Functional Analysis
Exercises 1/2009

1. Let (M, d) be a compact metric space and let

$$\mathcal{C}_{\mathbf{F}}(M) := \{\, f : M \to \mathbf{F} : f \text{ continuous} \,\}.$$

Show that $\|f\| := \sup\{\, |f(x)| : x \in M \,\}$ defines a norm on $\mathcal{C}_{\mathbf{F}}(M)$. (Hint! You are free to use the fact that elements in $\mathcal{C}_{\mathbf{F}}(M)$ are bounded functions.)

2. Prove the ℓ^p-version of the Minkowski inequality: Let $(a_n), (b_n) \in \ell^p$ with $1 < p < \infty$. Show that

$$\|(a_n) + (b_n)\|_p \le \|(a_n)\|_p + \|(b_n)\|_p.$$

(Hint. Use ℓ^p-version of the Hölder inequality, compare with DiBenedetto: Real Analysis, p. 224.)

3. Let $x_1, \ldots, x_k \in \mathbf{F}$, $y_1, \ldots, y_k \in \mathbf{F}$ and let $1 < p < \infty$. Show that

$$\sum_{j=1}^{k} |x_j y_j| \le \left(\sum_{j=1}^{k} |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{k} |y_j|^q \right)^{\frac{1}{q}}$$

for $q = \frac{p}{p-1}$.

4. Let X be a n-dimensional vector space over $\mathbf{F}$ with the basis $\{\, e_1, \ldots, e_n \,\}$. Show that

$$\|x\| := \left(\sum_{j=1}^{n} |\lambda_j|^2 \right)^{\frac{1}{2}}, \quad x = \sum_{j=1}^{n} \lambda_j e_j,$$

defines a norm on X.

5. Let $(X, \|\cdot\|)$ be a normed space such that $\lim_{n\to\infty} x_n = x$. Prove that

(i) $\lim_{i\to\infty} x_{n_i} = x$ for any subsequence (x_{n_i});

(ii) (x_n) is a Cauchy sequence.

6. Let $(X, \|\cdot\|)$ be a normed space. Prove that

(i) $|\|x\| - \|y\|| \le \|x - y\|$ for all $x, y \in X$;

(ii) $\lim_{n\to\infty} \|x_n\| = \|x\|$ if $\lim_{n\to\infty} x_n = x$.