Practice exam

Applied Mathematics and Physics in Programming ID00CS50-3003

Mathematics

The exam will probably have this kind of questions. The difficulty level can be about like this. The questions can still be modified.

1. Concepts about differential equations.

   (a) Which one of the following equations has order 3?

   $$y' + y^3 = x, \quad 4y'' + xy' + \sin(x)y = 3, \quad \boxed{y''' + x^2 y = \frac{1}{x}}.$$

   (b) Which one of the following equations is linear?

   $$y' + 2xy = \frac{1}{2x+3y}, \quad \boxed{y' + \sin(x)y = e^x}, \quad y'' + yy' + 2xy = 3.$$

   (c) Which one of the following equations is homogeneous?

   $$y' + y + x - 3 = 0, \quad \boxed{y' + \sin(x)y = 0}, \quad y'' + y' = x - y.$$

2. Show that $y = \frac{1}{1-x}$ is a particular solution for $y' = y^2$. Solution The derivative is

   $$y' = \frac{d}{dx}(1-x)^{-1} = -1 \cdot (1-x)^{-1-1}\frac{d}{dx}(1-x) = -1 \cdot (1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2} = y^2$$

3. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -30)$ (that is, satisfies $x = -3$, $y = -30$)? Solution We have $y(x) = y = \frac{4}{3}x^3 + C$. Set $x = -3$ and $y(-3) = -30$ to obtain

   $$-30 = \frac{4}{3}(-3)^3 + C,$$

   that is

   $$-30 = -36 + C.$$

   We have $C = 6$. The desired solution is $y(x) = y = \frac{4}{3}x^3 + 6$.

4. Solve

   $$y' + \frac{y}{x} = x^2$$

   by following the instructions.

   (a) Identify $p(x)$ and $q(x)$. Solution $p(x) = \frac{1}{x}$ and $q(x) = x^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Solution $\int p(x)dx = \ln(x)$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Solution $\mu(x) = x$ and $\frac{1}{\mu(x)} = \frac{1}{x}$

(d) Calculate $\int \mu(x)q(x)dx$. Solution $\frac{x^4}{4}$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Solution $y(x) = \frac{C}{x} + \frac{x^3}{4}$

5. Let $f(x) = \frac{1}{4}x^2$ for $-\pi \leq x \leq \pi$ and let $f(x)$ be periodic with period 2π. It's Fourier series is

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \cos(nx). \tag{1}$$

(a) Find the Fourier series of the 2π periodic function $g(x)$ for which

$$g(x) = \frac{x}{2}, \quad \text{when} \quad -\pi \leq x \leq \pi.$$

(b) Is $f(x)$ or $g(x)$ odd?
(c) Is $f(x)$ or $g(x)$ even?

Solution We have $Df(x) = g(x)$. Therefore, differentiate equation (1) on both sides to obtain

$$g(x) = -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin(nx).$$

6. Calculate by hand the discrete Fourier transform of $[2, 3]$. In other words, calculate by hand `fft`$([2, 3])$. Solution `fft`$([2, 3]) = [5, -1]$

Formulas

Differentiation and integration

Differentiation	Integration
$Dx^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
$De^x = e^x$	$\int e^x dx = e^x + C$
$Db^x = b^x \ln(b)$	$\int b^x dx = \frac{b^x}{\ln(b)}$
$D\ln(x) = \frac{1}{x}$	
$D\ln\|x\| = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$D\log_a(x) = \frac{1}{x\ln(a)}$	
$D\log_a\|x\| = \frac{1}{x\ln(a)}$	
$D\sin(x) = \cos(x)$	$\int \cos(x) dx = \sin(x) + C$
$D\cos(x) = -\sin(x)$	$\int \sin(x) dx = -\cos(x) + C$
$D\tan(x) = 1 + \tan^2(x)$	$\int 1 + \tan^2(x) dx = \tan(x) + C$
$Dx\ln(x) - x = \ln(x)$	$\int \ln(x) dx = x\ln(x) - x + C$
$D\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C$
$D\arccos(x) = \frac{1}{-\sqrt{1-x^2}}$	$\int \frac{1}{-\sqrt{1-x^2}} = \arccos(x) + C$
$D\arctan(x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} = \arctan(x) + C$
$D\sinh(x) = \cosh(x)$	
$D\cosh(x) = \sinh(x)$	
$D\tanh(x) = \frac{1}{\cosh^2(x)}$	

Differentiation	Integration
$Df(g(x)) = f'(g(x))g'(x)$	$\int f(g(x))g'(x)dx = f(g(x)) + C$
Special cases	
$D\ln(g(x)) = \frac{g'(x)}{g(x)}$	$\int \frac{g'(x)}{g(x)} dx = \ln(g(x)) + C$
$De^{g(x)} = e^{g(x)}g'(x)$	$\int g'(x)e^{g(x)} dx = e^{g(x)} + C$
$Dfg = f'g + fg'$	$\int f'g dx = fg - \int fg' dx$
$D(f/g) = (gf' - fg')/g^2$	

Solution formula

The solution of

$$y' + p(x)y = q(x)$$

is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$