Conformal mappings map small circles approximately to small circles

Let D be a domain and let $f : D \to \mathbb{C}$ be analytic. Let $a \in D$ such that $f'(a) \neq 0$. Then f can be represented as

$$f(z) = f(a) + f'(a)(z - a) + g(z)(z - a)$$

where g is an analytic map satisfying $g(z) \to 0$ as $z \to a$. Therefore

$$f(z) \approx f(a) + f'(a)(z - a)$$

when $z \approx a$. A small circle $D(a, r)$ with center a and radius r will be approximately mapped to a small circle $D(A, R)$ where $A = f(a)$ and $R = r|f'(a)|$. The mapping f magnifies the lengths near a with a factor $|f'(a)|$. The area will be magnified by a factor $|f'(a)|^2$.

Cauchy-Riemann equations

The number $|f'(a)|^2$ is the Jacobian determinant of f at a. To see this, let $z = x + iy$ and let u and v be the real and imaginary part of f respectively. Now $f(x, y) = u(x, y) + iv(x, y)$ and its Jacobian determinant is

$$J_f(a) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x,$$

where the subscripts denote partial derivatives. Because f is differentiable, the limits

$$\lim_{h \to 0} \frac{u(x + h, y) - u(x, y)}{h} + i\frac{v(x + h, y) - v(x, y)}{h} = u_x + iv_x$$

and

$$\lim_{h \to 0} \frac{u(x, y + h) - u(x, y)}{ih} + i\frac{v(x, y + h) - v(x, y + h)}{ih} = -iu_y + v_y$$

are equal to $f'(a)$. We obtain the Cauchy-Riemann equations

$$\begin{cases} u_x & = v_y \\ v_x & = -u_y. \end{cases}$$

Hence, the Jacobian satisfies

$$J_f(a) = u_x v_y - u_y v_x = u_x^2 + v_x^2 = |u_x + iv_x|^2 = |f'(a)|^2.$$

Real and imaginary parts are harmonic functions

Taking partial derivatives in the Cauchy-Riemann equations we obtain

$$\begin{cases} (u_x)_x &= (v_y)_x \\ (v_x)_y &= -(u_y)_y. \end{cases}$$

Because u and v have continuous partial derivatives of arbitrary order, we can change the order of differentiation and $(v_y)_x = (v_x)_y$. Consequently,

$$u_{xx} + u_{yy} = 0,$$

that is, u is a harmonic function.

Moebius transformations

Consider the mapping

$$f(z) = \frac{az+b}{cz+d}$$

where the denominator is not identically zero. Such mapping is called a Moebius transformation or a linear fractional transformation.

If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$ is a linear map, that is, a combination of a multiplication and a translation.

If $c \neq 0$, then

$$\frac{az+b}{cz+d} = \frac{acz + ad + bc - ad}{acz + ad} \cdot \frac{a}{c} = \frac{a}{c}\left(1 - \frac{ad - bc}{acz + ad}\right)$$

Hence, function f is a composition of multiplications, translations, and inversion $z \mapsto \frac{1}{z}$. Moreover, $f(z)$ is not constant whenever $ad - bc \neq 0$. In this case, f is a conformal map from $\hat{C}$ to $\hat{C}$.

Because multiplications, translations and inversion $z \mapsto \frac{1}{z}$ map circles to circles, so does f.

The cross-ratio

Define the cross-ratio as

$$cr(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

The cross-ratio is invariant under Moebius transformations. That is, we have $cr(T(z_1), T(z_2), T(z_3), T(z_4)) = cr(z_1, z_2, z_3, z_4)$ for

$$T(z) = \frac{az+b}{cz+d}.$$

The fact can be seen with direct calculation.

Mappings from $\mathbb{D}$ to $\mathbb{C}$

Let $\mathbb{D}$ be the unit disc of the complex plane. Let $f : D \to C$ be locally univalent with $f(0) = 0$ and $f'(0) = 1$. There are several criteria, when f is univalent.

By the Becker univalence criterion, if

$$|zP_f(z)|(1 - |z|^2) < 1, \quad z \in D,$$

then f is univalent. Here

$$P_f(z) = \frac{f''(z)}{f'(z)} = (\log f'(z))'$$

is the pre-Schwarzian derivative of f.

Let

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

be the Schwarzian derivative of f. The Schwarzian derivative of any Moebius map is zero. We have $S_T(z) = 0$ for $T(z) = \frac{az+b}{cz+d}$.

By the Nehari univalence criterion, if

$$|S_f(z)|(1 - |z|^2)^2 < 1, \quad z \in D,$$

then f is univalent.

Simple univalent maps

Mapping $f : D \to C$,

$$f(z) = z + \frac{z^2}{2}$$

is univalent. Namely, let $z, w \in D$ and consider

$$f(z) - f(w) = (z - w)\left(1 + \frac{z + w}{2}\right).$$

Since $\frac{z+w}{2} \in D$, having $f(z) = f(w)$ implies $z = w$.

Similarly,

$$f(z) = z + \frac{z^n}{n}$$

is univalent for any $n \in N$.

Let $a \in D$ and let $f : D \to C$,

$$f(z) = \frac{a - z}{1 - \bar{a}z}.$$

If $z \in \partial D$, then $\frac{1}{z} = \bar{z}$ and

$$\frac{a - z}{1 - \bar{a}z} = \frac{1}{z}\frac{a - z}{\frac{1}{z} - \bar{a}} = \bar{z}\frac{a - z}{\bar{a} - \bar{z}} \in \partial D.$$

Hence f maps ∂D to itself.

Let $z \in D$. Now $f(z) = a$. Therefore f maps D to itself.

Function f is its own inverse.