} \int_{E_1(r)} \log^+ |P_\lambda(z, f)| d\theta + O(1) = S(r, f).$$

To consider the complementary set $E_2(r)$, we now write

$$Q(z, f) = \sum_{\lambda \in J} Q_\lambda(z, f) = \sum_{\lambda \in J} b_\lambda(z) f^{l_0} (f')^{l_1} \cdots (f^{(\nu)})^{l_\nu}.$$

By assumption, $l_0 + \cdots + l_\nu \le n$ for all terms $P_\lambda(z,f)$, $\lambda \in J$. Therefore, for all $z \in E_2(r)$, we have

$$|P(z,f)| = \left| \frac{1}{f^n} \sum_{\lambda \in J} b_\lambda(z) f^{l_0} (f')^{l_1} \cdots (f^{(\nu)})^{l_\nu} \right| \le \sum_{\lambda \in J} |b_\lambda(z)| \left| \frac{f'}{f} \right|^{l_1} \cdots \left| \frac{f^{(\nu)}}{f} \right|^{l_\nu}.$$

Now, integrating over $E_2(r)$ and adding over all terms of $P(z,f)$, we obtain for some $K > 0$ that

$$\int_{E_2(r)} \log^+ |P(z,f)| d\theta \le \sum_{\lambda \in J} 2\pi m(r, b_\lambda) + K \sum_{j=1}^{\nu} 2\pi m\left(r, \frac{f^{(\nu)}}{f}\right) = S(r,f).$$

Adding the integrals of $|P(z,f)|$ over $E_1(r)$ and $E_2(r)$, the assertion follows. $\square$

Example. As a simple example of how to use the Clunie lemma, let us consider the Riccati differential equation

$$f' = a_0(z) + a_1(z) f + a_2(z) f^2, \qquad a_2 \ne 0,$$

with polynomial coefficients. It is not difficult to prove (but we are not proving it now) that all solutions of Riccati equation are meromorphic functions in the complex plane. Let now f be a non-rational solution. Since the coefficients $a_j(z)$, $j = 0,1,2$ are polynomials and f is non-rational, we have $m(r,a_j) = T(r,a_j) = O(\log r) = S(r,f)$. Therefore, we may apply the Clunie lemma. To this end, write the equation in the form

$$f(a_2(z) f) = f' - a_1(z) f - a_0(z).$$

Since the right hand side is of total degree one in f and its derivatives, we conclude by Clunie that

$$m(r, a_2 f) = S(r,f),$$

hence

$$m(r,f) \le m(a_2 f) + m(r, 1/a_2) = S(r,f) + O(\log r) = S(r,f).$$

If now f has finitely many poles only, then $N(r,f) = O(\log r)$, and so

$$T(r,f) = m(r,f) + N(r,f) = S(r,f) + O(\log r) = S(r,f),$$

a contradiction. Therefore, all non-rational solutions of the Riccati differential equation have infinitely many poles in the complex plane. However, this is not true for rational solutions, which may appear for suitable coefficients. For example,

$$f' = 2z - z^2 f + f^2$$

has a solution $f(z) = z^2$.

Next we prove a theorem due to A. Mohon'ko and V. Mohon'ko, see [2], Proposition 9.2.3:

Theorem 7.3. *Let f be a transcendental meromorphic function that satisfies an algebraic differential equation*

$$P(z, f, f', \ldots, f^{(n)}) = 0,$$

where P is a polynomial in all of its $n+2$ arguments. Suppose that a finite constant $c \in \mathbb{C}$ does not solve this equation, i.e. that $P(z, c, 0, \ldots, 0) \not\equiv 0$. Then

$$m\left(r, \frac{1}{f-c}\right) = S(r,f).$$

Proof. Clearly, $f-c$ satisfies another algebraic differential of the same type. Therefore, we may assume that $c=0$. Denoting

$$D(z) := P(z,0,0,\dots,0) \not\equiv 0,$$

we may write

$$P(z,f,f',\dots,f^{(n)}) = D(z) + Q(z,f,f',\dots,f^{(n)}).$$

Here

$$Q(z,f,f',\dots,f^{(n)}) = \sum_{\lambda=(j_0,\dots,j_n\in I)} Q_\lambda(z) f^{j_0}(f')^{j_1}\cdots(f^{(n)})^{j_n}$$

is a finite sum of terms with polynomial coefficients $Q_\lambda(z)$ such that $|\lambda| = j_0+\cdots+j_n \geq 1$ for all terms. To compute $m(r,1/f)$, it is sufficient to see what happens when $|f|\leq 1$, since $\log^+|1/f| = 0$ as soon as $|f|>1$. Now, if $|f|\leq 1$, then

$$\frac{1}{|f|} \leq \frac{1}{|f|^{j_0+\cdots+j_n}}$$

for all terms in Q, hence

$$\frac{1}{|f|}\left| f^{j_0}(f')^{j_1}\cdots(f^{(n)})^{j_n}\right| \leq \left|\frac{f'}{f}\right|^{j_1}\cdots\left|\frac{f^{(n)}}{f}\right|^{j_n}. \tag{7.5}$$

Since now $D(z)+Q(z,f,f',\dots,f^{(n)})$ vanishes identically, and $D(z)$ is a polynomial, elementary Nevanlinna properties imply that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &= m\left(\frac{D}{f}\frac{1}{D}\right) \leq m\left(r,\frac{D}{f}\right) + m\left(r,\frac{1}{D}\right) \\ &= m\left(r,\frac{Q}{f}\right) + m\left(r,\frac{1}{D}\right)) = S(r,f), \end{aligned}$$

since each term in Q/f is a sum of products of generalized logarithmic derivatives, see (7.5), and so Lemma 7.2 applies. □

Example. As a simple example of the Mohon'ko theorem, we consider the first Painlevé equation

$$f'' = z + 6f^2.$$

One may prove that all solutions of this equation are meromorphic functions. Let f be an arbitrary solutions of the Painlevé equation. Writing the equation in the form $f(6f) = -z + f''$, we conclude by the Clunie lemma that $m(r,f) = S(r,f)$. Denote now

$$P(z,f,f'') = z + 6f^2 - f'',$$

which vanishes identically for the solution f. Take now $c\in\mathbb{C}$. Then $P(z,c,0) = z+6c^2 \not\equiv 0$. Therefore, by the Mohon'ko theorem, $m(r,1/(f-c)) = S(r,f)$, and so, we have $m(r,c,f) = S(r,f)$ for all $c\in\widehat{\mathbb{C}}$.

We next proceed to the Valiron–Mohon'ko theorem, see [2], p. 29–34. The idea is to determine the characteristic function of the composed function $R(z,f)$, when R is rational in f with meromorphic coefficients small with respect to a meromorphic function f. Before the actual theorem, we need the following

Lemma 7.4. *Let*

$$\begin{aligned} A(z,f) &:= (\varphi_1(z)f + \cdots + \varphi_{p-1}(z)f^{p-1} + f^p)f^{p-2} \\ &= \varphi_1(z)f^{p-1} + \cdots + \varphi_{p-1}(z)f^{2p-3} + f^{2p-2}, \qquad p\geq 2, \end{aligned}$$

be a polynomial in f with meromorphic coefficients. Then there exist $u_0, \dots, u_{p-1}, q_0, \dots, q_{p-2}$ which are polynomials in $\varphi_1, \dots, \varphi_{p-1}$ with constant coefficients, such that

$$B(z,f) := u_0(z) + \cdots + u_{p-1}(z) f^{p-1}$$

satisfies the identity

$$B(z,f)^2 = A(z,f) + \sum_{j=0}^{p-2} q_j f^j.$$

Proof. See [2], p. 30–31. □

Theorem 7.5. *Let f be a meromorphic function and let*

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{j=0}^{p} a_j(z) f^j}{\sum_{j=1}^{q} b_j(z) f^j}$$

be an irreducible rational function in f with meromorphic coefficients a_j, b_j such that $T(r,a_j) = S(r,f)$ for $j = 1, \dots, p$ and $T(r,b_j) = S(r,f)$ for $j = 1, \dots, q$. Then we have for $R(z, f(z))$ that

$$T(r, R(z,f)) = dT(r,f) + S(r,f),$$

where $d = \max(p,q)$.

Proof. (1) First assume that $q = 0$. We may assume that $b_0(z) \equiv 1$. Since

$$\begin{aligned} T\left(r, \sum_{j=0}^{p} a_j f^j\right) &\le T\left(r, f\sum_{j=1}^{p} a_j f^{j-1}\right) + T(r,a_0) + O(1) \\ &\le T(r,f) + \sum_{j=1}^{p} T(r, a_j f^{p-1}) + T(r,a_0) + O(1), \end{aligned}$$

an immediate inductive argument results in

$$T\left(r, \sum_{j=0}^{p} a_j f^j\right) \le pT(r,f) + \sum_{j=0}^{p} T(r,a_j) + O(1) = pT(r,f) + S(r,f). \tag{7.6}$$

(2) Still assuming that $q = 0$, we have to prove the inequality converse to (7.6). To this end, first assume that $p = 1$. Then we have

$$T(r,f) = T\left(r, \frac{R - a_0}{a_1}\right) \le T(r,R) + S(r,f),$$

and so

$$T(r,R) = T(a_0 + a_1 f) \ge T(r,f) + S(r,f),$$

proving the case when $q = 0, p = 1$.

(3) Still keeping in the case $q = 0$, assume for induction that the assertion has been proved for all nominator polynomials $P(z,f)$ such that $\deg_f P(z,f) = s \le p-1$ so that we have

$$T(r, P(z,f)) = sT(r,f) + S(r,f). \tag{7.7}$$

To complete the inductive step, we may write

$$\frac{R - a_0}{a_p} = \frac{a_1}{a_p} f + \cdots + \frac{a_{p-1}}{a_p} f^{p-1} + f^p.$$

Now apply Lemma 7.4 to

$$A(z,f) := \frac{R-a_0}{a_p} f^{p-2} = (\varphi_1 f + \cdots + \varphi_{p-1} f^{p-1} + f^p) f^{p-2}.$$

Here now $\varphi_j := a_j/a_p$, $j = 1, \ldots, p-1$. Clearly, we have $T(r, \varphi_j) = S(r,f)$ for all $j = 1, \ldots, p-1$. Similarly, the functions $u_0, \ldots, u_{p-1}, q_0, \ldots, q_{p-2}$ determined by Lemma 7.4, all have their characteristic function of type $S(r,f)$. Therefore, by Lemma 7.4, we have

$$B(z,f)^2 = \frac{R-a_0}{a_p} f^{p-2} + \sum_{j=0}^{p-2} q_j f^j,$$

and $\deg_f B(r,f) = p-1$. Therefore, the inductive assumption (7.7) applies to $B(z,f)$ whose coefficients have, of course, their characteristic functions of type $S(z,f)$. By the inductive assumption, and by part (1) of the proof,

$$\begin{aligned} 2(p-1)T(r,f) + S(r,f) = 2T(r, B(z,f)) &= T(r, B(z,f)^2) \\ &\le (p-2)T(r,f) + T\left(r, \frac{R-a_0}{a_p}\right) + \sum_{j=0}^{p-2} T(r, q_j) + O(1) \\ &\le (p-2)T(r,f) + T(r,R) + S(r,f), \end{aligned}$$

hence

$$T(r,R) \ge pT(r,f) + S(r,f).$$

This completes the case $q = 0$.

(4) Proceeding to the general case of $q \ge 0$, the Jensen formula readily implies that $p \ge q$ may be assumed. But we may even assume that $p > q$. Indeed, suppose that $p = q$. Then consider

$$S := b_p P - a_p Q = (R - a_p b_p^{-1}) b_p Q.$$

Clearly, $\deg_f S \le p-1$. Then S/Q is irreducible. If not, then there exists a non-trivial common factor S_1, $S = S_1 S_2$, $Q = S_1 Q_2$. Then we have

$$\frac{S}{Q} = \frac{S_2}{Q_2} = b_p R - a_p = b_p \frac{P}{S_1 Q_2} - a_p.$$

But then $P = S_1 b_p^{-1}(S_2 + a_p Q_2)$, and so S_1 would be a common factor of P and Q as well, a contradiction. Since now $b_p^{-1} S/Q = R - a_p b_p^{-1}$, we see that

$$T(r,R) = T\left(r, b_p^{-1}\frac{S}{Q}\right) + S(r,f) = T\left(r, b_p \frac{Q}{S}\right) + S(r,f),$$

meaning that $p > q$ may be applied.

(5) We first prove the inequality

$$T(r,R) \le pT(r,f) + S(r,f) \tag{7.8}$$

for $p > q \ge 0$. Observe that the case $q = 0$ has been proved in part (1) above. Now make the following inductive assumption: Suppose that (7.8) has been proved for all functions $R(z,f)$ of the requited type such that $\deg_f Q = k \le q-1$, $\deg_f P = p > k$. By the division algorithm, write $R(z,f)$ in the form

$$R(z,f) = \sum_{j=0}^{p-q} c_j f^j + \frac{d_0 + \cdots + d_{q-1} f^{q-1}}{b_0 + \cdots + b_q f^q} = \sum_{j=0}^{p-q} c_j f^j + \frac{Q_1(z,f)}{Q(z,f)}.$$

By part (1) of the proof, and the present inductive assumption, we see that

$$\begin{aligned} T(r,R) &\le (p-q)T(r,f) + T(r,Q_1/Q) + S(r,f) = (p-q)T(r,f) + T(r,Q/Q_1) + S(r,f) \\ &\le (p-q)T(r,f) + qT(r,f) + S(r,f) = pT(r,f) + S(r,f). \end{aligned}$$

(6) Therefore, it remains to prove the inequality converse to (7.8), i.e.

$$T(r,R) \ge pT(r,f) + S(r,f) \tag{7.9}$$

for $p > q \ge 0$. Again, we may assume that $q > 0$. By the Euclidean algorithm, there exist two polynomials $U(z,f)$, $V(z,f)$ in f with meromorphic coefficients small in the $S(r,f)$-sense again so that

$$P(z,f)U(z,f) + Q(z,f)V(z,f) \equiv 1.$$

Denote $s :=\deg_f U$, $t :=\deg_f V$. Since $p > q$, we must have $t > s$. Clearly,

$$T\left(r, \frac{Q}{P} + \frac{U}{V}\right) = T\left(r, \frac{1}{PV}\right) = T(r,PV) + O(1) = (p+t)T(r,f) + S(r,f). \tag{7.10}$$

Since $t > s$, we also see that

$$T\left(r, \frac{Q}{P} + \frac{U}{V}\right) \le T\left(r, \frac{P}{Q}\right) + T\left(r, \frac{V}{U}\right) + O(1) \le T(r,R) + tT(r,f) + S(r,f). \tag{7.11}$$

Combining (7.10) and (7.11), we immediately obtain the required final inequality (7.9). □

Remark. Observe that the induction process in parts (3) and (5) actually contains a two-fold induction: We fist have to assume that p, resp. q, is just two, and then the induction immediately works. Assuming then $p = 3$, resp. $q = 3$, we may start with the induction assumption for $p - 1 = 2$, resp. $q - 1 = 2$, proving this case. Then we may continue inductively over all positive integer values of p, resp. q.

References

[1] Hayman, W., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[2] Laine, I., *Nevanlinna Theory and Complex Differential Equations*, W. de Gruyter, Berlin–New York, 1993.

8. Starting complex differential equations

In this section, we show how Nevanlinna theory may be applied to obtain information of the behavior of solutions of differential equations in the complex plane. Observe that we restrict ourselves to considering situations where a solution under consideration solves the differential equation in the whole complex plane. Since we are not going to prove this fact here, we always have to assume that the solution in question is meromorphic (entire) in $\mathbb{C}$. Observe, however, that the standard local existence theorem of solutions of first order differential equations of type $f' = F(z,f)$ carries over from the real axis case to the complex plane word by word. One needs only to replace absolute values by moduli. Therefore, it not difficult to prove, for example, that all solutions of linear differential equations

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0$$

with entire coefficients are entire functions as well.

The following theorem, originally due to Malmquist in 1913, was first proved, without relying on the Nevanlinna theory. In early 1930's, Yosida showed how this theorem may be easily proved, provided Nevanlinna theory will be invoked. Subsequent improvements have been made, for example, by Steinmetz, v. Rieth and Y. He (et IL).

Theorem 8.1. *Let $R(z,f)$ be rational in both arguments, irreducible in f. If the differential equation $f' = R(z,f)$ admits a transcendental meromorphic solution in the complex plane, then the equation reduces into*

$$f' = a_0(z) + a_1(z)f + a_2(z)f^2,$$

where at least one of the rational coefficients $a_j(z)$, $j = 0,1,2$, does not vanish identically.

Proof. Let d be the degree of $R(z,f)$ with respect to f. By the Valiron–Mohon'ko theorem, Theorem 7.5, and the remark (2) following Lemma 7.1, we immediately conclude that

$$dT(r,f) + S(r,f) = T(r,R(z,f)) = T(r,f') \leq 2T(r,f) + S(r,f),$$

hence $d \leq 2$. Therefore, the equation reduces into

$$f' = \frac{a_0(z) + a_1(z)f + a_2(z)f^2}{b_0(z) + b_1(z)f + b_2(z)f^2}.$$

To show that this reduced form still reduces to a polynomial in f, substitute $f = \alpha + 1/w$ with $\alpha \neq 0$. This results in

$$w' = Q(z,f) := -w^2 \frac{a_0 + (2\alpha a_0 + a_1)w + (\alpha^2 a_0 + \alpha a_1 + a_2)w^2}{b_0 + (2\alpha b_0 + b_1)w + (\alpha^2 b_0 + \alpha b_1 + b_2)w^2}.$$

This is of the same type as the original equation. By the first part of the proof, $\deg_w Q(z,f) \leq 2$ as well. But this may be the case only, if

$$\alpha^2 a_0 + \alpha a_1 + a_2 = 2\alpha a_0 + a_1 = 0$$

or

$$2\alpha b_0 + b_1 = b_0 = 0.$$

Since at least one of the coefficients a_0, a_1, a_2 does not vanish identically, we may find α so that the first of these two alternatives is not satisfied. But then $b_0 = b_1 = 0$, and the assertion follows. □

By the preceding theorem, the most interesting cases to consider solutions of differential equations in the whole complex planes are, apparently, linear differential equations and the Riccati differential equation.

As for the linear differential equations, the first order case is not too interesting, as this case may be explicitly solved. Therefore, let us first look at the second order case

$$f'' + a(z)f' + b(z)f = 0 \tag{8.1}$$

with entire coefficients. It is not difficult to prove (but will be omitted here) that all solutions of equation (8.1) are entire function in the complex plane. More precisely, all local solutions of this equation may be continued analytically over the whole complex plane, and the continuation still satisfies the same equation. moreover, it is elementary to see that substituting $f := g\exp(\int(-\frac{1}{2}a(\zeta)d\zeta$, then g satisfies a second order linear differential equation of type

$$f'' + A(z)f = 0, \tag{8.2}$$

where $A(z)$ is an entire function. If $A(z)$ is a polynomial of degree n, then it can be proved that all non-trivial solutions of (8.2) are entire functions of order $(n+2)/2$. However, we don't have tools available to prove this, as we have not the Wiman–Valiron theory available. Therefore, we proceed to consider (8.2), while assuming that $A(z)$ is a transcendental entire function, i.e. not a polynomial.

Proposition 8.2. *If $A(z)$ is transcendental entire, then all solutions of (8.2) are of infinite order.*

Proof. Suppose, contrary to the assertion, that f is a solution of finite order. By Remark on p. 51, $\rho(f') \leq \rho(f) < \infty$. Therefore, by Lemma 5.6,

$$m(r, f''/f) \leq m(r, f''/f') + m(r, f'/f) = O(\log r) + O(\log r) = O(\log r).$$

Now, writing (8.2) in the form $A = -f''/f$, we get

$$T(r, A) = m(r, A) = m(r, f''/f) = O(\log r).$$

By Theorem 4.4, A is rational, hence a polynomial, a contradiction. □

Let now f_1, f_2 be two linearly independent solutions of (8.2), Let $E := f_1 f_2$ be their product and let $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ be their Wronskian determinant. Differentiating the Wronskian, we immediately observe that $W(f_1, f_2)$ is constant. If the constant $= 0$, then f_1 is a multiple of f_2, contradicting the lilnear independence. Therefore, we may assume form now on that $W(f_1, f_2) = 1$. This implies that

$$\left(\frac{f_2}{f_1}\right)' = \frac{W(f_1, f_2)}{f_1^2} = \frac{1}{f_1^2},$$

and so

$$\frac{f_2'}{f_2} - \frac{f_1'}{f_1} = \frac{(f_2/f_1)'}{f_2/f_1} = \frac{f_1}{f_2}\frac{1}{f_1^2} = \frac{1}{f_1 f_2} = \frac{1}{E}.$$

Since the left hand side here is a difference of logarithmic derivatives, all poles there are simple. Therefore, E must have simple zeros only, meaning that a zero of f_1 never can be a zero of f_2. Now, from $E = f_1 f_2$ we obtain

$$\frac{f_2'}{f_2} + \frac{f_1'}{f_1} = \frac{E'}{E}.$$

Adding and subtracting, we see that

$$2\frac{f_2'}{f_2} = \frac{1}{E} + \frac{E'}{E}, \quad 2\frac{f_1'}{f_1} = -\frac{1}{E} + \frac{E'}{E}.$$

Now, it is elementary to see that

$$\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2,$$

hence

$$2\left(\frac{f_2'}{f_2}\right)' = 2\frac{f''}{f} - \frac{1}{2}\left(2\frac{f_2'}{f_2}\right)^2 = -2A - \frac{1}{2}\left(\frac{1}{E}\right)^2 - \frac{1}{2}\left(\frac{E'}{E}\right)^2 - \frac{1}{E}\frac{E'}{E}.$$

But now, by differentiation,

$$2\left(\frac{f_2'}{f_2}\right)' = \left(\frac{1}{E}\right)' + \left(\frac{E'}{E}\right)' = -\frac{1}{E}\frac{E'}{E} + \frac{E''}{E} - \left(\frac{E'}{E}\right)^2.$$

Equating the two expressions we have for $2(f_2'/f_2)'$ and simplifying, we get

$$4A = \left(\frac{E'}{E}\right)^2 - \left(\frac{1}{E}\right)^2 - 2\frac{E''}{E}, \tag{8.3}$$

which may also be written in the form

$$4AE^2 = (E')^2 - 1 - 2EE''.$$

Differentiating this last identity, we also get

$$E''' + 4AE' + 2A'E = 0.$$

We now give a few scattered results concerning the zeros of solutions of (8.2). Before these theorems, we first prove

Lemma 8.3. *Given two monotone increasing real-valued functions g, h in the positive real axis such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure, then, for each $\alpha > 1$, $g(r) \leq h(\alpha r)$ for all r sufficiently large.*

Proof. Denote $\sigma := \int_E dr$, and choose $r_0 := \sigma/(\alpha - 1)$. Then for each $r > r_0$, $[r, \alpha r] \setminus E \neq \emptyset$. Therefore, taking t in this non-empty set, we observe that

$$g(r) \leq g(t) \leq h(t) \leq h(\alpha r).$$

□

Theorem 8.4. *Suppose the coefficient $A(z)$ in (8.2) is of finite order of growth $\rho \notin \mathbb{N}$. Then, given two linearly independent solutions f_1, f_2, we have $\max(\lambda(f_1), \lambda(f_2)) \geq \rho$ for the exponent of convergences of their zero sequences.*

Proof. Recall first, from Complex Analysis II that

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log n(r, 1/f)}{\log r} = \inf\left(\alpha > 0; \int_0^\infty \frac{n(t, 1/f)dt}{t^{\alpha+1}} < \infty\right)$$

for all entire (in fact: meromorphic) functions.

Denoting $E = f_1 f_2$, we first observe that the inequality

$$\lambda(f_j) \leq \lambda(E)$$

is trivial for $j = 1, 2$, hence

$$\max(\lambda(f_1), \lambda(f_2)) \leq \lambda(E).$$

To prove the converse inequality, recall that the zero-sequences of f_1 and f_2 have no common points. Therefore, $n(r, 1/E) = n(r, 1/f_1) + n(r, 1/f_2)$. We may assume that $\lambda(E) > 0$, since otherwise there is nothing to prove. Since now $\int_0^\infty \frac{n(t,1/E}{t^{\lambda(E)+1-\varepsilon}}$ diverges for all $\varepsilon > 0$, we see that at least one of the integrals $\int_0^\infty \frac{n(t,1/f_j)}{t^{\lambda(E)+1-\varepsilon}}$, $j = 1, 2$ diverges. But this means that

$$\max(\lambda(f_1), \lambda(f_2)) \geq \lambda(E) - \varepsilon,$$

hence

$$\max(\lambda(f_1), \lambda(f_2)) = \lambda(E). \tag{8.4}$$

Assume now that we have

$$\max(\lambda(f_1), \lambda(f_2)) = \lambda(E) < \rho.$$

Taking $\lambda(E) < \beta < \rho$, we know that $\int_0^\infty \frac{n(t,1/E)}{t^{\beta+1}}$ converges. Therefore,

$$N(r, 1/E) = \int_0^r \frac{n(t,1/E)}{t} dt = \int_0^r t^\beta \frac{n(t,1/E)}{t^{\beta+1}} dt \le r^\beta \int_0^\infty \frac{n(t,1/E)}{t^{\beta+1}} dt = O(r^\beta). \tag{8.5}$$

Writing now (8.3) in the form

$$E^2 = \left((E'/E)^2 - 2E''/E - 4A\right)^{-1},$$

and applying elementary properties of Nevanlinna functions, we conclude that

$$T(r, E) = O\left(N(r, 1/E) + T(r, A)\right) + S(r, E) \tag{8.6}$$

outside of a possible exceptional set of finite linear measure. By Lemma 8.3,

$$T(r, E) \le K(N(r, 1/E) + T(r, A)) \le K r^{\rho+\varepsilon}$$

for all r sufficiently large, hence $\rho(E) \le \rho$. On the other hand, by elementary order considerations, $\rho(E) \ge \rho$, hence $\rho(E) = \rho$. Since ρ is not an integer, we have $\lambda(E) = \rho(E) = \rho$ by Complex Analysis II, a contradiction. $\square$

Remark. A well known conjecture is that if there are two linearly independent solutions f_1, f_2 of (8.2) such that

$$\max(\lambda(f_1), \lambda(f_2)) < \infty,$$

then $\rho(A)$ must be a natural number, or infinity. This has **not** been proved yet.

Theorem 8.5. *Equation (8.2), with $A(z)$ entire, admits two linearly independent solutions each having no zeros in the complex plane if and only if $A(z)$ may be represented in the form*

$$-4A(z) = h'(z)^2 + \varphi'(z)^2 - 2\varphi''(z),$$

where φ is a non-constant entire function and h is a primitive function of $\exp\varphi$.

Proof. Take first a non-constant entire function φ arbitrarily, and let h be a primitive of $\exp\varphi$. Moreover, denote $g := -\frac{1}{2}(h + \varphi)$. Then, an elementary computation shows that the two linearly independent functions $f_1 := e^g$ and $f_2 := e^{g+h}$ both satisfy the differential equation

$$f'' - \frac{1}{4}(h'(z)^2 + \varphi'(z)^2 - 2\varphi''(z))f = 0.$$

To prove the converse part, suppose that (8.2) admits two linearly independent zero-free solutions f_1, f_2, and denote $g = f_1/f_2$. Obviously, g is an entire function having no zeros in the complex plane. Similarly as in the proof of Proposition 8.2, we have

$$g' = -\frac{W(f_1, f_2)}{f_2^2} = -\frac{1}{f_2^2}.$$

Therefor, we may compute what is called the Schwarzian derivative of g:

$$S_g := \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 = -2\left(\frac{f_2'}{f_2}\right)' - 2\left(\frac{f_2'}{f_2}\right)^2 = -2\frac{f_2''}{f_2} = 2A.$$

On the other hand, since g is an entire function with no zeros, there exists an entire function h so that $g = e^h$, see Complex Analysis II. Therefore,

$$\frac{g''}{g'} = \frac{h''}{h'} + h',$$

hence

$$A = \frac{1}{2}S_g = \frac{1}{2}S_h - \frac{1}{4}(h')^2.$$

Let now z_0 be a zero of h', with the corresponding Taylor expansion

$$h'(z) = c_\alpha(z - z_0)^\alpha + \cdots .$$

Then z_0 is a pole g''/g', and one can see that its Laurent expansion around z_0 would be

$$\frac{g''(z)}{g'(z)} = \alpha(z - z_0)^{-1} + \cdots .$$

Now, making use of the definition of the Schwarzian derivative above, we obtain

$$2S_g(z) = -(2\alpha + \alpha^2)(z - z_0)^{-2} + \cdots .$$

This means that $4A = S_g$ has a pole at z_0, a contradiction since A was assumed to be entire. Therefore, h' is an entire function having no zeros in the complex plane, and so there exists an entire function φ so that $h' = \exp\varphi$. $\square$

Example. Take $\varphi(z) = z$. Then $\varphi(z) \equiv 1$, $h(z) = e^z$. By the preceding theorem, equation

$$f'' - \frac{1}{4}(1 + e^{2z})f = 0$$

admits two zero-free linearly independent solutions. Indeed, by the preceding proof, we see that

$$f_1(z) = e^{-\frac{1}{2}(z+e^z)}, \quad f_2 = e^{-\frac{1}{2}(z-e^z)}.$$

Similarly, taking $\varphi(z) = z + e^z$, we have $\varphi'(z) = 1 + e^z$, $\varphi''(z) = e^z$, $h(z) = \int e^{z+e^z} = e^{e^z}$. Therefore, equation

$$f'' - \frac{1}{4}\left(e^{2(z+e^z)} + e^{2z} + 1\right) f = 0$$

admits two linearly independent zero-free solutions, namely

$$f_1(z) = e^{-\frac{1}{2}(z+e^z+e^{e^z})}, \quad f_2(z) = e^{-\frac{1}{2}(z+e^z-e^{e^z})}.$$

Theorem 8.6. *Suppose that equation (8.2), $A(z)$ transcendental entire, admits two linearly independent solutions f_1, f_2 such that $\max(\lambda(f_1), \lambda(f_2)) < \infty$. Let then f an arbitrary solution of (8.2) such that f is not of the form αf_1 or αf_2, $\alpha \in \mathbb{C} \setminus \{0\}$. Then $\lambda(f) = \infty$.*

Proof. Suppose, contrary to the assertion that $\lambda(f) < \infty$ for such a solution of (8.2). Observe that f, f_1 are linearly independent. Denote $E := f_1 f_2$ and $F := f f_1$. Then we have $\lambda(E) < \infty$, $\lambda(F) < \infty$, see (8.4). Therefore, combining (8.5) and (8.6), we conclude that, for some finite $\beta > 0$,

$$T(r, E) = O(T(r, A) + r^\beta)$$

and

$$T(r, F) = O(T(r, A) + r^\beta)$$

outside of a possible exceptional set of finite linear measure.

Since all solutions of (8.2) form a two-dimensional linear space (with complex scalars), we find non-zero constants α_1, α_2 so that $f = \alpha_1 f_1 + \alpha_2 f_2$. Hence,

$$F = f f_1 = (\alpha_1 f_1 + \alpha_2 f_2) f_1 = \alpha_1 f_1^2 + \alpha_2 E.$$

But then

$$2T(r, f_1) = T(r, \alpha_1 f_1^2) + O(1) = T(r, F - \alpha_2 E) \le O(T(r, A) + r^\beta)$$
$$= O(m(r, A) + r^\beta) = O\left(m\left(r, \frac{f_1''}{f_1}\right) + r^\beta\right) = O(S(r, f_1) + r^\beta)$$

outside of a possible exceptional set of finite linear measure. But then $T(r, f_1) = O(r^\beta)$, again outside of a possible exceptional set. By Lemma 8.3, we conclude that $T(r, f_1) = O(r^\beta)$ holds for all r sufficiently large. Then we have, obviously, that $\rho(f_1) \le \beta < \infty$, a contradiction. □

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