Sisällys

1 Tämän monisteen kuvaus

Tässä monisteessa on matematiikan kurssien tehtäviä esimerkeillä höystettynä.

2 Sov: fysiikka (2.5op)

2.1 SI-yksiköt

1. Kirjoita käyttäen SI-yksiköitä:
   (a) 3 in, Ratkaisu: 0,0762 m
   (b) 1,4 lbs, Ratkaisu: 0,635 kg
   (c) 2,7 min, Ratkaisu: 162 s
   (d) 19,3 $\mathrm{g/cm^3}$, Ratkaisu: 19300 $\mathrm{kg/m^3}$
   (e) 5 km/h, Ratkaisu: 1,39 m/s
   (f) 327 kuutiotuumaa, Ratkaisu: 0,00536 $\mathrm{m^3}$
   (g) 10 leiviskää per jumpru. Ratkaisu: $1{,}04 \cdot 10^6\ \mathrm{kg/m^3}$
2. Muunna
   (a) 30 nm metreiksi, Ratkaisu: $3 \cdot 10^{-8}$ m
   (b) 150 μg milligrammoiksi, Ratkaisu: 0,15 mg
   (c) 20 cm/ns metreiksi sekunnissa. Ratkaisu: $2 \cdot 10^8$ m/s
3. Ilmaise lyhyemmin käyttäen sekä etuliitteitä että tieteellistä merkintätapaa (kymmenen potensseja):
   (a) 125000 m, Ratkaisu: $1{,}25 \cdot 10^5\ \mathrm{m} = 0{,}125\ \mathrm{Mm} = 125\ \mathrm{km}$ (

(b) 0,0000015 s. Ratkaisu: $1{,}5 \cdot 10^{-6}\,\mathrm{s} = 1{,}5\,\mu\mathrm{s}$

4. Jos mittaustulokseksi on annettu

   (a) 34 mm ± 2 mm, Ratkaisu: 32 mm - 36 mm

   (b) 15,14 m/s ± 2%, Ratkaisu: 14,84 m/s - 15,44 m/s

   (c) 4,367(5) Ω, Ratkaisu: 4,362 Ω - 4,372 Ω

   (d) 7,4 W, Ratkaisu: 7,35 W - 7,45 W

   miltä väliltä voit odottaa todellisen arvon löytyvän?

2.2 merkitsevät numerot

5. Oletetaan, että sinulla on suorakulmion mallinen metallilevy. Mittaat sen pituudeksi viivoittimella 12 mm ja leveydeksi mikrometriruuvilla 5,98 mm. Huomaa, että mittaustulosten epätarkkuudet on nyt ilmaistu merkitsevien numeroiden avulla. Laske ja ilmaise vastaukset seuraaviin kysymyksiin oikealla tarkkuudella.

   (a) Mikä on suorakulmion pinta-ala? Ratkaisu: $72\,\mathrm{mm}^2$

   (b) Mikä on suorakulmion piiri eli kaikkien sivujen summa? Ratkaisu: 36 mm

   (c) Mikä on pituuden ja leveyden suhde? Ratkaisu: 2,0

   (d) Mikä on pituuden ja leveyden erotus? Ratkaisu: 6 mm

6. Vastuksen resistanssia R mitattaessa saatiin seuraavat tulokset:

   473 Ω, 468 Ω, 469 Ω, 471 Ω, 475 Ω, 469 Ω.

   Laske arvio $\overline{R}$ todelliselle resistanssille. Ilmaise tulos oikealla tarkkuudella. Ratkaisu: 471 Ω.

7. Kirjassa mainitaan, että yksi tapa arvioida mittausvirhettä on laskea datajoukon vaihteluvälin pituuden puolikas. Siis yleiselle suureelle x

   $\Delta x \approx \frac{x_{\max} - x_{\min}}{2}$,

   missä $x_{\max}$ on suurin mitattu arvo ja $x_{\min}$ pienin mitattu arvo.

   Laske tätä menetelmää käyttäen edellisen tehtävän vastusmittausten virhe ΔR. Ratkaisu: 3,5 Ω.

   Huom. Näin laskemalla saadaan virheelle yleensä liian suuri arvo. Luennolla esitellään hieman parempi tapa.

2.3 liikelaskuja

8. Pyöräilijä ajaa suoralla tiellä. Minuutin ajan sen keskinopeus on 8,5 m/s. Kuinka pitkän matkan pyöräilijä tuona aikana etenee?

9. Orava juoksee ensin suoraan 20 metrin matkan 5 sekunnissa. Sitten se pysähtyy, kääntyy ympäri ja juoksee 10 metriä takaisin 3 sekunnissa.

   (a) Mikä on oravan keskinopeus? Ratkaisu: 1,25m/s

   (b) Mikä on oravan keskivauhti? Ratkaisu: 3,75m/s

10. Kappaleen paikka ajan funktiona on $x(t) = 10t - 2t^2$. Tässä paikka on ilmaistu metreissä ja aika sekunneissa.

    (a) Mikä on kappaleen paikka ajanhetkellä $t = 1$? Ratkaisu: $x(1) = 8$

    (b) Millä kahdella ajanhetkellä kappale on origossa? Ratkaisu: $t = 0$ ja $t = 5$

    (c) Mikä on kappaleen keskinopeus ajanhetkien $t = 1$ ja $t = 3$ välillä? Ratkaisu: 2

    (d) Laske kappaleen hetkellinen nopeus $v(t) = \frac{dx}{dt}(t)$. Käytä matematiikan puolella opittuja derivoimissääntöjä. Ratkaisu:
    $x'(t) = 10 - 4t$

(e) Laske kappaleen hetkellinen kiihtyvyys $a(t) = \frac{dv}{dt}(t)$. Käytä jälleen matematiikan puolella opittuja derivoimissääntöjä. Ratkaisu: $x''(t) = 10$

11. Sadan metrin pikajuoksussa juoksija lähtee liikkeelle levosta ja kiihdyttää ensin vakiokiihtyvyydellä 2 sekunnin ajan kunnes hänen nopeutensa on 10 m/s. Loput matkasta hän juoksee tällä saavuttamallaan vakionopeudella.

    (a) Mikä on juoksijan kiihtyvyys ensimmäisen kahden sekunnin aikana? Ratkaisu: $5\text{m}/s^2$

    (b) Mikä on juoksijan keskinopeus kiihdytysvaiheen aikana? Ratkaisu: 5m/s

    (c) Kuinka pitkän matkan juoksija etenee kiihdytysvaiheen aikana? Ratkaisu: 10m

    (d) Mikä on juoksijan aika maalissa? Ratkaisu: 11s

2.4 vino heittoliike

12. Ammutaan tykillä tasaisella maalla maanpinnan tasolta 60° kulmaan alkunopeudella 100 m/s. Unohdetaan ilmanvastus. Voit käyttää putoamiskiihtyvyydelle arvoa $g = 10\,\text{m/s}^2$.

    (a) Mitkä ovat alussa nopeuden vaaka- ja pystysuorat komponentit v_{0x} ja v_{0y}? Ratkaisu: $v_{0x} = 50\,\text{m/s}$; $v_{0y} \approx 86{,}60\,\text{m/s}$

    (b) Kauanko kestää, että ammus on saavuttanut maksimikorkeutensa? Ratkaisu: 8,66 s

    (c) Mikä on ammuksen maksimikorkeus? Ratkaisu: 375 m

    (d) Kuinka pitkälle ammus lentää? Ratkaisu: 866 m

Vihje. Vaakasuunnassa tasainen liike $\begin{cases} v_x(t) &= v_{0x} \\ x(t) &= v_{0x}t \end{cases}$

Pystysuunnassa tasaisesti kiihtyvä liike $\begin{cases} v_y(t) &= v_0 - gt \\ y(t) &= y_0 + v_0 t - \frac{1}{2}gt^2 \end{cases}$

Luennolla mainitut kaavat

(a) $v_{0x} = v_0 \cos(\alpha)$, $v_{0y} = v_0 \sin(\alpha)$

(b) ehdosta $v_y(t) = 0$ saadaan $t_{\text{LAKI}} = \frac{v_{0y}}{g}$

(c) sijoita t_{LAKI} korkeuden $y(t)$ kaavaan

(d) sijoita $t_{\text{LAKI}} = 2t_{\text{LAKI}}$ matkan $x(t)$ kaavaan

13. Kappaleeseen, jonka massa on $m = 2\,\text{kg}$, kohdistuu kaksi voimaa, $\mathbf{F}_1 = 3\,\text{N}\hat{\mathbf{i}} - 10\,\text{N}\hat{\mathbf{j}}$ ja $\mathbf{F}_2 = -1\,\text{N}\hat{\mathbf{i}} + 6\,\text{N}\hat{\mathbf{j}}$. Mikä on kappaleen kiihtyvyys? Ratkaisu: $\mathbf{a} = 1\,\text{m/s}^2\hat{\mathbf{i}} - 2\,\text{m/s}^2\hat{\mathbf{j}}$

14. Kappaleeseen (massa 100 kg) kohdistuu vakiovoima, jonka suuruus on 50 N. Oletetaan, että kappale lähtee liikkeelle levosta. Kuinka kauan kestää, että kappale on liikkunut 25 metrin matkan? Mikä on tuolloin kappaleen nopeus? Ratkaisu: 10 s; 5 m/s

2.5 voima

15. Laatikko, jonka massa on 40 kg, liukuu jäällä. Kun sille annetaan alkunopeus 10 m/s, se pysähtyy tasaisesti hidastuen kitkan vaikutuksesta viidessä sekunnissa. Kuinka suuri kitkavoima laatikkoon vaikuttaa? Entä mikä on laatikon ja jään välinen liikekitkakerroin? Käytä putoamiskiihtyvyydelle arvoa $g = 10\,\text{m/s}^2$. Ratkaisu: $F_\mu^k = 80\,\text{N}$, $\mu_k = 0{,}2$.

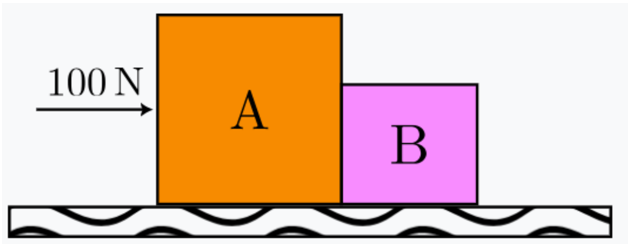
16. Putoamiskiihtyvyyden g suuruus Maassa on $9{,}81\,\text{m/s}^2$ ja Kuussa $1{,}62\,\text{m/s}^2$. Oletetaan, että golfpallon paino Maassa on 0,441 N.

    (a) Mikä on golfpallon massa Maassa? Ratkaisu: 45,0 g

    (b) Mikä on golfpallon massa Kuussa? Ratkaisu: 45,0 g

(c) Mikä on golfpallon paino Kuussa? Ratkaisu: 0,0728 N

17. Kaksi laatikkoa makaa vieri vieressä vaakasuoralla kitkattomalla pinnalla. Laatikkoon A kohdistetaan 100 N:n pinnan suuntainen ulkoinen voima oheisen kuvan mukaisesti:



Laatikon A massa on $m_A = 20\,\mathrm{kg}$ ja laatikon B puolestaan $m_B = 5\,\mathrm{kg}$. Kuinka suuren voiman laatikko A kohdistaa laatikkoon B? Mikä on laatikoiden kiihtyvyys? Ratkaisu: $F_{A\to B} = 20\,\mathrm{N}$, $a = 4\,\mathrm{m/s^2}$

3 Applied mathematics ... in programming (2.5op)

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Rules for differentiation and integration were discussed in the course "Calculus and Physics in Data Analysis ID00CS42-3003". Let's recall them.

3.1 Recall differentiation

1. Differentiate
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^2 + 3$ Ratkaisu: $2x$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}x^3 + x$ Ratkaisu: $3x^2 + 1$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(2x)$ Ratkaisu: $2\cos(2x)$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x}e^{3x}$ Ratkaisu: $3e^{3x}$
   (e) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x)$ Ratkaisu: $-4\sin(4x)$
   (f) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x)$ Ratkaisu: $\frac{1}{x}$

2. Often, it is smart to modify the expression of the function before differentiation. Differentiate
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2}$ Ratkaisu: $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}4^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln(4)}$ Ratkaisu: $\ln(4)e^{x\ln(4)} = \ln(4)4^x$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(2) + \ln(x)$ Ratkaisu: $\frac{1}{x}$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x}\log_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\ln(2)}\ln(x)$ Ratkaisu: $\frac{1}{x\ln(2)}$
   (e) $\frac{\mathrm{d}}{\mathrm{d}x}x^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln(x)}$ Ratkaisu: $e^{\ln(x)}(\ln(x) + 1)$, that is, $x^x(\ln(x) + 1)$

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^3\sin(2x)$ Ratkaisu: $3x^2 \cdot \sin(2x) + x^3 \cdot 2\cos(2x)$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x)e^{3x}$ Ratkaisu: $-4\sin(4x) \cdot e^{3x} + \cos(4x) \cdot 3e^{3x}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}x\ln(x) - x$ Ratkaisu: $\ln(x)$
   (d) $\sin(x)e^{-x}$ Ratkaisu: $\cos(x) \cdot e^{-x} + \sin(x) \cdot (-1)e^{-x}$

4. A quotient is differentiated by the rule $(f/g)' = (gf' - fg')/g^2$. Differentiate
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{x}$ Ratkaisu: $\frac{x\cdot\cos(x) - \sin(x)\cdot 1}{x^2}$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{\cos(x)}$ Ratkaisu: $\frac{1}{\cos^2(x)} = 1 + \tan^2(x) = \sec^2(x)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{e^x}$ Ratkaisu: $\frac{e^x\cos(x)-\sin(x)e^x}{e^{2x}} = \frac{\cos(x)-\sin(x)}{e^x}$

5. A composed function can be differentiated by the rule $\frac{\mathrm{d}}{\mathrm{d}x}f(g(x))) = f'(g(x))g'(x)$. Differentiate

   (a) $\sin(x^2)$ Ratkaisu: $2x\cos(x^2)$

   (b) $\sin(1/x)$ Ratkaisu: $-\frac{1}{x^2}\cos(1/x)$

   (c) e^{x^2} Ratkaisu: $2xe^{x^2}$

   (d) $e^{\sin(x)}$ Ratkaisu: $e^{\sin(x)}\cos(x)$

6. An inverse function can be differentiated by the rule

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Differentiate $\arcsin(x)$ Ratkaisu: Denote $f(x) = \sin(x)$ which implies $f'(x) = \cos(x)$ and $f^{-1}(x) = \arcsin(x)$. By the given formula, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}.$$

In the denominator, we have composed cos and arcsin and it looks complicated. However, we can express $\cos(x) = \sqrt{1-(\sin(x))^2}$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-(\sin(\arcsin(x)))^2}} = \frac{1}{\sqrt{1-x^2}}.$$

3.2 Recall integration

7. Integration is the inverse operation to differentiation. Because $\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$, we have $\int 3x^2 \mathrm{d}x = x^3 + C$, where C is a constant. Integrate

   (a) $\int \cos(x)\mathrm{d}x$ Ratkaisu: $\sin(x) + C$

   (b) $\int -\sin(x)\mathrm{d}x$ Ratkaisu: $\cos(x) + C$

   (c) $\int \sin(x)\mathrm{d}x$ Ratkaisu: $-\cos(x) + C$

8. A monomial can be integraded by the rule $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$. Integrate

   (a) $\int x^3 \mathrm{d}x$ Ratkaisu: $\frac{1}{4}x^4 + C$

   (b) $\int x^{1/2} \mathrm{d}x$ Ratkaisu: $\frac{1}{(3/2)}x^{3/2} + C = \frac{2}{3}x^{3/2} + C$

9. A power of a function can be integrated by the rule $\int f'(x)f(x)^n \mathrm{d}x = \frac{1}{n+1}f(x)^{n+1} + C$. Integrate

   (a) $\int \cos(x)(\sin(x))^3 \mathrm{d}x$ Ratkaisu: $\frac{1}{4}(\sin(x))^4 + C$

   (b) $\int \cos(2x)(\sin(2x))^3 \mathrm{d}x = \frac{1}{8}(\sin(2x))^4 + C$

10. Recall $\int \frac{1}{x}\mathrm{d}x = \ln(x) + C$. Integrate $\int \frac{2}{x}\mathrm{d}x$. Ratkaisu: $2\ln(x) + C$

11. Recall that $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln(f(x)) + C$. Integrate

    (a) $\int \frac{1}{3x+1}\mathrm{d}x$ Ratkaisu: $\frac{1}{3}\ln(3x+1) + C$

    (b) $\int \frac{2x}{1+x^2}\mathrm{d}x$ Ratkaisu: $\ln(1+x^2) + C$

    (c) $\int \tan(x)\mathrm{d}x$. (Express the tangent in terms of sine and cosine.) Ratkaisu: $-\ln(\cos(x)) + C$

3.3 Basics of differential equations

Ratkaisu: Topics

- order
- showing that a given function is a solution by substituting
- finding the value of the constant C

Basics of differential equations are discussed in the videos “week1video1.mov” and “week1video2.mov”, which can be found from the OneDrive folder.

Let’s study a function $y(x)$. To make the notation easier, we often write just y instead of $y(x)$. (Sometimes the variable can be t or some other variable.)

If we differentiate y, we obtain the first derivative y'. If we differentiate again, we obtain the second derivative y''.

An ordinary differential equation (an ODE) is an equation in terms of y and its derivatives. The order of the differential equation is the largerst order of differentiation.

12. Find the order of the following differential equations [3, 4.1.1]

    (a) $y' + y = 4y^2$ Ratkaisu: Function y can be found in three terms. In the first one y' is differentiated once, so its order of differentiation is 1. The second (y) and the third (y^2) are not differentiated, order is 0. The equation is of order 1.

    (b) $(y')^2 = y' + 2y$ Ratkaisu: Order is 1.

    (c) $y''' + y''y' = 3x^2$ Ratkaisu: The highest derivative is of order 3. The equation has order 3.

    (d) $y' = y'' + 3t^2$ Ratkaisu: Order is 2.

    (e) $\frac{dy}{dt} = t$ Ratkaisu: This can be written as $y'(t) = t$ or $y' = t$. The order is 1.

    (f) $\frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^4$ Ratkaisu: This can be written $y' + y'' = 3x^4$. The order is 2.

    (g) $\left(\frac{dy}{dt}\right)^2 + 8\frac{dy}{dt} + 3y = 4t$ Ratkaisu: This can be written $(y')^2 + 8y' + 3y = 4t$. The order is 1.

13. Show that the functions y are solutions to the corresponding differential equations. [?, 4.1.8–4.1.17] Vihje. It is enough to calculate the derivative and to substitute it to the equation. **(Probably in the Exam, there is a question like one of (a)–(i).)**

    (a) Show that $y = x^3/3$ is a particular solution for $y' = x^2$. Ratkaisu: We have $y = x^3/3$. By differentiating, we obtain $y' = 3x^2/3 = x^2$. We see that the equation $y' = x^2$ is satisfied.

    (b) Show that $y = 2e^{-x} + x - 1$ is a particular solution for $y' = x - y$.

    (c) Show that $y = e^{3x} - e^x/2$ is a particular solution for $y' = 3y + e^x$.

    (d) Show that $y = \frac{1}{1-x}$ is a particular solution for $y' = y^2$.

    (e) Show that $y = e^{x^2/2}$ is a particular solution for $y' = xy$.

    (f) Show that $y = 4 + \ln(x)$ is a particular solution for $xy' = 1$.

    (g) Show that $y = 3 - x + x\ln(x)$ is a particular solution for $y' = \ln(x)$.

    (h) Show that $y = e^x + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$ is a particular solution for $y' = \cos(x) + y$.

    (i) Show that $y = \pi e^{-\cos(x)}$ is a particular solution for $y' = y\sin(x)$.

    Ratkaisu: is not available yet. Need 1 hour to write it down nicely.

14. From [3, 4.1.18]. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -30)$ (that is, satisfies $x = -3$, $y = -30$)? **(In the exam, there will be one initial value problem. The difficulty is not clear yet.)** Ratkaisu: We have $y(x) = y = \frac{4}{3}x^3 + C$. Set $x = -3$ and $y(-3) = -30$ to obtain

$$-30 = \frac{4}{3}(-3)^3 + C,$$

that is

$$-30 = -36 + C.$$

We have $C = 6$. The desired solution is $y(x) = y = \frac{4}{3}x^3 + 6$.

3.4 Integrable equations

15. From [3, 4.1.22]. The general solution of the differential equation $y' = (2xy)^2$ is $y(x) = -\frac{3}{C+4x^3}$. Which particular solution passes through the point $(1, -0.5)$? Vihje. We have $-0.5 = y(1) = \frac{3}{C+4(1)^3}$. Solve C.

- An integrable diff. equation is of the form $y' = f(x)$ and can be integrated on both sides to obtain $y = \int f(x)dx$. Equations (a)-(e) are integrable.
- A separable diff. equation can be rearranged so that it is integrable. Equations (f)-(g) are separable.

16. From [3, 4.1.28–4.1.37]. Find the general solution for the following ODE's.
    (a) $y' = 3x + e^x$ Ratkaisu: By integrating on both sides, we obtain $y = \int 3x + e^x dx$. The integral of the right hand side is $\int 3x + e^x dx = 3x^2/2 + e^x + C$. The general solution of the equation is therefore $y = 3x^2/2 + e^x + C$.
    (b) $y' = \ln(x) + \tan(x)$
    (c) $y' = \sin(x)e^{\cos(x)}$
    (d) $y' = 4^x$
    (e) $y' = 2t\sqrt{t^2 + 16}$
    (f) $y' = y$ Vihje. Dividing by y we obtain $\frac{y'}{y} = 1$ which can be integrated on both sides. Ratkaisu: We obtained $\frac{y'(x)}{y(x)} = 1$. By integrating we obtain $\ln(y(x) = x + C$. By taking the exponential on both sides, we obtain $e^{\ln(y(x))} = e^{x+C} = e^C e^x$. Because the exponential and logarithm are inverse functions, they cancel out, and we obtain $y(x) = e^C e^x$. Here e^C is just a constant, write $e^C = A$. We obtain $y(x) = Ae^x$.
    (g) $y' = \frac{y}{x}$ Vihje. Dividing by y, we obtain $\frac{y'}{y} = \frac{1}{x}$ which can be integrated on both sides.

3.5 Separable equations

Ratkaisu:

- The diff. equations in the next problem are separable. That is, they can be reorganized and then integrated on both sides. They are called initial value problems, because an initial value is given and can be used to solve the integration constant C.

17. Solve the initial value problems

    (a) $y' = 5x$, $y(0) = 1$ Ratkaisu: General solution of the equation is $y(x) = \frac{5}{2}x + C$. The solution of the initial value problem is $y(x) = \frac{5}{2}x + 1$.

    (b) $y' = \frac{3\sin(x)}{y}$, $y(\pi) = 4$ Ratkaisu: General solution of the equation is $y(x) = \sqrt{C - 6\cos(x)}$, initial condition gives $C = 10$, solution of the initial value problem is $y(x) = \sqrt{10 - 6\cos(x)}$

    (c) $y^2 y' = e^{-x}$, $y(0) = 0$ Ratkaisu: General solution of the equation is $y(x) = 3^{1/3}(C - e^{-x})^{1/3}$, initial condition gives $C = 0$, solution of the initial value problem is $y(x) = -3^{1/3}e^{-x/3}$

Ratkaisu:

- A differential equation is linear if y and its derivatives are multiplied with just functions of x. There must not be powers (e.g. $(y')^7$), products (e.g. yy'' or function expressions of y (e.g. $\cos(y')$) or its derivatives.

18. Are the following differential equations linear?

    (a) $y' + 3y = \sin(x)$ Ratkaisu: Yes, the equation is linear.

    (b) $y' + x^2 y = 0$ Ratkaisu: Yes, the equation is linear.

    (c) $y' + xy^2 = 0$ Ratkaisu: No, the equation is not linear due to the term xy^2 which has a power of y.

    (d) $\sin(y') + 3y = 3x$ Ratkaisu: No, the equation is not linear, because of $\sin(y')$.

Ratkaisu:

- The next equations are "linear first order homogeneous equations". **Relax.** The equations are also separable, they can be solved by reorganizing and integrating on both sides.

19. Solve the homogeneous equations

    (a) $y' + 7y = 0$ Ratkaisu: Rearranged $y'/y = -7$ yields with integration $\ln(y) = -7x + C$ giving $y(x) = Ae^{-7x}$, where C and A are constant.

    (b) $y' + \cos(x)y = 0$ Ratkaisu: Rearranged $y'/y = -\cos(x)$ yields $y(x) = Ce^{\sin(x)}$

    (c) $xy' + 3x^2 y = 0$ with initial condition $y(0) = 2$ Ratkaisu: Dividing by x, we have $y' + 3xy = 0$, rearranged to $y'/y = -3x$. Integrating on both sides, we get finally $y(x) = Ce^{-\frac{3x^2}{2}}$. Solution of the initial value problem is $y(x) = 2e^{-\frac{3x^2}{2}}$

3.6 Solution formula for first order linear equations

Ratkaisu: It is a happy thing that for the equation $y' + p(x)y = q(x)$ there exists a solution formula

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu = e^{\int p(x)dx}.$$

In case of differential equations, solution formulas are very rare.

Let's discuss this more.

* * *

First order linear differential equation is of the form

$$a(x)y'(x) + b(x)y(x) = c(x).$$

By dividing with $a(x)$, we obtain the standard form

$$y'(x) + \underbrace{\frac{b(x)}{a(x)}}_{p(x)} y(x) = \underbrace{\frac{c(x)}{a(x)}}_{q(x)},$$

that is

$$y' + p(x)y = q(x).$$

In the book [3, pp. 411–413] it is explained how this equation can be solved.

(a) Identify $p(x)$ and $q(x)$.

(b) Calculate $\int p(x)dx$. Don't add a constant C yet.

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$

(d) Calculate $\int \mu(x)q(x)dx$.

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

Example. Let's solve $y' + 2xy = x$.

1. We have $p(x) = 2x$ and $q(x) = x$.
2. We have $\int p(x)dx = \int 2xdx = x^2$. (We don't add C.)
3. We have $\mu(x) = e^{x^2}$ and $\frac{1}{\mu(x)} = \frac{1}{e^{x^2}} = e^{-x^2}$.
4. We have
$$\int \mu(x)q(x)dx = \frac{1}{2} \int 2xe^{x^2} dx = \frac{1}{2} e^{x^2}.$$
5. The solution is
$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx = Ce^{-x^2} + e^{-x^2} \frac{1}{2} e^{x^2}$$

that is

$$y(x) = Ce^{-x^2} + \frac{1}{2}.$$

20. **Exercise. (Possibly an Exam question.)**

Solve

$$y' + \frac{y}{x} = x^2$$

by following the instructions.

(a) Identify $p(x)$ and $q(x)$. Ratkaisu: $p(x) = \frac{1}{x}$ and $q(x) = x^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Ratkaisu: $\int p(x)dx = \ln(x)$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Ratkaisu: $\mu(x) = x$ and $\frac{1}{\mu(x)} = \frac{1}{x}$

(d) Calculate $\int \mu(x)q(x)dx$. Ratkaisu: $\frac{x^4}{4}$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Ratkaisu: $y(x) = \frac{C}{x} + \frac{x^3}{4}$

21. **Exercise.** Solve

$$y'(x) + \tan(x)y = (\cos(x))^2.$$

(a) Identify $p(x)$ and $q(x)$. Ratkaisu: $p(x) = \tan(x)$ and $q(x) = (\cos(x))^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Ratkaisu: In an earlier exercise, it was found that $\int p(x)dx = -\ln(\cos(x)) = \ln \frac{1}{\cos(x)}$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Ratkaisu: $\mu(x) = \frac{1}{\cos(x)}$ and $\frac{1}{\mu(x)} = \cos(x)$

(d) Calculate $\int \mu(x)q(x)dx$. Ratkaisu: $\sin(x)$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Ratkaisu: $y(x) = C\cos(x) + \cos(x)\sin(x)$

22. **Exercise.** Write $y' = 3y + 2$ in standard form $y' + p(x)y = q(x)$ and solve by same instructions as above. That is, use the solution formula

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Ratkaisu: $y = Ce^{3x} - \frac{2}{3}$

23. **Exercise.** Solve $y' = 2y - x^2$. Ratkaisu: $y = Cx^3 + 6x^2$

(a) Identify $p(x)$ and $q(x)$. Ratkaisu: The equation is

$$y' - 2y = -x^2$$

so $p(x) = -2$ and $q(x) = -x^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Ratkaisu: $\int p(x)dx = -2x$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Ratkaisu: $\mu(x) = e^{-2x}$ and $\frac{1}{\mu(x)} = e^{2x}$

(d) Calculate $\int \mu(x)q(x)dx$. Ratkaisu: We obtain

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx = Ce^{2x} + e^{2x} \int e^{-2x}(-x^2)dx.$$

The integral is too difficult for this course and can be left as it is.

For the interested, the integral can be done by integration by parts, which means the formula

$$\int f'gdx = fg - \int fg'dx.$$

We obtain

$$\begin{aligned}
\int e^{-2x}(-x^2)dx &= \frac{-1}{2}e^{-2x}(-x^2) - \int \frac{-1}{2}e^{-2x}(-2x)dx \\
&= \frac{x^2}{2}e^{-2x} - \int e^{-2x}xdx \\
&= \frac{x^2}{2}e^{-2x} - \frac{-1}{2}e^{-2x}x + \int \frac{-1}{2}e^{-2x} \cdot 1dx \\
&= \frac{x^2}{2}e^{-2x} + \frac{1}{2}e^{-2x}x + \frac{1}{4}e^{-2x} \\
&= \frac{e^{-2x}}{4}(2x^2 + 2x + 1).
\end{aligned}$$

We obtain

$$y(x) = Ce^{2x} + e^{2x} \cdot \frac{e^{-2x}}{4}(2x^2 + 2x + 1),$$

that is,

$$y(x) = Ce^{2x} + \frac{1}{4}(2x^2 + 2x + 1).$$

Need more exercise? See the course book [3, p. 420, problems 225–232.]

3.7 Basics of Fourier analysis

24. Calculate the partial derivatives
    (a) $\frac{\partial}{\partial x} x^2 t + e^x + 7$
    (b) $\frac{\partial}{\partial t} x^2 t + e^x + 7$
    (c) $\frac{\partial}{\partial x} \sin(2x) + \sin(3t)$
    (d) $\frac{\partial}{\partial t} \sin(2x)\sin(3t)$
25. Show that

$$y(x,t) = \sin(nx)\sin(nct),$$

where n is a natural number, is a solution of the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2} y = c^2 \frac{\partial^2}{\partial x^2} y \\ y(0,t) = 0 \\ y(\pi, t) = 0. \end{cases}$$

26. For the sine wave $\frac{3}{2}\sin(7t - 1)$, what is the
    (a) amplitude
    (b) angular frequency
    (c) phase
    (d) period
27. Express $6\sin(4t + \pi/3)$ as the sum of sine and cosine, that is, in the form $C\sin(\omega t) + D\cos(\omega t)$. Vihje. Remember $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.
28. Express $5\sin(2t) + 2\cos(2t)$ by one sine wave, that is, in the form $A\sin(\omega t + \phi)$

    Vihje. By using the vector dot product, we have $5\sin(2t) + 2\cos(2t) = (5, 2) \cdot (\sin(2t), \cos(2t))$. Let's express $(x, y) = (5, 2)$ in polar coordinates $(x, y) = r(\cos(\alpha), \sin(\alpha))$, where $r = \sqrt{x^2 + y^2}$ and $\alpha = \arctan(y/x)$.

    Ratkaisu: We obtain $r = \sqrt{29}$ and $\alpha = \arctan(2/5)\frac{180^\circ}{\pi} = 21.8^\circ$. Thus

$$(5, 2) = \sqrt{29}(\cos(21.8^\circ), \sin(21.8^\circ)).$$

Therefore

$$5\sin(2t) + 2\cos(2t) = \sqrt{29}(\sin(2t)\cos(21.8^\circ) + \cos(2t)\sin(21.8^\circ)) = \sqrt{29}\sin(2t + 21.8^\circ).$$

The result can be obtained directly with the formulas $A = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$.

29. Solve the following nonhomogeneous equations
    (a) $y' - 5y = 2$
    (b) $y' + 2y = 5e^{-x}$ with initial condition $y(0) = 1$

3.8 Fourier series

Ratkaisu: Fourier series are a method to study e.g. differential equations. Functions $\sin(nx)$ is a solution of

$$y'' + n^2 y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

The equation is related to a vibrating string attached on both ends at 0 and π. These sine functions can be summed to form a Fourier series.

Book [4, pp. 1–39] contains an introduction to Fourier analysis. (Skip the too difficult formulas.)

30. For the sine wave $\frac{3}{2}\sin(7t-1)$, what is the
    (a) amplitude Ratkaisu: $\frac{3}{2}$
    (b) angular frequency Ratkaisu: 7 rad/s
    (c) phase Ratkaisu: -1
    (d) period Ratkaisu: $\frac{2\pi}{7}$

31. Express $5\sin(2t) + 2\cos(2t)$ by one sine wave, that is, in the form $A\sin(\omega t + \phi)$
    Vihje. By using the vector dot product, we have $5\sin(2t) + 2\cos(2t) = (5, 2)\cdot(\sin(2t), \cos(2t))$. Let's express $(x, y) = (5, 2)$ in polar coordinates $(x, y) = r(\cos(\alpha), \sin(\alpha))$, where $r = \sqrt{x^2+y^2}$ and $\alpha = \arctan(y/x)$.
    Ratkaisu: We obtain $r = \sqrt{29}$ and $\alpha = \arctan(2/5)\frac{180^\circ}{\pi} = 21.8^\circ$. Thus

    $$(5, 2) = \sqrt{29}(\cos(21.8^\circ), \sin(21.8^\circ)).$$

    Therefore

    $$5\sin(2t) + 2\cos(2t) = \sqrt{29}(\sin(2t)\cos(21.8^\circ) + \cos(2t)\sin(21.8^\circ)) = \sqrt{29}\sin(2t + 21.8^\circ).$$

    The result can be obtained directly with the formulas $A = \sqrt{x^2+y^2}$ and $\phi = \arctan(y/x)$.

32. Express $6\sin(4t+\pi/3)$ as the sum of sine and cosine, that is, in the form $C\sin(\omega t) + D\cos(\omega t)$. Ratkaisu:

    $$6\sin(4t+\pi/3) \approx 3\sin(4t) + 5.2\cos(4t)$$

33. Consider a square wave $g(t)$ whose amplitude is 3 and the period is $T = 2$. When $0 \le t \le 2$, the function is defined piece-wise as

    $$g(t) = \begin{cases} 3, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } 1 \le t \le 2 \end{cases}$$

    and outside this time interval the function repeats itself periodically. Find the Fourier coefficients a_0, a_n and b_n. Vihje. Find the Fourier coefficients of

    $$f(x) = \begin{cases} 3, & \text{if } 0 \le t \le \pi, \\ 0, & \text{if } \pi \le t \le 2\pi \end{cases}$$

    to obtain

    $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

    Then substitute $x = \pi t$ to obtain $g(t) = f(\pi t) = \ldots$. Ratkaisu:

    $$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} 3\frac{1+(-1)^{n+1}}{n}\sin(nx)$$

    and by substituting $x = \pi t$ we obtain

    $$g(t) = f(\pi t) = \frac{3}{2} + \sum_{n=1}^{\infty} 3\frac{1+(-1)^{n+1}}{n}\sin(\pi n t).$$

34. **(Possible exam question.)** For the function

$$f(t) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ -1, & \text{if } -\pi \le x \le 0 \end{cases},$$

find its Fourier series. Ratkaisu:

$$f(t) = \sum_{n=1}^{\infty} 2\frac{1+(-1)^n}{n}\sin(nx)$$

3.9 Discrete Fourier transformation / FFT

35. **(Possible exam question.)** Let $[x_0, x_1] = [2, 3]$. Calculate the discrete Fourier transformation

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}.$$

Check the answer in OctaveOnline with `y=fft([x_0, x_1])`.

Ratkaisu: $[y_0, y_1] = [5, -1]$.

36. Let $[y_0, y_1]$=[5,-1]. Calculate the inverse discrete Fourier transformation

$$\begin{cases} x_0 &= \frac{1}{2}(x_0 + x_1) \\ x_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}.$$

Check the answer in OctaveOnline with `x=ifft([y_0,y_1])`.

Ratkaisu: $[x_0, x_1] = [2, 3]$.

37. Let $[x_0, x_1, x_2, x_3] = [1, 2, 1, 2]$. Calculate the discrete Fourier transformation

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}.$$

Check the answer in OctaveOnline with `y=fft([x_0,x_1,x_2,x_3])`.

Ratkaisu: $[y_0, y_1, y_2, y_3] = [6, 0, -2, 0]$.

38. Let $[y_0, y_1, y_2, y_3] = [6, 0, -2, 0]$. Calculate the inverse discrete Fourier transformation

$$\begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}.$$

Check the answer in OctaveOnline with `x=ifft([x_0,x_1,x_2,x_3])`.

Ratkaisu: $[x_0, x_1, x_2, x_3] = [1, 2, 1, 2]$.

3.10 Numerical solutions (not asked in exam)

1. Solve numerically with Python's Scipy library, and analytically with Python's Sympy library. Use the initial condition $y(0) = 2$ in the numerical solution. Draw the graphs of the numerical and analytic solutions. (Use same initial condition for the analytic solutions.)

   (a) $y' + 7y = 0$

   (b) $y' + \cos(x)y = 0$

   (c) $xy' + 3x^2y = 0$

2. Write a Python program which removes the noise from a function which is the sum of two sine waves. You can use, for example, the wave

```
noisy_signal = np.sin(a)+np.sin(3*a-1)+10*np.random.rand(len(a))
```

3.11 Formulas

3.11.1 Differentiation and integration

Differentiation		**Integration**	
Dx^n	$= nx^{n-1}$	$\int x^n dx$	$= \frac{x^{n+1}}{n+1} + C$
De^x	$= e^x$	$\int e^x dx$	$= e^x + C$
Db^x	$= b^x \ln(b)$	$\int b^x dx$	$= \frac{b^x}{\ln(b)}$
$D\ln(x)$	$= \frac{1}{x}$		
$D\ln\|x\|$	$= \frac{1}{x}$	$\int \frac{1}{x} dx$	$= \ln\|x\| + C$
$D\log_a(x)$	$= \frac{1}{x\ln(a)}$		
$D\log_a\|x\|$	$= \frac{1}{x\ln(a)}$		
$D\sin(x)$	$= \cos(x)$	$\int \cos(x) dx$	$= \sin(x) + C$
$D\cos(x)$	$= -\sin(x)$	$\int \sin(x) dx$	$= -\cos(x) + C$
$D\tan(x)$	$= 1 + \tan^2(x)$	$\int 1 + \tan^2(x) dx$	$= \tan(x) + C$
$Dx\ln(x) - x$	$= \ln(x)$	$\int \ln(x) dx$	$= x\ln(x) - x + C$
$D\arcsin(x)$	$= \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}$	$= \arcsin(x) + C$
$D\arccos(x)$	$= \frac{1}{-\sqrt{1-x^2}}$	$\int \frac{1}{-\sqrt{1-x^2}}$	$= \arccos(x) + C$
$D\arctan(x)$	$= \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}$	$= \arctan(x) + C$
$D\sinh(x)$	$= \cosh(x)$		
$D\cosh(x)$	$= \sinh(x)$		
$D\tanh(x)$	$= \frac{1}{\cosh^2(x)}$		

Differentiation		**Integration**	
$Df(g(x))$	$= f'(g(x))g'(x)$	$\int f(g(x))g'(x) dx$	$= f(g(x)) + C$
Special cases			
$D\ln(g(x))$	$= \frac{g'(x)}{g(x)}$	$\int \frac{g'(x)}{g(x)} dx$	$= \ln(g(x)) + C$
$De^{g(x)}$	$= e^{g(x)} g'(x)$	$\int g'(x) e^{g(x)} dx$	$= e^{g(x)} + C$
Dfg	$= f'g + fg'$	$\int f'g dx$	$= fg - \int fg' dx$
$D(f/g)$	$= (gf' - fg')/g^2$		

3.11.2 Solution formula

The solution of

$$y' + p(x)y = q(x)$$

is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

3.11.3 Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

3.11.4 Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$

Viitteet

[1] A. Croft, R. Davison, M. Hargreaves, and J. Flint, Engineering Mathematics, 5th ed. (Pearson, Harlow, 2017)

[2] Calculus Volume 1, OpenStax (2016). `https://openstax.org/details/books/calculus-volume-1`

[3] Calculus Volume 2, OpenStax (2016). `https://openstax.org/details/books/calculus-volume-2`

[4] Stein, E.M., and R. Shakarchi Fourier Analysis: An Introduction. Princeton University Press (2003).