information

maanantai 8. tammikuuta 2024 11.57

TODAY

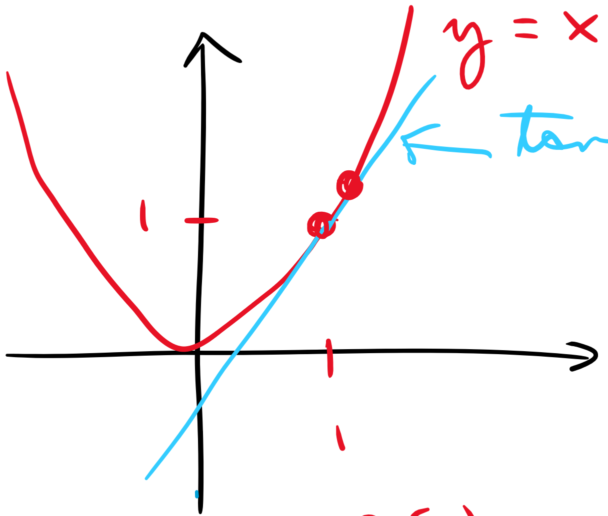
- COMING FROM LUNCH
- HELLO / TEACHER INTRODUCING HISSELF
- COURSE CONTENT
  - YOU HAVE STUDIED { DIFFERENTIATION, INTEGRATION } WITH ILPO
  - LET'S PROGRESS TO DIFFERENTIAL EQUATIONS
- CHECKING THE MOODLE PAGE
- LET'S WORK ON EXERCISES
- RECALLING { DIFFERENTIATION, INTEGRATION } - DIFF. EQUATIONS

SUGGESTIONS HOW TO IMPROVE

- VIDEOS ALSO TO YOUTUBE

Recalling differentiation and integration

maanantai 8. tammikuuta 2024 12.31



$y = x^2$

← tangent, EQUATION?

x	$y = x^2$
1	$1^2 = 1$
1.1	$1.1^2 = 1.21$

(1, 1)

(1, 1.21)

SOLUTION.

$f(x) = x^2$

GROWTH SPEED $f'(x) = 2x$

$x = 1$ $f'(1) = 2$

FORMULA $y - \underbrace{y_0}_{1} = k(x - \underbrace{x_0}_{1})$

$y - 1 = 2(x - 1)$

$y = 2x - 2 + 1$

$\underline{\underline{y = 2x - 1}}$

DIFFERENTIATION RULES

$\frac{d}{dx} \sin(x) = \cos(x)$

$\frac{d}{dx} \cos(x) = -\sin(x)$

FORMULA

$\ldots + \cos(x)\sin(y)$

FORMULA

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$\frac{d}{dx}$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\frac{d}{dx} x^{3} = 3x^{2}$$

$3-1$

$$\Rightarrow \int 3x^2\,dx = x^3 + C$$

CONSTANT

INTEGRATION CANCELS OUT DIFFERENTIATION

$$\frac{d}{dx} e^{x} = e^{x}$$

$$\frac{d}{dx} e^{2x} = e^{2x} \cdot \frac{d}{dx} 2x = e^{2x} \cdot 2$$

INVERSE FUNCTION OF e^x IS $\ln(x)$

INVERSE OF x^2 IS $\sqrt{x}$

$$\frac{d}{dx} 2^{7} = 0$$

$$\frac{d}{dx} 2^{x} = \frac{d}{dx} e^{\ln(2^{x})} = \frac{d}{dx} e^{x\ln(2)}$$

$$= \ln(2)\, e^{x\ln(2)}$$

$$= 2^{x}\ln(2)$$

INTEGRATION

$+1$

$$\int 2x^{2}\,dx = x^{3}$$

$$\frac{d}{\cdot} 3x^{2} = 3\cdot 2\cdot x^{1}$$

$\int^{1} 3x^{2}\,dx = x^{3}$

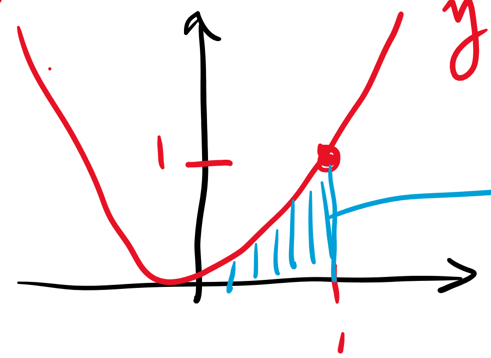
$\frac{d}{dx} 3x^{2} = 3\cdot 2\cdot x^{1}$

$\int \cos(x)\,dx = +\sin(x) + C$

$\int \sin(x)\,dx = -\cos(x) + C$

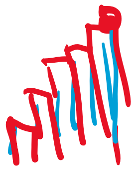
AREA AS AN INTEGRAL

$y = x^{2}$



AREA

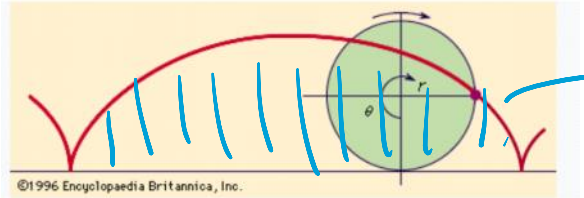
USE RIEMANN SUMS



$$\int_0^1 x^2\,dx = \Big|_0^1 \frac{1}{3}x^{3}$$

$$= \frac{1}{3}\Big[x^{3}\Big]_{x=0}^{x=1}$$

$$= \frac{1}{3}\left(1^{3} - 0^{3}\right) = \frac{1}{3}$$

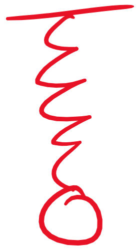


CYCLOID

GALILEI WANTED TO KNOW THIS AREA → HE MADE THE SHAPE

FROM LEAD
AND WEIGHTED IT

NEWTON $F = ma = mx''$



POSITION $x(t)$

VELOCITY $x'(t)$

ACCELERATION $x''(t)$

1.1 Recall differentiation

1. Differentiate = FIND THE DERIVATIVE

   (a) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 + 3$ $= 2x + 0 = 2x$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x} x^3 + x$ $= 3x^2 + 1$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(2x)$ $= 2\cos(2x)$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x} e^{3x}$ $= 3e^{3x}$
   (e) $\frac{\mathrm{d}}{\mathrm{d}x} \cos(4x)$ $= -4\sin(4x)$
   (f) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x)$ $= \frac{1}{x}, \; x \neq 0$

$\frac{d}{dx} x^{7} = 7x^{6}$ -1

2. Often, it is smart to modify the expression of the function before differentiation. Differentiate

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/2}$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x} 4^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(4)}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(2) + \ln(x)$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x}\log_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\ln(2)}\ln(x)$
   (e) $\frac{\mathrm{d}}{\mathrm{d}x} x^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(x)}$

$\frac{d}{dx} e^{ax} = a e^{ax}$

$\frac{d}{dx} x \ln(x) = \ldots$

3. A product is differentiated by the rule $\boxed{(fg)' = f'g + fg'}$. Differentiate

   (a) $\frac{\mathrm{d}}{\mathrm{d}x} x^3 \sin(2x)$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x)e^{3x}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x} x\ln(x) - x$
   (d) $\sin(x)e^{-x}$

4. A quotient is differentiated by the rule $(f/g)' = (gf' - fg')/g^2$. Differentiate

   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{x}$
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{\cos(x)}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(x)}{e^x}$

2. Often, it is smart to modify the expression of the function before differentiation. Differentiate

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   (b) $\frac{\mathrm{d}}{\mathrm{d}x} 4^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(4)}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(2) + \ln(x)$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x}\log_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\ln(2)}\ln(x)$
   (e) $\frac{\mathrm{d}}{\mathrm{d}x} x^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(x)}$ $= e^{x\ln(x)} \cdot \left(\left[\frac{d}{dx} x\right] \ln(x) + x \frac{d}{dx}\ln(x) \right)$

$\frac{d}{dx} c = 0$	Constant Rule
$\frac{d}{dx} x^n = nx^{n-1}$	Power Rule
$\frac{d}{dx}\sin(x) = \cos(x)$ $\frac{d}{dx}\cos(x) = -\sin(x)$	Trigonometric Rules
$\frac{d}{dx} b^x = b^x \ln(b)$	Exponential Rule
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	Logarithmic Rule

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate

(a) $\frac{\mathrm{d}}{\mathrm{d}x} x^3 \sin(2x)$ $= \left(\frac{d}{dx} x^3\right) \sin(2x) + x^3 \frac{d}{dx} \sin(2x) = \ldots + \ldots$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \cos(4x) e^{3x}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} x \ln(x) - x$

(d) $\sin(x) e^{-x}$

Calculating together

perjantai 12. tammikuuta 2024 12.14

$\frac{d}{dx} x^{n} = n x^{n-1}$

1. Differentiate

(a) $\frac{\mathrm{d}}{\mathrm{d}x} x^2+3$ $= 2x^1 + 0 = 2x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^3+x$ $= 3x^2+1$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(2x)$ $= 2\cos(2x)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x} e^{3x}$ $= 3e^{3x}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x} \cos(4x)$ $= -4\sin(4x)$

(f) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x)$ $= \frac{1}{x}$

$\frac{d}{dx} 3\underbrace{x^0}_{=1} = 0 \cdot 3x^{-1} = 0$

$\frac{d}{dx} x^{1} = 1 \underbrace{x^0}_{=1} = 1$

$f(x) = \sin(x)$ $f'(x) = \cos(x)$

$g(x) = 2x$ $g'(x) = 2$

$\sin(2x) = f(g(x))$

$\sin(2x)' = \boxed{f'(g(x))} g'(x)$

$= \cos(2x) \cdot 2$

2. Often, it is smart to modify the expression

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/2}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} 4^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(4)}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(2) + \ln(x)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\log_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\ln(2)}\ln(x)$

(e) $\frac{\mathrm{d}}{\mathrm{d}x} x^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(x)}$

2. Often, it is smart to modify the expression

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/2}$ $= \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} 4^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(4)}$ $= e^{x\ln(4)} \cdot \ln(4) = \ln(4)\, 4^x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(2) + \ln(x)$ $= 0 + \frac{1}{x} = \frac{1}{x}$ OR $\frac{d}{dx}\ln(2x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\log_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\ln(2)}\ln(x)$ $= \frac{1}{\ln(2)} \frac{d}{dx}\ln(x) = \frac{1}{\ln(2)} \frac{1}{x} = \frac{1}{x\ln(2)}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x} x^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(x)}$

$= e^{x\ln(x)} \frac{d}{dx}(x\ln(x))$

$= e^{x\ln(x)} \left[\ln(x) \underbrace{\frac{d}{dx} x}_{=1} + \underbrace{\cancel{x} \underbrace{\frac{d}{dx}\ln(x)}_{=\frac{1}{\cancel{x}}}}_{=1}\right]$

$= e^{x\ln(x)} \cdot [\ln(x) + 1]$

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate

(a) $\frac{\mathrm{d}}{\mathrm{d}x} x^3\sin(2x)$ $= 3x^2\sin(2x) + x^3 \underbrace{\frac{d}{dx}\sin(2x)}_{= \cos(2x) \cdot 2}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x)e^{3x}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} x\ln(x) - x$

(d) $\sin(x)e^{-x}$

DERIVATIVE OF A PRODUCT

CHAIN RULE

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate

(a) $\frac{\mathrm{d}}{\mathrm{d}x} x^3\sin(2x)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(4x)e^{3x}$ $= -4\sin(4x) \cdot e^{3x} + \cos(4x) \cdot 3e^{3x}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} x\ln(x) - x$

(d) $\sin(x)e^{-x}$

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate

(a) $\frac{d}{dx} x^3 \sin(2x)$

(b) $\frac{d}{dx} \cos(4x)e^{3x}$

(c) $\frac{d}{dx} x \ln(x) - x$ $= 1 \cdot \ln(x) + \underbrace{\cancel{x} \cdot \frac{1}{\cancel{x}}}_{=1} - 1 = \ln(x)$

(d) $\sin(x)e^{-x}$

3. A product is differentiated by the rule $(fg)' = f'g + fg'$. Differentiate

(a) $\frac{d}{dx} x^3 \sin(2x)$

(b) $\frac{d}{dx} \cos(4x)e^{3x}$

(c) $\frac{d}{dx} x \ln(x) - x$

(d) $\sin(x)e^{-x}$ $= \cos(x) \cdot e^{-x} + \sin(x) \cdot (-1) e^{-x}$

$-1 \cdot x$

$= \cos(x) e^{-x} - \sin(x) e^{-x}$

$$\boxed{\frac{f}{g} = \frac{gf' - fg'}{g^2}}$$

4. A quotient is differentiated by the rule $(f/g)' = (gf' - fg')/g^2$. Differentiate

(a) $\frac{d}{dx} \frac{\sin(x)}{x}$ $= \frac{x \frac{d}{dx} \sin(x) - \sin(x) \frac{d}{dx} x}{x^2}$ $\frac{d}{dx}x = 1$

$= \frac{x \cos(x) - \sin(x)}{x^2}$

$= \frac{\cancel{x} \cos(x)}{x^{\cancel{2}}} - \frac{\sin(x)}{x^2}$

(b) $\frac{d}{dx} \tan(x) = \frac{d}{dx} \frac{\sin(x)}{\cos(x)}$

(c) $\frac{d}{dx} \frac{\sin(x)}{e^x}$

JUST THESE TOPICS ARE NEEDED FOR QUIZ 1 ?

5. A composed function can be differentiated by the rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$. Differentiate

(a) $\sin(x^2)$

(b) $\sin(1/x)$

(c) e^{x^2}

(d) $e^{\sin(x)}$

NOT SO IMPORTANT. A BIT DIFFICULT.

6. An inverse function can be differentiated by the rule

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Differentiate $\arcsin(x)$.

INVERSE OPERATIONS

$2 \underset{:3}{\overset{\cdot 3}{\rightleftarrows}} 6$

$100 \underset{-5}{\overset{+5}{\rightleftarrows}} 105$

$3 \xrightarrow{(\,)^2} 3^2 = 9 \xrightarrow{\sqrt{\;}} \sqrt{9} = 3$

$7 \quad 7^2 = 49 \quad \sqrt{49} = 7$

$2 \quad 2^2 = 4 \quad \sqrt{4} = 2$

$\cancel{\ln}(\cancel{e}^3) = 3 \qquad \sqrt{2^{\cancel{2}}} = 2$

EXPONENTIAL FUNCTION e^x
AND LOGARITHM $\ln(x)$
CANCEL OUT → INVERSE FUNCTIONS OF EACH OTHER

$5 \xrightarrow{(\,)^2} 25 \xrightarrow{\sqrt{\,}} 5$

$5 \qquad 100 \qquad 10$

$\sin(x) \xrightarrow{\frac{d}{dx}} \cos(x) \xrightarrow{\int dx} \sin(x) + C$

$x^3 + 2 \qquad 3x^2 + 0 \qquad x^3 + C,$ C CONSTANT

$\ln(x) \qquad \frac{1}{x} \qquad \ln(x) + C$

$e^{2x} \qquad 2e^{2x} \qquad e^{2x} + C$

$e^{2x} + C \qquad \frac{1}{2} \cdot 2e^{2x} \qquad \frac{1}{2}e^{2x} + C$

https://www.wolframalpha.com/input?i=integral+of+e%5E%282x%29

integral of e^(2x)

NATURAL LANGUAGE | MATH INPUT

Indefinite integral

$\int e^{2x}\,dx = \frac{e^{2x}}{2}$ + constant

7. Integration is the inverse operation to differentiation. Because $\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$, we have $\int 3x^2 \mathrm{d}x = x^3 + C$, where C is a constant. Integrate

   (a) $\int \cos(x)\mathrm{d}x$

   (b) $\int -\sin(x)\mathrm{d}x$

   (c) $\int \sin(x)\mathrm{d}x$

8. A monomial can be integraded by the rule $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$. Integrate

   (a) $\int x^3 \mathrm{d}x$

   (b) $\int x^{1/2} \mathrm{d}x$

Thursday 18.1.

torstai 18. tammikuuta 2024 12.22

TODAY

- EXAMPLE COMBINING FACTS SO FAR

  TOPIC OF QUIZ 2

- WE CAN CHECK 12 AND 13 AT LEAST PARTIALLY

- CALCULATE 13 - 16 ?

- LET'S END AT 13.50 (JUHA GOES TO EXAM)

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

$\cos(x)$

EXAMPLE: $\frac{d}{dx}\ln(\sin(x)) = \frac{1}{\sin(x)}\cos(x)$

$$\frac{d}{dx}\sin(2x) = 2\cos(2x)$$

$$\frac{d}{dx}\sin(x^3) = \cos(x^3)\cdot \overbrace{\frac{d}{dx}x^3}^{=3x^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$$

SUBSTITUTE

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$g(x) = \sin(x)$

$g'(x) = \cos(x)$

$$\frac{d}{dx}\ln(\sin(x)) = \underbrace{f'(g(x))}_{=\frac{1}{g(x)} = \frac{1}{\sin(x)}}\ \underbrace{g'(x)}_{=\cos(x)}$$

$$= \frac{1}{\sin(x)}\cos(x)$$

ANOTHER WAY TO THINK

$$\boxed{\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}}$$

THINK

$$\Rightarrow \boxed{\int \boxed{\frac{f'(x)}{f(x)}}\,dx = \ln(f(x)) + C}$$

EXAMPLE. $\frac{1}{2}\int \frac{2x}{1+x^2}\,dx$? ?

$f(x) = 1 \times x^2$

$f'(x) = 2x$?

$$= \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx = \frac{1}{2}\ln(f(x)) + C$$

$$= \frac{1}{2}\ln(1+x^2) + C$$

EXAMPLE. SOLVE $\frac{y'}{y} = x \quad \|\int dx$

$$\Rightarrow \int \frac{y'}{y} = \int x\,dx$$

$$\ln(y) = \frac{1}{2}x^2 + C \quad \| e^{(\ )}$$

$$\boxed{e^{a+b} = e^a e^b}$$

$$\cancel{e}^{\cancel{\ln}(y)} = \underbrace{e^{\frac{1}{2}x^2 + C}}$$

$$= e^{\frac{1}{2}x^2}\underbrace{e^C}_{= A}$$

$= A =$ CONSTANT

$$\underline{y = Ae^{\frac{1}{2}x^2}}$$

EXAMPLE SOLVE $\boxed{y' = yx^2} \quad \| : y$

OK, WE CAN INTEGRATE ON BOTH SIDES $\rightarrow$ $\underbrace{\frac{y'}{y}}_{\text{JUST } y} = \underbrace{x^2}_{\text{JUST } x}$

JUST y JUST x

THESE KIND OF EQUATIONS ARE CALLED SEPARABLE

12. Find the order of the following differential equations

(a) $y' + y = 4y^2$ — 1

(b) $(y')^2 = y' + 2y$ — 1

(c) $y''' + y''y' = 3x^2$ → ORDER 3

(d) $y' = y'' + 3t^2$ 2

(e) $\frac{dy}{dt} = t$ → 1

(f) $\frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^4$ → 2

(g) $\left(\underbrace{\frac{dy}{dt}}_{1}\right)^2 + 8\underbrace{\frac{dy}{dt}}_{1} + 3y = 4t$ → 1 ~~2~~

13. Show that the functions y are solutions to the corresponding differential equations. Hint. It is enough to calculate the derivative and to substitute it to the equation.

(a) Show that $y = x^3/3$ is a particular solution for $y' = x^2$. Solution We have $y = x^3/3$. By differentiating, we obtain $y' = 3x^2/3 = x^2$. We see that the equation $y' = x^2$ is satisfied.

(b) Show that $y = 2e^{-x} + x - 1$ is a particular solution for $y' = x - y$.

(c) Show that $y = e^{3x} - e^x/2$ is a particular solution for $y' = 3y + e^x$.

(d) Show that $y = \frac{1}{1-x}$ is a particular solution for $y' = y^2$.

(e) Show that $y = e^{x^2/2}$ is a particular solution for $y' = xy$.

(f) Show that $y = 4 + \ln(x)$ is a particular solution for $xy' = 1$.

(g) Show that $y = 3 - x + x\ln(x)$ is a particular solution for $y' = \ln(x)$.

(h) Show that $y = e^x + \frac{\sin(x)}{2} - \frac{\cos(x)}{2}$ is a particular solution for $y' = \cos(x) + y$.

(i) Show that $y = \pi e^{-\cos(x)}$ is a particular solution for $y' = y\sin(x)$.

Solution is not available yet. Need 1 hour to write it down nicely.

e) $y = e^{\frac{x^2}{2}} \Rightarrow y' = e^{\frac{x^2}{2}} \underbrace{\frac{d}{dx}\frac{x^2}{2}}_{= \frac{2x}{2} = x} = xe^{\frac{x^2}{2}} = xy$

→ OK $y' = xy$ IS SATISFIED

OK, LET'S CALCULATE (WE CAN CONTINUE ALSO TOMORROW IF YOU LIKE)

OK, LET'S CALCULATE (WE CAN CONTINUE ALSO TOMORROW IF YOU LIKE)

14. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -30)$ (that is, satisfies $x = -3$, $y = -30$)? Solution We have $y(x) = y = \frac{4}{3}x^3 + C$. Set $x = -3$ and $y(-3) = -30$ to obtain

$$-30 = \frac{4}{3}(-3)^3 + C,$$

that is

$$-30 = -36 + C.$$

We have $C = 6$. The desired solution is $y(x) = y = \frac{4}{3}x^3 + 6$.

15. The general solution of the differential equation $y' = (2xy)^2$ is $y(x) = -\frac{3}{C+4x^3}$. Which particular solution passes through the point $(1, -0.5)$? Hint. We have $1 = y(-0.5) = \frac{3}{C+4(-0.5)^3}$. Solve C.

16. Find the general solution for the following ODE's.

(a) $y' = 3x + e^x$ Solution By integrating on both sides, we obtain $y = \int 3x + e^x dx$. The integral of the right hand side is $\int 3x + e^x dx = 3x^2/2 + e^x + C$. The general solution of the equation is therefore $y = 3x^2/2 + e^x + C$.

(b) $y' = \ln(x) + \tan(x)$

(c) $y' = \sin(x)e^{\cos(x)}$

(d) $y' = 4^x$

(e) $y' = 2t\sqrt{t^2 + 16}$

(f) $y' = y$ Hint. Dividing by y we obtain $\frac{y'}{y} = 1$ which can be integrated on both sides. Solution We obtained $\frac{y'(x)}{y(x)} = 1$. By integrating we obtain $\ln(y(x) = x + C$. By taking the exponential on both sides, we obtain $e^{\ln(y(x))} = e^{x+C} = e^C e^x$. Because the exponential and logarithm are inverse functions, they cancel out, and we obtain $y(x) = e^C e^x$. Here e^C is just a constant, write $e^C = A$. We obtain $y(x) = Ae^x$.

(g) $y' = \frac{y}{x}$ Hint. Dividing by y, we obtain $\frac{y'}{y} = \frac{1}{x}$ which can be integrated on both sides.

First order linear homogeneous equations

perjantai 26. tammikuuta 2024 8.52

$$a(x)y' + b(x)y = 0 \quad \|:a(x)$$

$$y' + \underbrace{\frac{b(x)}{a(x)}}_{p(x)} y = 0$$

$$\boxed{y' + p(x)y = 0}$$

LINEAR

$\cancel{(y')^2}$

$\cancel{\sin(y')}$

$\cancel{yy'}$

$$\boxed{a(x)y' + b(x)y}$$

2 WAYS TO SOLVE

1. THE EQUATION IS SEPARABLE

$$y' = -p(x)y \quad \|:y$$

$$\frac{y'}{y} = -p(x)$$

$$\int \frac{y'}{y}\,dx = -\int p(x)\,dx$$

$$\ln y = -\int p(x)\,dx + C \quad \|e()$$

$$y = e^{\cancel{\ln} y} = Ae^{-\int p(x)\,dx \cancel{+C}}$$

19. Solve the homogeneous equations

(a) $y'+7y=0$ Solution Rearranged $y'/y=-7$ yields with integration $\ln(y)=-7x+C$ giving $y(x)=Ae^{-7x}$, where C and A are constant.

(b) $y'+\cos(x)y=0$ Solution Rearranged $y'/y=-\cos(x)$ yields $y(x)=Ce^{\sin(x)}$

(c) $xy'+3x^2y=0$ with initial condition $y(0)=2$ Solution Dividing by x, we have $y'+3xy=0$, rearranged to $y'/y=-3x$. Integrating on both sides, we get finally $y(x)=Ce^{-\frac{3x^2}{2}}$. Solution of the initial value problem is $y(x)=2e^{-\frac{3x^2}{2}}$

$$\boxed{y'+p(x)y=0}$$

2 WAYS TO SOLVE

1. THE EQUATION IS SEPARABLE

$$y'+p(x)y=0$$

$$\frac{y'}{y}=-p(x)$$

$$\vdots$$

$$y(x)=Ce^{-\int p(x)\,dx}$$

EXAMPLES. $y'+17y=0$

$y'=-17y \quad \| : y$

$\int \frac{y'}{y}dx=\int -17\,dx \Rightarrow \ln(y)=-17x+A \quad \| e^{(\,)}$

$\Rightarrow y(x)=Ce^{-17x}$

$y'+\sin(x)y=0$

$\int \frac{y'}{y}dx=\int -\sin(x)dx=\cos(x) \Rightarrow \ln(y)=\cos(x)+A$

$\Rightarrow y(x)=Ce^{\cos(x)}$

$xy'+x^4y=0 \quad \| : x$

$$y' + x^3 y = 0 \qquad \| : y$$

$$\int \frac{y'}{y}\,dx = \int -x^3\,dx = -\frac{x^4}{4} \Rightarrow \ln(y) = -\frac{x^4}{4} + A \Rightarrow \underline{y(x) = Ce^{-\frac{x^4}{4}}}$$

HOMOGENEOUS

$$\boxed{y' + p(x)y = \underset{q(x)}{0}}$$

2 WAYS TO SOLVE

① THE EQUATION IS SEPARABLE

② USE THE SOLUTION FORMULA

$$y' + p(x)y = \cancel{q(x)}^{\,0}$$

$$\Rightarrow y = \frac{C}{\mu(x)} + \cancel{\frac{1}{\mu(x)}\int \mu(x)\underbrace{q(x)}_{=0}dx}$$

WHERE $\mu(x) = e^{\int p(x)dx}$

SOLUTION FORMULA

$$y' + p(x)y = q(x)$$

$$\Rightarrow y = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx$$

WHERE $\mu(x) = e^{\int p(x)dx}$

$$y' + \underbrace{17}_{p(x)}y = \underbrace{0}_{q(x)}$$

- $\int p(x)dx = \int 17\,dx = 17x \cancel{+C}$
- $\mu(x) = e^{\int p(x)dx} = e^{17x}$ AND $\frac{1}{\mu(x)} = \frac{1}{e^{17x}} = e^{-17x}$
- $y(x) = \frac{C}{\mu(x)} + \cancel{\frac{1}{\mu(x)}\int \mu(x)\underbrace{q(x)}_{=0}dx} = \underline{Ce^{-17x}}$

$$y' + \underbrace{\sin(x)}y = \underbrace{0}_{q(x)}$$

$$y' + \underbrace{\sin(x)}_{=p(x)} y = \underbrace{0}_{q(x)}$$

- $\int p(x)\,dx = -\int -\sin(x)\,dx = -\cos(x) \cancel{+C}$
- $\mu(x) = e^{\int p(x)dx} = e^{-\cos(x)}$ AND $\frac{1}{\mu(x)} = \boxed{e^{\cos(x)}}$

$$y(x) = \frac{c}{\mu(x)} + \cancel{\frac{1}{\mu(x)} \int \mu(x) \underbrace{q(x)}_{=0}}\,dx$$

$$= \underline{\underline{c\, e^{\cos(x)}}}$$

$$xy' + x^4 y = 0 \quad \| : x$$

$$y' + \underbrace{x^3}_{=p(x)} y = \underbrace{0}_{=q(x)}$$

- $\int p(x)dx = \int x^3 dx = \frac{x^4}{4}$
- $\mu(x) = e^{\int p(x)\,dx} = e^{\frac{x^4}{4}}$ AND $\frac{1}{\mu(x)} = e^{-\frac{x^4}{4}}$
- $y(x) = \frac{c}{\mu(x)} + 0 = \underline{\underline{c\,e^{-\frac{x^4}{4}}}}$

First order linear non-homogeneous equations

perjantai 26. tammikuuta 2024 9.33

$\cancel{(y')^2}$ $\cancel{yy'}$ $\cancel{\sin(y')}$

$$y' + p(x)y = q(x)$$

$$\Rightarrow y = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx$$

WHERE $\mu(x) = e^{\int p(x)\,dx}$

EXAMPLE. $y' + \underbrace{2x}_{=p(x)} y = \underbrace{x}_{=q(x)}$

①

② $\int p(x)\,dx = \int 2x\,dx = x^2 \cancel{+C}$

③ $\mu(x) = e^{\int p(x)\,dx} = e^{x^2}$

$\frac{1}{\mu(x)} = \frac{1}{e^{x^2}} = e^{-x^2}$

④ $\int \mu(x) q(x)\,dx$ $f(x) = x^2$, $f'(x) = 2x$

$$= \frac{1}{2} \int e^{x^2} \cdot 2x\,dx$$

$$= \frac{1}{2} \int e^{f(x)} f'(x)\,dx = \frac{1}{2} e^{f(x)} = \boxed{\frac{1}{2} e^{x^2}}$$

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \underbrace{\int \mu(x) q(x)\,dx}$$

$e^{-x^2} \cdot e^{x^2} = e^{-x^2 + x^2} = e^0 = 1$

$$= Ce^{-x^2} + \cancel{e^{-x^2}}_{1} \cdot \frac{1}{2} \cancel{e^{x^2}}_{1}$$

$$\underline{\underline{y(x) = Ce^{-x^2} + 1}}$$

Example 4.16

Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

$$xy' + 3y = 4x^2 - 3x \quad \| : x$$

$$x y' + 3y = 4x^2 - 3x \quad \| : x$$

$$y' + \underbrace{\frac{3}{x}}_{=p(x)} y = \underbrace{4x - 3}_{=q(x)}$$

①

② $\int p(x)\,dx = 3\int \frac{1}{x}\,dx = 3\ln(x) \cancel{+C}$

③ $\mu(x) = e^{\int p(x)dx} = \cancel{e}^{\cancel{\ln}(x^3)} = x^3$

$$\frac{1}{\mu(x)} = \frac{1}{x^3}$$

④ $\int \mu(x)\, q(x)\,dx = \int x^3(4x-3)\,dx$

$$= \int 4x^4 - 3x^3 \;dx$$

$$= \underbrace{\frac{4}{5}x^5 - \frac{3}{4}x^4}$$

⑤ $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \underbrace{\int \mu(x)\, q(x)\, dx}$

$$= \frac{C}{x^3} + \frac{1}{\cancel{x^3}}\left(\frac{4}{5}x^{5-3} - \frac{3}{4}x^{4-3}\right)$$

$$y(x) = \frac{C}{x^3} + \frac{4}{5}x^2 - \frac{3}{4}x$$

Talking about practice exam

tiistai 30. tammikuuta 2024 10.14

1. Concepts about differential equations.

(a) Which one of the following equations has order 3?

$$y' + y^3 = x, \quad 4y'' + xy' + \sin(x)y = 3, \quad \boxed{y''' + x^2 y = \frac{1}{x}}.$$

(b) Which one of the following equations is linear? IN TERMS OF y?

$$y' + 2xy = \frac{1}{2x + \underbrace{3y}}, \quad \boxed{y' + \sin(x)y = e^x}, \quad y'' + \underbrace{yy'} + 2xy = 3.$$

NOT LINEAR

NOT LINEAR

(c) Which one of the following equations is homogeneous?

$$y' + y + \underbrace{x} - 3 = 0, \quad \boxed{y' + \sin(x)y = 0}, \quad y'' + y' = \underbrace{x} - y.$$

NOT HOMOGENEOUS

NOT HOMOG.

HOMOG. = EACH TERM CONTAINS y

LINEAR = OF FORM $y' + p(x)y = q(x)$

Solution formula for first order linear diff. equations

tiistai 30. tammikuuta 2024 10.05

$y' + p(x)y = q(x)$

$\mu(x) = e^{\int p(x)dx}$

$\frac{1}{\mu(x)} = \ldots$

$\int \mu(x)q(x)dx = \ldots$

$p(x) + q(x) = y'$

$y' = 3x + e^x$

$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$

CAN BE USED FOR EXERCISES 17 →

$\frac{d}{dx}$

$\ln(x) \to \frac{1}{x}$

$\int dx$

EXAMPLE.

$y' + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{x^2}_{q(x)}$

$\int p(x)dx = \int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2\ln(x) \cancel{+C}$

$\mu(x) = e^{\int p(x)dx} = e^{2\ln(x)} = \cancel{e}^{\cancel{\ln}(x^2)} = x^2$

$\frac{1}{\mu(x)} = \frac{1}{x^2}$

$\int \underbrace{\mu(x)}_{=x^2} \underbrace{q(x)}_{x^2} dx = \int x^4 dx = \frac{x^5}{5} \cancel{+C}$

$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$

$= \frac{C}{x^2} + \frac{\cancel{1}}{\cancel{x^2}} \frac{x^{\cancel{5}3}}{5} = \frac{C}{x^2} + \frac{x^3}{5}$ ANSWER

CAN BE USED FOR

$y' = 3x + e^x \Rightarrow y' + \underbrace{0}_{=p(x)} \cdot y = \underbrace{3x + e^x}_{=q(x)}$

SOLUTION.

$\int p(x)dx = \int 0\, dx = 0$

$\mu(x) = e^{\int p(x)dx} = e^0 = 1 \qquad \frac{1}{\mu(x)} = 1$

$\int \mu(x)q(x)dx = \int 3x + e^x dx = \frac{3x^2}{2} + e^x \cancel{+C}$

$y(x) = \frac{C}{\mu(x)} + \overbrace{\frac{1}{\mu(x)}}^{=1} \int \mu(x)q(x)dx$

$= C + \frac{3x^2}{2} + e^x$ ANSWER

COURSE 64 PARTICIPANTS
9 IN CAMPUS
55 AT HOME/WORK/ETC.

FOR US 9 PEOPLE AT CAMPUS
EXERCISES 17 - 19 ARE ENOUGH THIS WEEK.

CONTENT OF THE QUIZ:
- QUESTIONS ABOUT THE FORMULA

$y' + p(x)\,y = q(x)$

$\mu(x) = e^{\int p(x)\,dx}$ ① = INTEGRATING FACTOR

$\frac{1}{\mu(x)} = \ldots$

$\int \mu(x) q(x)\,dx = \ldots$

② $p(x) = 0 \Rightarrow \mu(x) = 1$

③ $q(x) = 0 \Rightarrow y(x) = \frac{C}{\mu(x)}$

$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx$

17. Solve the initial value problems

(a) $y' = 5x$, $y(0) = 1$ Solution Gener

value problem is $y(x) = \frac{5}{2}x + 1$.

(b) $y' = \frac{3\sin(x)}{y}$, $y(\pi) = 4$ Solution

condition gives $C = 10$, solution of

(c) $y^2 y' = e^{-x}$, $y(0) = 0$ Solution

condition gives $C = 0$, solution of

SORRY, COULD NOT USE THE FORMULA

(b) $y\,y' = 3\sin(x) \Rightarrow \int 2y\,y'\,dx = 6\int \sin(x)\,dx \Rightarrow y^2 = -6\cos(x) + C$

(c) $y^2 y' = e^{-x} \Rightarrow \int 3y^2 y'\,dx = \int 3e^{-x}$

$\Rightarrow y^3 = \int 3e^{-x} = \ldots = -3e^{-x} + C = C - 3e^{-x}$

$\int x^2\,dx$

$\boxed{\frac{d}{dx}\, x^3 = 3x^2}$

$= \frac{1}{3}\int 3x^2\,dx = \frac{1}{3}x^3 + C$

$\frac{1}{3}\int \underbrace{3y^2\,y'}_{\frac{d}{dx}\,y^3}\,dx$

$$\boxed{\int y^p\, y'\,dx = \frac{y^{p+1}}{p+1} + C \quad \text{FOR } p > -1}$$

MISPRINTS

CAN USE $y' + p(x)\,y^{\cancel{2}} = q(x) \Rightarrow y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx$

WHERE $\mu(x) = e^{\int p(x)\,dx}$

17. Solve the initial value problems

(a) $y' = 5x$, $y(0) = 1$ Solution General solution of the equation is $y(x) = \frac{5}{2}x^{2} + C$. The solution of the initial value problem is $y(x) = \frac{5}{2}x^{2} + 1$.

(b) $y' = \frac{3\sin(x)}{y}$, $y(\pi) = 4$ Solution General solution of the equation is $y(x) = \sqrt{C - 6\cos(x)}$, initial condition gives $C = 10$, solution of the initial value problem is $y(x) = \sqrt{10 - 6\cos(x)}$

(c) $y^2 y' = e^{-x}$, $y(0) = 0$ ~~Solution General solution of the equation is $y(x) = 3^{1/3}(C - e^{-x})^{1/3}$, initial condition gives $C = 0$, solution of the initial value problem is $y(x) = -3^{1/3}e^{-x/3}$~~ $C = 1$

BETTER WAY TO WRITE $y(x) = \sqrt[3]{3C - 3e^{-x}}$

$y(0) = \sqrt[3]{3C - 3\underbrace{e^0}_{=1}} = \underbrace{\sqrt[3]{3C - 3}}_{} = 0 \rightarrow \boxed{C = 1}$

$\Rightarrow y(x) = \sqrt[3]{3 - 3e^{-x}}$

$\Rightarrow y(x) = \ldots$

17. (c) $\underbrace{y^2 y'}_{\text{JUST } y} = \underbrace{e^{-x}}_{\text{JUST } x}$ AND $y(0)=0$

SOLUTION. WE CAN INTEGRATE ON BOTH SIDES

$\int y^2 y' dx = ?$

$\int x^2 dx = \frac{1}{3}x^3$

$$\boxed{\int y^2 y' dx = \frac{1}{3}y^3}$$

$\int e^{-x} dx = ?$

$De^x = e^x$

$De^{ax} = e^{ax} \underbrace{\frac{d}{dx}(ax)}_{=a} = ae^{ax}$

$$\boxed{\int a e^{ax} dx = e^{ax} + C}$$

$\Rightarrow \int e^{-1 \cdot x} dx = -1 \cdot \underbrace{\int -1 \cdot e^{-1 \cdot x} dx}_{e^{-x}} = \boxed{-e^{-x}}$

$y^2 y' = e^{-x}$ CANNOT USE FORMULAS WITH $M(x)$

$\Rightarrow \int y^2 y' dx = \int e^{-x} dx$

$\Rightarrow \frac{1}{3}y^3 = -e^{-x} + C \quad || \cdot 3$

$y^3 = 3C - 3e^{-x} \quad || \sqrt[3]{\ }$

$y = \sqrt[3]{3C - 3e^{-x}}$ AND $y(0)=0$

$\Rightarrow$ WE CAN SOLVE $C=1$

SUBSTITUTE

$y(0)=0$ ($0 = x$) $\longrightarrow 0 = y(0) = \sqrt[3]{3C - 3\underbrace{e^0}_{=1}} = \sqrt[3]{3\underbrace{(C-1)}_{=0}}$

MUST BE $\Rightarrow C=1$

16. (f) $y' = y$

$\Rightarrow y' - 1 \cdot y = 0$

THIS IS OF THE FORM

$y' + p(x)y = q(x)$

WITH $\begin{cases} p(x) = -1 \\ q(x) = 0 \end{cases}$

WE CAN USE 2 METHODS

• METHOD ① SEPARATION

• METHOD ② FORMULA

$$\boxed{y' + p(x)\,y = q(x) \Rightarrow y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx \quad \text{WHERE } \mu(x) = e^{\int p(x)dx}}$$

① BY SEPARATION

IDEA $y' = f(x)g(y) \quad || : g(y)$

$$\int \frac{y'}{g(y)} = \int f(x) \Rightarrow$$ WE GET THE SOLUTION

$y' = y \quad || : y$

$$\frac{y'}{y} = \frac{y}{y} = 1$$

$$\boxed{\int \frac{y'}{y}\,dx} = \int 1\,dx$$

$$\int \frac{1}{x}\,dx = \ln(x)$$

$$\boxed{\ln(y)} = x + c \quad || \; e^{(\;)}$$

FORMULA

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + C$$

REMEMBER TO ADD ALSO USELESS FORMULAS

$$y = \cancel{e}^{\ln(y)} = e^{x+c} = e^x \underbrace{e^c}_{=A}$$

ANSWER $\boxed{y = Ae^x}$ AND $y(0) =$

$$\int \frac{y'}{y}\,dx = \ln(y)$$

CHECK

$$D\,\ln(y(x)) = \frac{1}{y(x)} \cdot y'(x) = \frac{y'(x)}{y(x)}$$

OR $y = e^c e^x$ OR

16. (f) $y' = y$ CAN ALSO BE SOLVED BY SOLUTION FORMULA

$$\Rightarrow y' \underbrace{-1}_{=p(x)} \cdot y = \underbrace{0}_{q(x)}$$

$$y' + p(x)\,y = q(x) \Rightarrow \boxed{y(x) = \frac{C}{\mu(x)}} + \cancel{\frac{1}{\mu(x)}\int \mu(x)q(x)\,dx}^{=0}$$

WHERE $\mu(x) = e^{\int p(x)dx}$

• $\int p(x)\,dx = \int -1\,dx = -x \; \cancel{+C}$

• $\mu(x) = e^{\int p(x)\,dx} = e^{-x}$

• $y(x) = \frac{C}{\mu(x)} = \frac{C}{e^{-x}} = Ce^x$

ANSWER $y(x) = Ce^x$

16. (b) $y' = \ln(x) + \tan(x)$

(c) $y' = \sin(x)e^{\cos(x)}$

SOLUTION. 16(b)

$$y' = \ln(x) + \tan(x)$$

$$\Rightarrow \quad y(x) = \int \ln(x) + \tan(x)\,dx$$

$$= \int \ln(x)\,dx + \int \tan(x)\,dx$$

$$= x\ln(x) - x - \ln(\cos(x)) + C$$

11(c) WE FOUND

$$\int \tan(x)\,dx = -\ln(\cos(x)) + C$$

FORMULA

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

CHECK

$$D\left(x\ln(x) - x\right) = 1\cdot\ln(x) + \underbrace{x\cdot\frac{1}{x}}_{=1} - 1$$

$$= \ln(x)$$

$$\int \tan(x)\,dx = -\int \frac{-\sin(x)}{\cos(x)}\,dx = -\int \frac{f'(x)}{f(x)}\,dx = -\ln f(x)$$

$$= -\ln(\cos(x)) + C$$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

FORMULA $\int f'(x)\boxed{e^{f(x)}}\,dx = e^{f(x)}$

$y' = \ln(x) + \tan(x)$

16. (c) $y' = \sin(x)e^{\cos(x)}$

SOLUTION.

$$y' = \sin(x)\, e^{\cos(x)}$$

$$\Rightarrow \quad y(x) = -\int \overbrace{-\sin(x)}^{f'(x)}\, e^{\overbrace{\cos(x)}^{=f(x)}}\,dx$$

$$= -e^{\cos(x)} + C$$

ANSWER

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$\boxed{+ C}$

Exercises using the solution formula

keskiviikko 7. helmikuuta 2024 11.59

$$y' + p(x)y = q(x).$$

In the book [?, pp. 411–413] it is explained how this equation can be solved.

(a) Identify $p(x)$ and $q(x)$.

(b) Calculate $\int p(x)dx$. Don't add a constant C yet.

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$

(d) Calculate $\int \mu(x)q(x)dx$.

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$.

Example. Let's solve $y' + 2xy = x$.

1. We have $p(x) = 2x$ and $q(x) = x$.
2. We have $\int p(x)dx = \int 2xdx = x^2$. (We don't add C.)
3. We have $\mu(x) = e^{x^2}$ and $\frac{1}{\mu(x)} = \frac{1}{e^{x^2}} = e^{-x^2}$.
4. We have

$$\int \mu(x)q(x)dx = \frac{1}{2} \int 2xe^{x^2} dx = \frac{1}{2} e^{x^2}.$$

5. The solution is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx = Ce^{-x^2} + e^{-x^2} \frac{1}{2} e^{x^2}$$

that is

$$y(x) = Ce^{-x^2} + \frac{1}{2}.$$

20. **Exercise. (Possibly an Exam question.)**

Solve
$$y' + \frac{y}{x} = x^2$$
by following the instructions.

(a) Identify $p(x)$ and $q(x)$. Solution $p(x) = \frac{1}{x}$ and $q(x) = x^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Solution $\int p(x)dx = \ln(x)$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Solution $\mu(x) = x$ and $\frac{1}{\mu(x)} = \frac{1}{x}$

(d) Calculate $\int \mu(x)q(x)dx$. Solution $\frac{x^2}{2}$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Solution $y(x) = \frac{C}{x} + \frac{x}{2}$

21. **Exercise.** Solve
$$y'(x) + \tan(x)y = (\cos(x))^2.$$

(a) Identify $p(x)$ and $q(x)$. Solution $p(x) = \tan(x)$ and $q(x) = (\cos(x))^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Solution In an earlier exercise, it was found that $\int p(x)dx = -\ln(\cos(x)) = \ln \frac{1}{\cos(x)}$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$ Solution $\mu(x) = \frac{1}{\cos(x)}$ and $\frac{1}{\mu(x)} = \cos(x)$

(d) Calculate $\int \mu(x)q(x)dx$. Solution $\sin(x)$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Solution $y(x) = C\cos(x) + \cos(x)\sin(x)$

22. **Exercise.** Write $y' = 3y + 2$ in standard form $y' + p(x)y = q(x)$ and solve by same instructions as above. That is, use the solution formula
$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

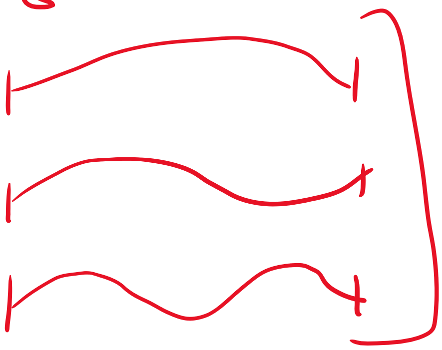
Solution $y = Ce^{3x} - \frac{2}{3}$

23. **Exercise.** Solve $y' = 2y - x^2$. Solution $y = Cx^3 + 6x^2$

Need more exercise? See the course book [?, p. 420, problems 225–232.]

FOURIER SERIES

GUITAR



ADD → REAL VIBRATION

plot y=x/2 and plot y=sin(x)-(1/2)*sin(2*x)+(1/3)*sin(3*x)-(1/4)*sin(4*x)+(1/5)*sin(5*x)

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

Assuming "plot" is a plotting function | Use as referring to geometry instead

Input interpretation

plot $y = \frac{x}{2}$ — OUR ORIGINAL FUNCTION $f(x)$

$y = \sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \frac{1}{5}\sin(5x)$ ← APPROXIMATION BY TRIGONOMETRIC FUNCTIONS

FOURIER SERIES

HOW TO FIND THESE NUMBERS, NAMED FOURIER COEFFICIENTS

Plot

$\cos(x)$, $\cos(2x)$, $\cos(3x)$

$\sin(x)$, $\sin(2x)$, ...

FORMULA

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)}$$

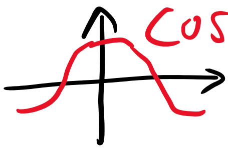
WHERE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

2 · AVERAGE OF f OVER $[-\pi, \pi]$

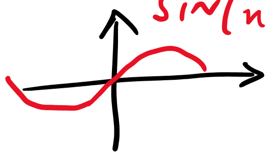
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

EVEN $\cos(nx)$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

ODD $\sin(nx)$

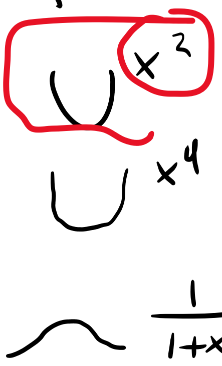


$f(x)$

x^2

∞

$f(x)$



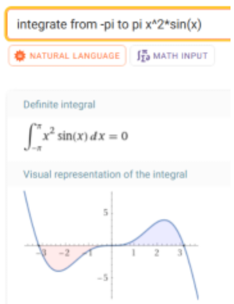
x^2

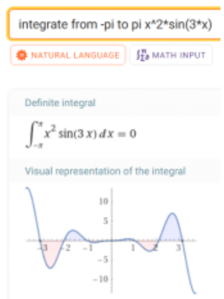
x^4

$\frac{1}{1+x^4}$

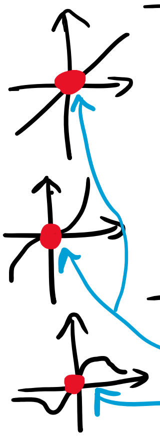
EVEN FUNCTIONS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\boxed{\cos(nx)}}_{\text{NEEDED}} + \underbrace{b_n}_{=0} \cancel{\sin(nx)}$$





$f(x)$ ODD FUNCTIONS



x

x^3

$\frac{x}{1+x^4}$

$$f(x) = \underbrace{\cancel{\frac{a_0}{2}}}_{=0} + \sum_{n=1}^{\infty} \underbrace{a_n}_{=0} \cancel{\cos(nx)} + \underbrace{b_n \sin(nx)}_{\text{NEEDED}}$$

GO THROUGH ORIGIN ⇒ CANNOT PUT $+1$ OR $+2$ OR...

EXAMPLE. FIND THE FOURIER SERIES OF $f(x)=x$.

SOLUTION $f(x)=x$ IS ODD $\Rightarrow \begin{cases} a_0 = 0 \\ a_n = 0 \end{cases}$.

JUST NEED TO CALCULATE $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(nx)\,dx$.

LET'S PRACTICE MORE NEXT WEEK. WATCH VIDEOS OR READ BEFOREHAND IF YOU LIKE.

$\| -2y$

23. **Exercise.** Solve $y' = 2y - x^2$. Solution $y = Cx^3 + 6x^2$

Need more exercise? See the course book [?, p. 420, problems 225–232.]

SOLUTION. LET'S WRITE IN THE FORM

$$y' + p(x)y = q(x)$$

$$y' + p(x)y = q(x)$$

$$y' \underbrace{-2}_{p(x)} y = \underbrace{-x^2}_{q(x)}$$

$$\int p(x)\,dx = \int -2\,dx = -2x \cancel{+C} \Rightarrow \mu(x) = e^{\int p(x)dx} = e^{-2x}$$

$$\frac{1}{\mu(x)} = e^{2x}$$

$$\int \mu(x) q(x)\,dx = \int e^{-2x}(-x^2)\,dx$$ CANNOT SIMPLIFY

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) q(x)\,dx$$

$$= Ce^{2x} + e^{2x} \underbrace{\int e^{-2x}(-x^2)\,dx}_{=?}$$

TODAY WE CAN CALCULATE IN 2021, 22 THIS WAY

$$\int e^{-2x}(-x^2)\,dx$$ OK, WE CAN CALCULATE BY PARTIAL INTEGRATION

$$\neq \int e^{-2x}dx + \int -x^2 dx$$

$$-x^2 \xrightarrow{D} -2x \xrightarrow{D} -2 \xrightarrow{D} \cancel{0}$$

$$x = \int 1\,dx$$
$$= \int 1\cdot 1\cdot 1\,dx$$
$$= \int 1dx + \int 1dx + \int 1dx$$
$$= x + x + x = 3x$$

$$\int \underline{e^{-2x}}\,\underline{(-x^2)}\,dx = \underline{-\tfrac{1}{2}e^{-2x}}(-x^2) - \int \underline{-\tfrac{1}{2}e^{-2x}}(-2x)\,dx$$

INT

$$= \frac{x^2}{2} e^{-2x} + \underbrace{\int e^{-2x} x \, dx}$$

$$= +\frac{1}{2} e^{-2x} x + \underbrace{\int -\frac{1}{2} e^{-2x} \cdot 1 \, dx}$$

$$= -\frac{1}{2}\left(-\frac{1}{2}\right) e^{-2x}$$

$$= \frac{x^2}{2} e^{-2x} + \frac{x}{2} e^{-2x} + \frac{1}{4} e^{-2x} = \frac{1}{2}\left(x^2 e^{-2x} + x e^{-2x} + \frac{1}{2} e^{-2x}\right)$$

$$= \frac{e^{-2x}}{4}\left(2x^2 + 2x + 1\right)$$

WolframAlpha

integral e^(-2*x)*(-x^2) dx

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD

Indefinite integral | Approximate

$$\int e^{-2x}(-x^2)\,dx = \frac{1}{4} e^{-2x}(2x^2 + 2x + 1) + \text{constant}$$

ABOVE WE USED THE FORMULA

https://en.wikipedia.org/wiki/Integration_by_parts

where we neglect writing the constant of integration. This yields the formula for **integration by parts**:

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx,}$$

$$y(x) = \frac{c}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x) q(x) \, dx$$

$$= c e^{2x} + \cancel{e^{2x}} \cdot \frac{\cancel{e^{-2x}}}{4}\left(2x^2 + 2x + 1\right)$$

ENGLISH A B C ... X Y Z ~~Å Ä Ö~~

FINNISH A B C ... ~~X~~ Y ~~Z~~ Å Ä Ö

XURITO ?

XYLITOL = KSYLITOL

20.

20. **Exercise. (Possibly an Exam question.)**
Solve
$$y' + \frac{y}{x} = x^2 \Rightarrow y' + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{x^2}_{q(x)}$$
by following the instructions.

(a) Identify $p(x)$ and $q(x)$. Solution $p(x) = \boxed{\frac{1}{x}}$ and $q(x) = x^2$

(b) Calculate $\int p(x)dx$. Don't add a constant C yet. Solution $\int p(x)dx = \boxed{\ln(x)}$

7

$e^{\ln(x)} = x$

(c) Simplify $\mu(x) = e^{\int p(x)dx}$ and $\frac{1}{\mu(x)}$. Solution $\mu(x) = x$ and $\frac{1}{\mu(x)} = \frac{1}{x}$

(d) Calculate $\int \mu(x)q(x)dx$. Solution $\frac{x^4}{4}$

(e) The solution is $y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \int \mu(x)q(x)dx$. Solution $y(x) = \frac{C}{x} + \frac{x^3}{4}$

$$\int x \cdot x^2 \, dx = \int x^3 \, dx = \frac{x^4}{4}$$

(e) $$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)} \underbrace{\int \mu(x) q(x) \, dx}_{= \frac{x^4}{4}}$$
$$= \frac{C}{x} + \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{4}}}{4} = \frac{C}{x} + \frac{x^3}{4}$$

$$\int \tan(x) \, dx$$

Discrete Fourier transformation

lauantai 10. helmikuuta 2024 11.44

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{a_n \boxed{\cos(nx)}}_{①} + \underbrace{b_n \boxed{\sin(nx)}}_{②}$$

WHERE

$$\begin{cases} a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\boxed{\cos(nx)}\,dx \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx \end{cases}$$

$\boxed{e^{it}} \leftrightarrow \begin{cases}\cos(t)\\ \sin(t)\end{cases}$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

EULER $e^{it} = \cos(t) + i\sin(t)$

$+\ e^{-it} = \cos(t) - i\sin(t)$

$e^{it} + e^{-it} = 2\cos(t)$

EXAMPLE

$$f(x) = 0.7\sin(x) + 0.4\sin(7x)$$

$b_3 = 0$

$\sin(3x)$

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi}(0 + 0)\quad \sin(3x)\,dx$$

b_1

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} 0.7\sin(x)\sin(3x)\,dx = 0$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} 0.4\sin(7x)\sin(3x)\,dx = 0$$

hence 1, 2, 3, 4, ...

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx =: \hat{f}(k).$$

For a general function f we seek for the corresponding representation

EASY MULAS +

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

MINUS ?

EASY FORMULAS

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

+

$$f(x) = \sum_{n=-\infty}^{\infty} \int c_n \boxed{e^{inx}} \, dn$$

MINUS ?

WHERE

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \boxed{e^{-inx}} \, dx$$

$$\boxed{e^a e^b = e^{a+b}}$$

$$e^{inx} e^{-inx} = e^{inx - inx} = e^0 = \boxed{1}$$

Fourier transform

We define

TRANSFORM ORIG $\boxed{\text{SIGNAL}}$

TIME

FREQUENCY

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt,$$

$$\sum_{n=0}^{N} \quad N = 2, 4$$

which implies

+

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw.$$

$N = 100$

USED IN THEORETICAL PHYSICS

ORIG

TRANSFORM

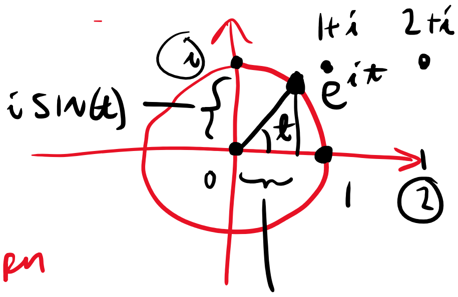
https://en.wikipedia.org/wiki/Discrete_Fourier_transform

Discrete Fourier transform

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n} \quad \textbf{(Eq.1)}$$

TRANSFORM ORIGINAL

$e^{it} = \cos(t) + i\sin(t)$



The inverse transform is given by:

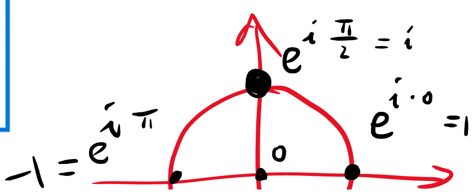
TRANSFORM

ORIG

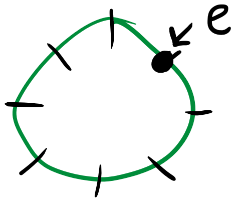
Inverse transform

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i2\pi \frac{k}{N} n} \quad \textbf{(Eq.2)}$$

$e^{i\frac{2\pi}{8}}$

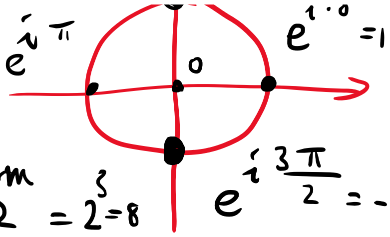


$k=0$



$e^{i\frac{2\pi}{8}}$

$\left(e^{i\frac{2\pi}{N}}\right)^{kn}$ $N = 2^m = 2^3 = 8$



https://en.wikipedia.org/wiki/Discrete_Fourier_transform

BEST WAY TO CALCULATE

Discrete Fourier transform

$$\cancel{X_k} = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n} \quad \textbf{(Eq.1)}$$

y_R

TRANSFORM ORIGINAL

MINUS

$\leftarrow$ FFT

The inverse transform is given by:

Inverse transform

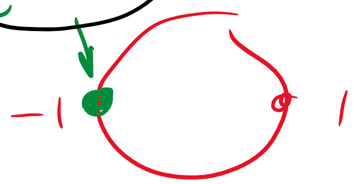
$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} \cancel{X_k} \cdot e^{i2\pi \frac{k}{N} n} \quad \textbf{(Eq.2)}$$

ORIG TRANSFORM y_R

IFFT

kn

$e^{i\frac{2\pi}{N}}$



$e^{i\frac{2\pi}{8}}$ -1

$2\pi \quad kn$

$$e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$$

EXAMPLE

$N = 2$

ORIGINAL SIGNAL $\quad x_n: \quad \underbrace{x_0 \quad x_1} \quad \underbrace{x_0 \quad x_1} \quad \underbrace{x_0 \quad x_1}$

TRANSFORM $\quad y_R: \quad \underbrace{y_0 \quad y_1} \quad \underbrace{y_0 \quad y_1} \quad \underbrace{y_0 \quad y_1}$

$x_0 = 2 \quad x_1 = 3$

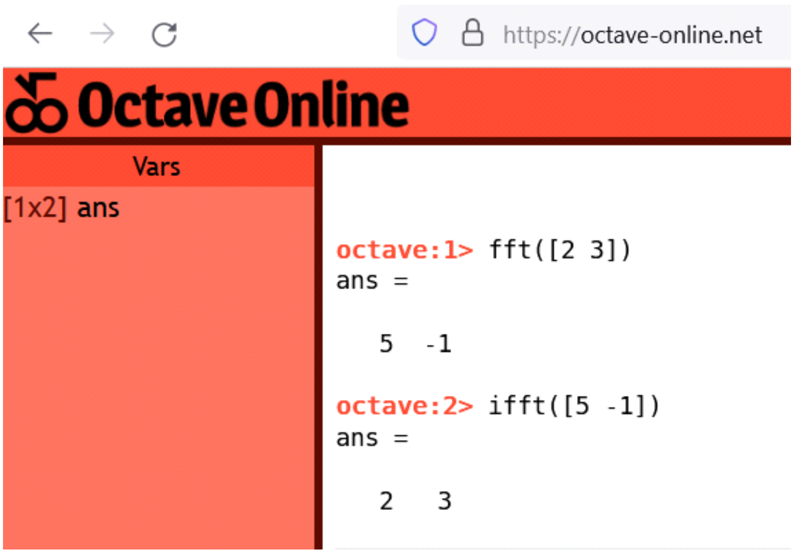
$$\left\{ y_0 = x_0 (-1)^{0\cdot 0 = 0} + x_1 (-1)^{0\cdot 1 = 0} = x_0 + x_1 = 5 \right.$$

$$(-1)^{1\cdot 0 = 0} \qquad (-1)^{1\cdot 1 = 1} \qquad = -1$$

$$y_1 = x_0 (-1)^{\overbrace{1\cdot 0}^{=0}} + x_1 \underbrace{(-1)^{\overbrace{1\cdot 1}^{=1}}}_{=-1} = x_0 - x_1 = -1$$

$y_0 \quad y_1$

$$\begin{cases} x_0 = \frac{1}{2}\left(y_0 (-1)^{+0\cdot 0} + y_1 (-1)^{+0\cdot 1} \right) = \frac{1}{2}(y_0 + y_1) = 2 \\ x_1 = \frac{1}{2}\left(y_0 (-1)^{+1\cdot 0} + y_1 (-1)^{+1\cdot 1} \right) = \frac{1}{2}(y_0 - y_1) = 3 \end{cases}$$

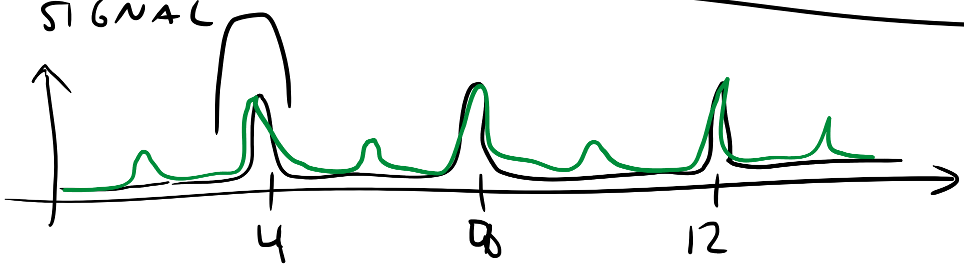


x_n ODD $\longrightarrow$ y_n ? REAL VALUES

x_n EVEN $\longrightarrow$ y_n ? ~~JUST~~ PURELY IMAGINARY VALUES

DISCRETE SIGNAL

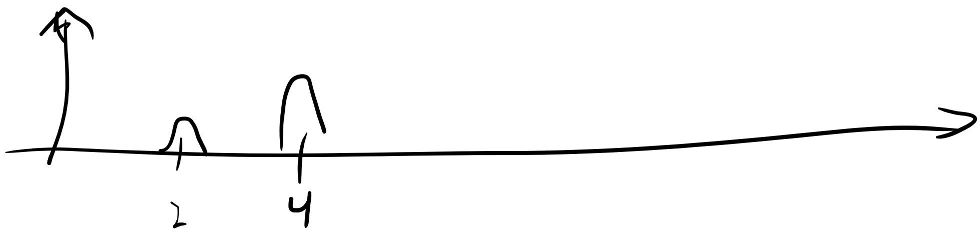
x_n :



FFT

y_n :

ABS(y_n)



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CONSTANT WITH RESPECT TO x

$$\frac{d}{dx} x^2 y^3 = y^3 \frac{d}{dx} x^2 = \underline{\underline{y^3 \cdot 2x}}$$

$$\frac{d}{dy} \underset{\text{CONSTANT}}{x^2} y^3 = \underline{\underline{x^2 \cdot 3y^2}}$$

24. Calculate the partial derivatives

(a) $\frac{\partial}{\partial x} x^2 t + e^x + 7$ $= 2xt + e^x + 0$

(b) $\frac{\partial}{\partial t} x^2 t + e^x + 7$ $= x^2 \cdot 1 + 0 + 0$

(c) $\frac{\partial}{\partial x} \sin(2x) + \sin(3t)$ $= \cos(2x) \cdot 2 + 0$

(d) $\frac{\partial}{\partial t} \sin(2x) \sin(3t)$ $= \sin(2x) \cdot \cos(3t) \cdot 3$

CONSTANT

NEXT COUPLE EXAMPLES
ODE
FFT

MORE PARTIAL DERIVATIVES

$$\boxed{\frac{d}{dx} x^{n} = n x^{n-1}}$$

$$\frac{d}{dx} x^t = t x^{t-1}$$

$$\frac{d}{dt} x^t = x^t \ln(x)$$

$$\frac{d}{dt} e^t = e^t$$

$$\frac{d}{dt} 2^t = 2^t \ln(2)$$

$$\boxed{\frac{d}{dx} x = 1}$$

$$\frac{d}{dx} \frac{x}{t} = \frac{1}{t}$$

$$\frac{d}{dt} \frac{1}{t} = \frac{d}{dt} t^{-1} = -1 \cdot t^{-1-1} = -\frac{1}{t^2}$$

$$\frac{d}{dt} \frac{x}{t} = -\frac{x}{t^2}$$

$$\frac{d}{dx} \sin(x + t^2) = \cos(x + t^2) \cdot \underbrace{\frac{d}{dx}(x + t^2)}_{=1+0}$$

$$\frac{d}{dt} \sin(x + t^2) = \cos(x + t^2) \cdot \frac{d}{dt}(x + t^2)$$

a $\cos(x+t^2)\cdot \underbrace{\frac{d}{dt}(x+t^2)}_{=0+2t}$

ODE EXAMPLE

$$y' + p(x)y = q(x) \rightarrow y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx$$

WHERE $\mu(x) = e^{\int p(x)\,dx}$

(WORKS ALMOST ALWAYS IN THIS COURSE)

EXAMPLE. FIRST ORDER LINEAR EQ

$$y' + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{x^5}_{q(x)}$$

1 → CAN USE SOLUTION FORMULA

2 $p(x)$

3 $\int p(x)\,dx = \int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln(x) \cancel{+C}$

4 $\mu(x) = e^{\int p(x)dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$

5 $\int \mu(x)q(x)\,dx = \int x^2 \cdot x^5\,dx = \int x^7\,dx = \frac{x^8}{8} \cancel{+C}$

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx$$

$$= \frac{C}{x^2} + \frac{1}{\cancel{x^2}}\cdot\frac{x^{\cancel{8}\,6}}{8} \Rightarrow y(x) = \frac{C}{x^2} + \frac{x^6}{8}$$

WE CAN CHECK BY SUBSTITUTING TO THE EQ. (NO NEED TO DO IN EXAM)

$$y'(x) = \frac{-2C}{x^3} + \frac{6x^5}{8}$$

TO THE EQ. EXAM)

CHECK $y' + \frac{2}{x} y = x^5$

$$\frac{-2\cancel{c}}{\cancel{x^3}} + \frac{6x^5}{8} + \frac{2}{x}\left(\frac{\cancel{c}}{\cancel{x^2}} + \frac{x^6}{8}\right) = x^5$$

$$\frac{6x^5}{8} + \frac{2}{\cancel{x}} \cdot \frac{x^{\cancel{6}\,5}}{8} = x^5$$

$$\frac{6}{8} + \frac{2}{8} \qquad \frac{6}{8}x^5 + \frac{2}{8}x^5 = 1 \cdot x^5 \quad \text{OK}$$

$$= \frac{6+2}{8} = \frac{8}{8} = 1$$

$$y' + \frac{2}{x} y = x^5$$

(*)

HOW TO SOLVE WITHOUT THE FORMULA?

CAN BE DONE

DIFFICULT/CONFUSING?

(NO NEED TO KNOW IN EXAM)

$y' + \frac{2}{x} y = x^5 \longrightarrow$ ONE SOLUTION $y_{NH}(x)$

$y' + \frac{2}{x} y = 0 \longrightarrow$ GENERAL SOLUTION $y_H(x)$

(*) GENERAL SOLUTION $y(x) = y_H(x) + y_{NH}(x)$

① $y' + \frac{2}{x} y = x^5 \longrightarrow$ ONE SOLUTION $y_{NH}(x)$

GUESS $y(x) = C x^6$

$y'(x) = 6C x^5$

MUST BE $C = \frac{1}{8}$

$$y' + \frac{2}{x} y = 6C x^5 + \frac{2}{\cancel{x}} C x^{\cancel{6}\,5} = C\underbrace{(6+2)}_{8} x^5 = 1 \cdot x^5$$

$$y_{NH}(x) = \frac{x^6}{8}$$

② $y' + \frac{2}{x} y = 0 \Rightarrow y' = -\frac{2}{x} y \quad \| : y$

$$\Rightarrow \frac{y'}{y} = -\frac{2}{x}$$

$y' = \frac{dy}{dx}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{2}{x} \quad \Big| \cdot dx$$

$$\Rightarrow \int \frac{1}{y} dy = \int -\frac{2}{x} dx$$

$$\Rightarrow \ln(y) = -2\ln(x) + C \quad \| e^{(\ )}$$

$$y = \cancel{e}^{\cancel{\ln}(y)} = e^C e^{-2\ln(x)} = e^C x^{-2} = \frac{1}{x^2}\underbrace{e^C}_{A}$$

$$\Rightarrow y_H(x) = \frac{A}{x^2}$$

$y' + \frac{2}{x} y = x^5$

$$y(x) = y_H + y_{NH} = \frac{A}{x^2} + \frac{x^6}{8}$$

NEXT

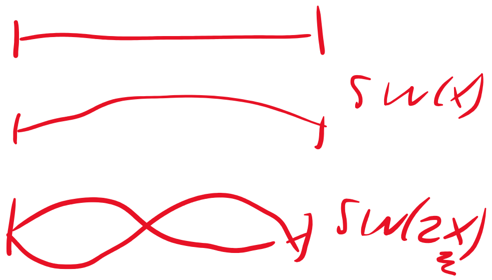
SHAPE TIME FFT

25. Show that

$$\cancel{y}(x,t) = \overbrace{\sin(nx)}\,\underbrace{\sin(nct)},$$

where n is a natural number, is a solution of the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y = c^2 \dfrac{\partial^2}{\partial x^2} y \\ y(0,t) = 0 \\ y(\pi,t) = 0. \end{cases}$$



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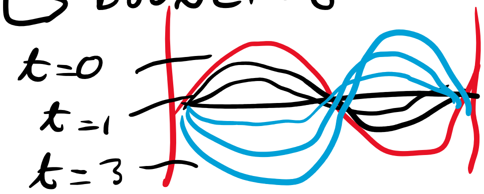


SOLUTION.

$$y\cancel{(x,t)} = \sin(nx)\,\underline{\sin(nct)}$$

$$\frac{\partial}{\partial t} y = \sin(nx)\cos(nct)\cdot nc$$

$$\frac{\partial^2}{\partial t^2} y = \sin(nx)\left(-\sin(nct)\right)\cdot nc\cdot nc = \underline{-\sin(nx)\sin(nct)n^2c^2}$$

$$\frac{\partial}{\partial x} y = \cos(nx)\cdot n\sin(nct)$$

$$\frac{\partial^2}{\partial x^2} y = -\sin(nx)\,\underline{n^2}\sin(nct) \,\|\cdot \underline{c^2}$$

$$c^2\frac{\partial^2}{\partial x^2} = \underline{-\sin(nx)\sin(nct)n^2c^2}$$

SAME!

$$\boxed{\frac{\partial^2}{\partial t^2} y = c^2 \frac{\partial^2}{\partial x^2}}$$

25. Show that

$$y(x,t) = \sin(nx)\sin(nct),$$

where n is a natural number, is a solution of the problem

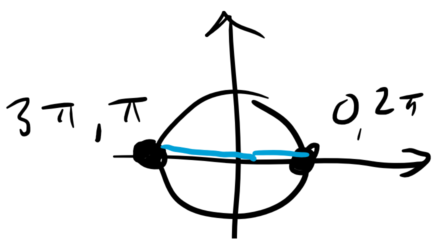
$$\begin{cases} \dfrac{\partial^2}{\partial t^2} y = c^2 \dfrac{\partial^2}{\partial x^2} y \\ y(0,t) = 0 \\ y(\pi,t) = 0. \end{cases}$$

y AT $x=0$ ALWAYS 0

y AT $x=\pi$ ALWAYS 0

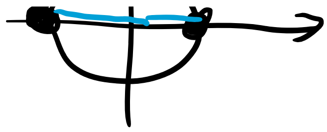
$$y(x,t) = \sin(nx)\sin(nct)$$

$$y(0,t) = \underbrace{\sin(0)}_{=0}\sin(nct) = 0$$



$$y(\pi,t) = \sin(n\pi)\sin(nct) = 0$$

$y(\pi, t) = \underbrace{\sin(n\pi)}_{=0} \sin(nct) = 0$



NOT HARD?

26. For the sine wave $\frac{3}{2}\sin(7t-1)$, what is the
    (a) amplitude
    (b) angular frequency
    (c) phase
    (d) period

27. Express $6\sin(4t+\pi/3)$ as the sum of sine and cosine, that is, in the form $C\sin(\omega t)+D\cos(\omega t)$. Hint. Remember $\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)$.

28. Express $5\sin(2t)+2\cos(2t)$ by one sine wave, that is, in the form $A\sin(\omega t+\phi)$
Hint. By using the vector dot product, we have $5\sin(2t)+2\cos(2t)=(5,2)\cdot(\sin(2t),\cos(2t))$. Let's express $(x,y)=(5,2)$ in polar coordinates $(x,y)=r(\cos(\alpha),\sin(\alpha))$, where $r=\sqrt{x^2+y^2}$ and $\alpha=\arctan(y/x)$.
Solution We obtain $r=\sqrt{29}$ and $\alpha=\arctan(2/5)\frac{180^\circ}{\pi}=21.8^\circ$. Thus

$$(5,2)=\sqrt{29}(\cos(21.8^\circ),\sin(21.8^\circ)).$$

Therefore

$$5\sin(2t)+2\cos(2t)=\sqrt{29}(\sin(2t)\cos(21.8^\circ)+\cos(2t)\sin(21.8^\circ))=\sqrt{29}\sin(2t+21.8^\circ).$$

The result can be obtained directly with the formulas $A=\sqrt{x^2+y^2}$ and $\phi=\arctan(y/x)$.

34. Consider a square wave $g(t)$ whose amplitude is 3 and the period is $T=2$. When $0\le t\le 2$, the function is defined piece-wise as

$$g(t)=\begin{cases}3, & \text{if } 0\le t\le 1,\\ 0, & \text{if } 1\le t\le 2\end{cases}$$

and outside this time interval the function repeats itself periodically. Find the Fourier coefficients a_0, a_n and b_n.

35. For the function

$$f(t)=\begin{cases}1, & \text{if } 0\le x\le \pi\end{cases}$$

NOT

35. For the function

$$f(t) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ -1, & \text{if } -\pi \le x \le 0 \end{cases},$$

find its Fourier series.

NOT EASY ?

FFT ?

30. Solve numerically with Python's Scipy library, and analytically with Python's Sympy library. Use the initial condition $y(0) = 2$ in the numerical solution. Draw the graphs of the numerical and analytic solutions. (Use same initial condition for the analytic solutions.)

(a) $y' + 7y = 0$

(b) $y' + \cos(x)y = 0$

(c) $xy' + 3x^2y = 0$

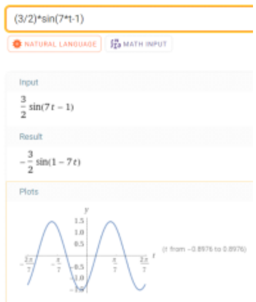
NO. ILPO KNOWS PYTHON BETTER.

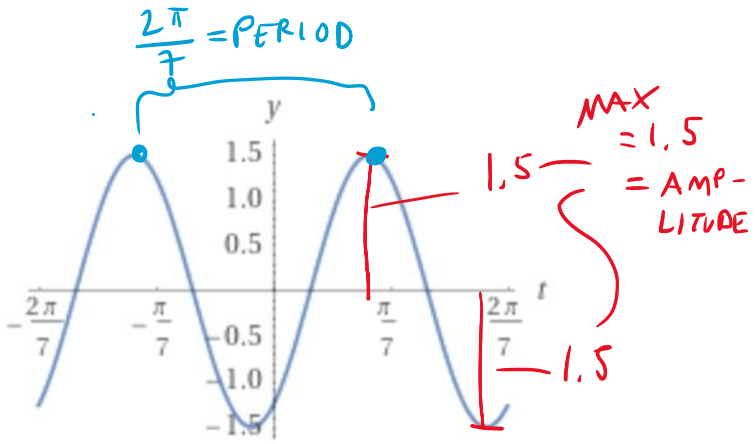
36. Write a Python program which removes the noise from a function which is the sum of two sine waves. You can use, for example, the wave

```
noisy_signal = np.sin(a)+np.sin(3*a-1)+10*np.random.rand(len(a))
```

TODAY
- YOU CAN STILL ASK ABOUT DIFF. EQUATIONS
- WE CAN CALCULATE 26-28 AND 34-35
- THERE ARE ALSO EXERCISES ABOUT FFT (CHECK IF YOU ARE INTERESTED)

NEXT WEEK
- 28.2. YOU CAN ASK ABOUT THINGS
- 1.3. EXAM





26. For the sine wave $\frac{3}{2}\sin(7t-1)$, what is the
(a) amplitude $= \frac{3}{2}$
(b) angular frequency $= 7$ (rad/s)
(c) phase $= -1$
(d) period $= \frac{2\pi}{7}$

SIN(X) PERIOD IS $2\pi \approx 6.3$
SIN(3X) PERIOD IS $\frac{2\pi}{3} \approx 2.1$
SIN(9X) PERIOD IS $\frac{2\pi}{9} \approx 0.7$

EXAMPLE. $\frac{4}{3}\cos(4t-2) = \frac{4}{3}\sin(4t \underbrace{-2 + \frac{\pi}{2}}_{\text{PHASE}})$

AMPLITUDE $= \frac{4}{3}$
ANGULAR FREQUENCY $= 4$
PHASE $= -2 + \frac{\pi}{2}$
PERIOD $= \frac{2\pi}{4}$

IF YOU CHANGE COS TO SIN, NEED TO ADD $+\frac{\pi}{2}$

$\cos(2.1) = \sin(2.1 + \frac{\pi}{2})$

WANT TO WRITE USING SIN? NEED TO ADD

27. Express $6\sin(4t+\pi/3)$ as the sum of sine and cosine, that is, in the form $C\sin(\omega t)+D\cos(\omega t)$. Hint. Remember $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

SOLUTION.

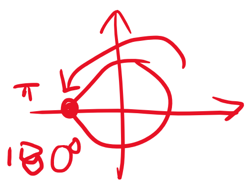
$6\sin(4t + \frac{\pi}{3})$

$= 6\left(\sin(4t)\cos(\frac{\pi}{3}) + \cos(4t)\sin(\frac{\pi}{3})\right)$

$= \underbrace{6\cos(\tfrac{\pi}{3})}_{C = 3} \sin(4t) + \underbrace{6\sin(\tfrac{\pi}{3})}_{D \approx 5.2} \cos(4t)$, $\cos(\frac{\pi}{3}) = \frac{1}{2}$

$\approx 3\sin(4t) + 5.2\cos(4t)$

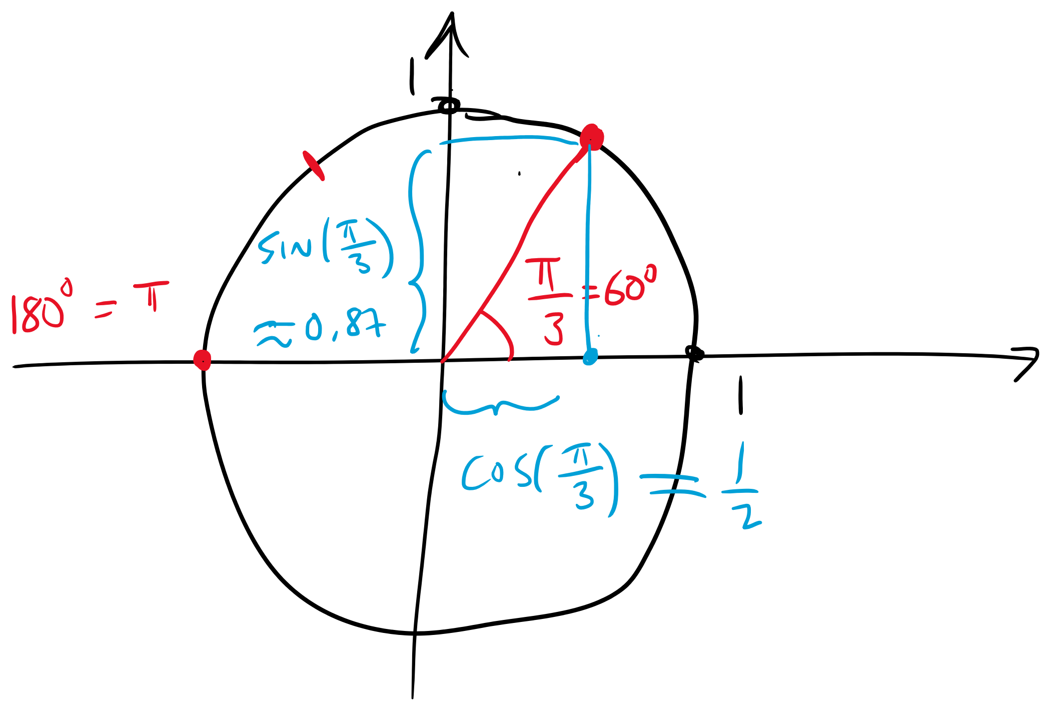
$\pi = 3.14$



$\sin(\frac{\pi}{3}) = \sin(\frac{3.1415}{…})$

$$\approx 3\sin(4t) + 5.2\cos(4t)$$

$$\sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{3.1415}{3}\right) = \sin(1.047) = \sin\left(\frac{\pi}{3}\cdot\frac{180^\circ}{\pi}\right) = \sin(60^\circ)$$



NOT IN THE EXAM

$\omega = 2$

28. Express $5\sin(2t) + 2\cos(2t)$ by one sine wave, that is, in the form $A\sin(\omega t + \phi)$

Hint. By using the vector dot product, we have $5\sin(2t) + 2\cos(2t) = (5, 2)\cdot(\sin(2t), \cos(2t))$. Let's express $(x, y) = (5, 2)$ in polar coordinates $(x, y) = r(\cos(\alpha), \sin(\alpha))$, where $r = \sqrt{x^2 + y^2}$ and $\alpha = \arctan(y/x)$.

Solution We obtain $r = \sqrt{29}$ and $\alpha = \arctan(2/5)\frac{180^\circ}{\pi} = 21.8^\circ$. Thus

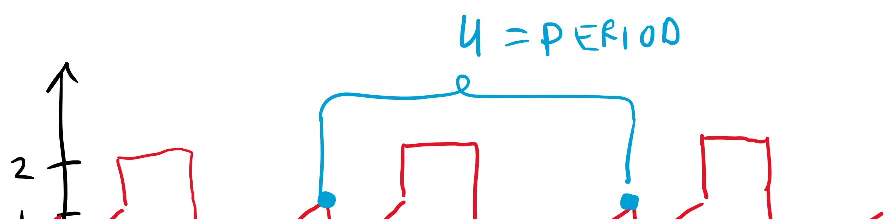
$$(5, 2) = \sqrt{29}(\cos(21.8^\circ), \sin(21.8^\circ)).$$

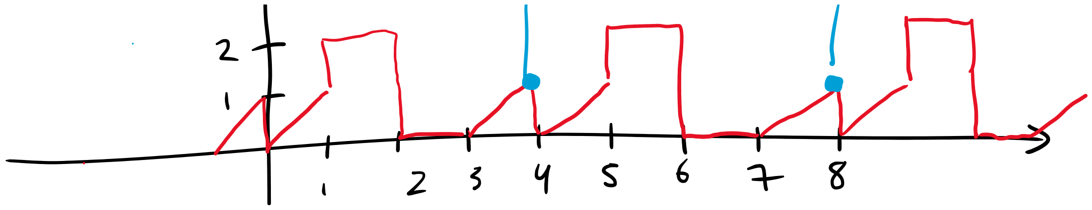
Therefore

$$5\sin(2t) + 2\cos(2t) = \sqrt{29}(\sin(2t)\cos(21.8^\circ) + \cos(2t)\sin(21.8^\circ)) = \sqrt{29}\sin(2t + 21.8^\circ).$$

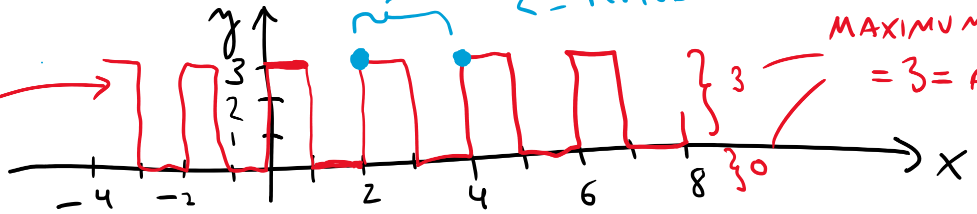
The result can be obtained directly with the formulas $A = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$

EXAMPLE.





PERIOD ? ① CHOOSE TOTALLY TWO SIMILAR PLACES
② FIND THEIR DIFFERENCE IN X



34. Consider a square wave $g(t)$ whose amplitude is 3 and the period is $T = 2$. When $0 \le t \le 2$, the function is defined piece-wise as

$$g(t) = \begin{cases} 3, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } 1 \le t \le 2 \end{cases}$$

and outside this time interval the function repeats itself periodically. Find the Fourier coefficients a_0, a_n and b_n.

HINT. LET $f(x) = \begin{cases} 3 & \text{if } 0 \le x \le \pi \\ 0 & \text{if } -\pi \le x \le 0 \end{cases}$ $\rightarrow$ $f(\underbrace{\pi t}_{x}) = g(t)$

USE

$$\boxed{\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\ \text{WHERE} \quad a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{aligned}}$$

THEN SUBSTITUTE $x = \pi t$ TO THE SERIES

EXAMPLE. CALCULATE BY HAND $FFT([2,3])$.

SOLUTION

$y_0 = 2 + 3 = 5$

$y_1 = 2 - 3 = -1$.

ANSWER: $[5, -1]$

35. For the function

$f(t) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \end{cases}$ ← DEFINED

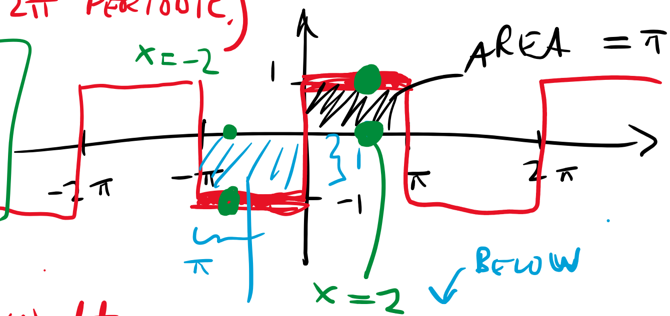
35. For the function

$$f(t) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ -1, & \text{if } \underbrace{-\pi \le x \le 0}, \end{cases}$$ ⟸ DEFINED PIECEWISE

find its Fourier series. (ASSUME $f(t)$ IS 2π PERIODIC)

SOLUTION. METHOD 1

$f(-x) = -f(x) \Rightarrow f$ ODD



AREA $= -\pi \cdot 1 = -\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\underbrace{\int_{-\pi}^{0} -1\,dt} + \frac{1}{\pi}\underbrace{\int_{0}^{\pi} 1\,dt}_{=\pi} = \frac{-\pi}{\pi} + \frac{\pi}{\pi} = \underline{\underline{0}}$$

$$= \Big|_{-\pi}^{0} -t = -(0) - [-(-\pi)] = \underline{\underline{-\pi}}$$

$\boxed{a_0 = 0}$

METHOD 2 BECAUSE THE FUNCTION IS ODD $\left\{ \boxed{\begin{matrix} a_0 = 0 \\ a_n = 0 \end{matrix}} \right.$

f ODD $\Leftrightarrow f(-x) = -f(x)$

$b_n = ?$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

DRAW $f(t)\sin(2t)$ TO SEE THIS

$$= \frac{2}{\pi}\int_{0}^{\pi} \underbrace{f(t)}_{=1}\sin(nt)\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin(nt)\,dt$$

$$= \frac{2}{\pi}\Big|_{0}^{\pi} \frac{-\cos(nt)}{n} = \frac{2}{\pi}\left[\frac{\overbrace{-\cos(n\pi)}^{=(-1)^{n+1}} \overbrace{-(-\cos(0))}^{=1}}{n}\right] = \frac{2}{\pi}\,\frac{1+(-1)^{n+1}}{n}$$

FOURIER SERIES

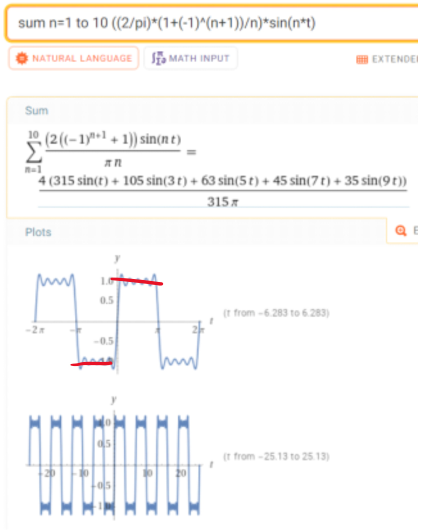
$$f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{1+(-1)^{n+1}}{n}\sin(nt)$$

sum n=1 to 10 ((2/pi)*(1+(-1)^(n+1))/n)*sin(n*t)

NATURAL LANGUAGE MATH INPUT EXTENDE

OK, IT WORKS

−1, 1, −1, 1



Extra material

maanantai 19. helmikuuta 2024 12.13

The exam is based on the exercises above. The following exercises are just in case you are interested about the topic.

(There might be some mistakes in the formulas.)

36. Let $[x_0, x_1] = [2, 3]$. Calculate

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}.$$

Check the answer in OctaveOnline with
y=fft($[x_0, x_1]$).

Solution Solution $[y_0, y_1]$=[5,-1].

37. Let $[y_0, y_1]$=[5,-1]. Calculate

$$\begin{cases} x_0 &= \frac{1}{2}(x_0 + x_1) \\ x_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}.$$

Check the answer in OctaveOnline with
x=fft($[y_0, y_1]$).

Solution Solution $[x_0, x_1] = [2, 3]$.

38. Let $[x_0, x_1, x_2, x_3] = [1, 2, 1, 2]$. Calculate

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - \cancel{i}x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}.$$

— NO i HERE

Check the answer in OctaveOnline with
y=fft($[x_0, x_1, x_2, x_3]$).

Solution Solution $[y_0, y_1, y_2, y_3] = [6, 0, -2, 0]$.

39. Let $[y_0, y_1, y_2, y_3] = [6, 0, -2, 0]$. Calculate

$$\begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - \cancel{i}x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}.$$

— NO i HERE

Check the answer in OctaveOnline with
x=ifft($[x_0, x_1, x_2, x_3]$).

Solution Solution $[x_0, x_1, x_2, x_3] = [1, 2, 1, 2]$.

Practice exam

keskiviikko 28. helmikuuta 2024 11.53

Applied Mathematics and Physics in Programming ID00CS50-3003

Mathematics, teacher: Juha-Matti Huusko, juha-matti.huusko@oamk.fi

Answer to all six questions. In the end of the pdf file, there are some formulas.

* * *

1. Give the requested examples.
   - (a) A differential equation of order 3.
   - (b) A separable differential equation.
   - (c) A non-separable differential equation.
   - (d) A non-linear differential equation.
   - (e) A linear and homogeneous differential equation.
   - (f) A linear and non-homogeneous differential equation.
2. Show that the functions y are solutions to the corresponding differential equations.
   - (a) Show that $y = \frac{1}{1-x}$ is a particular solution for $y' = y^2$.
   - (b) Show that $y = 3 - x + x\ln(x)$ is a particular solution for $y' = \ln(x)$.
3. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -30)$?
4. Solve
$$y' - \tan(x)y = 3(\sin(x))^2.$$
5. For the function
$$f(t) = \begin{cases} 1, & \text{if } 0 \le x \le \pi \\ -1, & \text{if } -\pi \le x \le 0 \end{cases}$$
find its Fourier series.
6. Find the discrete Fourier transform of $[2, 3]$. In other words, calculate by hand `fft([2,3])`.

In the following pages, there are some formulas.

Formulas

Differentiation and integration

Differentiation

$$\begin{aligned} Dx^n &= nx^{n-1} \\ De^x &= e^x \\ Db^x &= b^x\ln(b) \\ D\ln(x) &= \frac{1}{x} \\ D\ln|x| &= \frac{1}{x} \\ D\log_a(x) &= \frac{1}{x\ln(a)} \\ D\log_a|x| &= \frac{1}{x\ln(a)} \\ D\sin(x) &= \cos(x) \\ D\cos(x) &= -\sin(x) \\ D\tan(x) &= 1+\tan^2(x) \\ Dx\ln(x) - x &= \ln(x) \end{aligned}$$

$$\begin{aligned} D\arcsin(x) &= \frac{1}{\sqrt{1-x^2}} \\ D\arccos(x) &= \frac{1}{-\sqrt{1-x^2}} \\ D\arctan(x) &= \frac{1}{1+x^2} \\ D\sinh(x) &= \cosh(x) \\ D\cosh(x) &= \sinh(x) \\ D\tanh(x) &= \frac{1}{\cosh^2(x)} \end{aligned}$$

Integration

$$\begin{aligned} \int x^n dx &= \frac{x^{n+1}}{n+1} + C \\ \int e^x dx &= e^x + C \\ \int b^x dx &= \frac{b^x}{\ln(b)} \\ \int \frac{1}{x} dx &= \ln|x| + C \\ \int \cos(x)dx &= \sin(x) + C \\ \int \sin(x)dx &= -\cos(x) + C \\ \int 1+\tan^2(x)dx &= \tan(x) + C \\ \int \ln(x)dx &= x\ln(x) - x + C \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} &= \arcsin(x) + C \\ \int \frac{1}{-\sqrt{1-x^2}} &= \arccos(x) + C \\ \int \frac{1}{1+x^2} &= \arctan(x) + C \end{aligned}$$

Differentiation

$$Df(g(x)) = f'(g(x))g'(x)$$

Special cases

$$\begin{aligned} D\ln(g(x)) &= \frac{g'(x)}{g(x)} \\ De^{g(x)} &= e^{g(x)}g'(x) \end{aligned}$$

$$\begin{aligned} Dfg &= f'g + fg' \\ D(f/g) &= (gf' - fg')/g^2 \end{aligned}$$

Integration

$$\int f(g(x))g'(x)dx = f(g(x)) + C$$

$$\begin{aligned} \int \frac{g'(x)}{g(x)}dx &= \ln(g(x)) + C \\ \int g'(x)e^{g(x)}dx &= e^{g(x)} + C \end{aligned}$$

$$\int f'g dx = fg - \int fg' dx$$

Solution formula

The solution of

$$y' + p(x)y = q(x)$$

is

$$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx, \quad \text{where} \quad \mu(x) = e^{\int p(x)dx}.$$

Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

Moreover, if f is odd, that is, $f(-x) = -f(x)$, then

$$f(x) = \sum_{n=1}^{\infty} b_n\sin(nx),$$

and if f is even, that is, $f(-x) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$$

Discrete Fourier transform / FFT

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

Transform and inverse transform

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$

$$\begin{cases} y_0 &= x_0 + x_1 + x_2 + x_3 \\ y_1 &= x_0 - ix_1 - x_2 + ix_3 \\ y_2 &= x_0 - x_1 + x_2 - x_3 \\ y_3 &= x_0 + ix_1 - x_2 - ix_3 \end{cases}, \quad \begin{cases} y_0 &= \frac{1}{4}(x_0 + x_1 + x_2 + x_3) \\ y_1 &= \frac{1}{4}(x_0 + ix_1 - x_2 - ix_3) \\ y_2 &= \frac{1}{4}(x_0 - x_1 + x_2 - x_3) \\ y_3 &= \frac{1}{4}(x_0 - ix_1 - x_2 + ix_3) \end{cases}$$

LET'S PRACTICE BY SOLVING THESE

1. Give the requested examples.
   (a) A differential equation of order 3. $y''' + \sin(x) y = 0$
   (b) A separable differential equation. $y' = y^2 \sin(x)$
   (c) A non-separable differential equation. $y' = \sin(xy)$
   (d) A non-linear differential equation. $y' + y^2 = 0$
   (e) A linear and homogeneous differential equation. $y' + y = 0$
   (f) A linear and non-homogeneous differential equation. $y' + y = x^3$

SEPARABLE

$$\boxed{y' = f(x)\,g(y)} \quad \| : g(y) \qquad y' = \frac{dy}{dx}$$

$$\frac{y'}{g(y)} = f(x)$$

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x) \quad \| \cdot dx$$

$$\underbrace{\frac{1}{g(y)}\,dy}_{\text{JUST } y} = \underbrace{f(x)\,dx}_{\text{JUST } x}$$

$$(y')^1 + p(x)\, y^1 = q(x) \quad \text{LINEAR} \to \text{SOLUTION FORMULA EXISTS} \to \text{WE CAN USE THE FORMULA}$$

$$(y')^2 + p(x)\, y^3 = q(x) \quad \text{NONLINEAR} \to \text{NO SOLUTION FORMULA} \to \text{NEED PYTHON/NUMERICAL METHODS}$$

$$q(x) = 0 \iff \text{HOMOGENEOUS}$$

2. Show that the functions y are solutions to the corresponding differential equations.
   (a) Show that $y = \frac{1}{1-x}$ is a particular solution for $y' = y^2$.
   (b) Show that $y = 3 - x + x\ln(x)$ is a particular solution for $y' = \ln(x)$.

SOLUTION (a)

(b) $y = 3 - x + x\ln(x)$

$y' = 0 - 1 + 1 \cdot \ln(x) + x \cdot \frac{1}{x}$

$y' = \ln(x)$ OK

$$y = \frac{1}{1-x}$$

$$y = (1-x)^{-1} \qquad D\,x^n = n x^{n-1}$$

$$y' = -1 \cdot (1-x)^{-2} \cdot \underbrace{D(1-x)}_{= 0 - 1}$$

$$y' = (1-x)^{-2} = \frac{1}{(1-x)^2} = y \quad \text{OK}$$

3. The general solution of $y' = 4x^2$ is $y = \frac{4}{3}x^3 + C$, where C is any constant. Which particular solution passes through the point $(-3, -30)$? "x" "y"

SOLUTION

$$y(-3) = \frac{4}{3}(-3)^3 + C = -\frac{4}{\cancel{3}} \cdot 3^{\cancel{3}\,2} + C$$

$$= -4 \cdot 9 + C$$

ANSWER:

ANSWER:

$$y = \frac{4}{3}x^3 + 6$$

$= -4.9 + c$

$= -36 + c = -30 \quad \| +36$

$\underline{\underline{C = 6}}$

4. Solve

$$\underbrace{y' - \tan(x)}_{p(x)}y = \underbrace{3(\sin(x))^2}_{q(x)}.$$

SOLUTION.

$\int p(x)dx = \int -\tan(x)dx$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$= \int \frac{-\sin(x)}{\cos(x)}dx$

$= \int \frac{f'(x)}{f(x)}dx = \ln(f(x)) \cancel{+C}$

$= \ln(\cos(x))$

$$\boxed{\int \frac{g'(x)}{g(x)}dx = \ln(g(x)) + C}$$

$\ln(x)$

$= \log_e(x)$

e^x AND $\ln(x)$ ARE EACH OTHERS INVERSE FUNCTIONS

$\mu(x) = e^{\int p(x)dx} = \cancel{e}^{\cancel{\ln}(\cos(x))} = \cos(x)$

$\Rightarrow \frac{1}{\mu(x)} = \frac{1}{\cos(x)}$

$\int \mu(x)q(x)dx = 3\int \cos(x)(\sin(x))^2 dx = \cancel{3}\frac{(\sin(x))^3}{\cancel{3}}$

$= (\sin(x))^3$

$$\boxed{\int f'(x)f(x)^n dx = \frac{f(x)^{n+1}}{n+1}}$$

$y(x) = \frac{C}{\mu(x)} + \frac{1}{\mu(x)}\int \mu(x)q(x)dx$

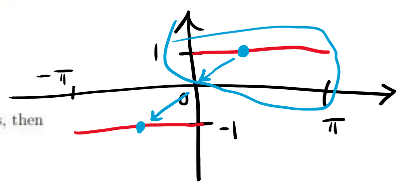
$= \frac{C}{\cos(x)} + \frac{1}{\cos(x)}(\sin(x))^3$

5. For the function

$$f(t) = \begin{cases} 1, & \\ -1, & \text{if } -\pi \le x \le 0 \end{cases}$$

find its Fourier series.

6. Find the discrete Fourier transform of [2, 3]. In other words, calculate by hand **fft**([2, 3]).



Fourier series

If f is periodic with period 2π and f, f' and f'' are piece-wise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad = 0$$

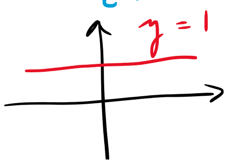
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad = 0 \; ?$$

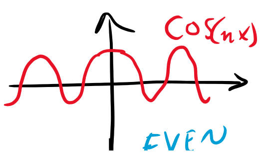
f IS ODD ?

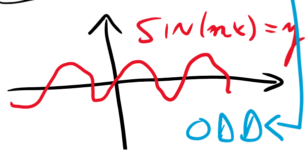
f IS EVEN ?

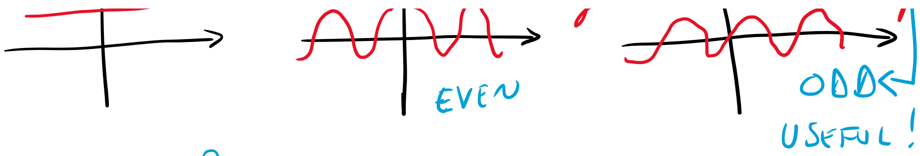
$$\boxed{f(-x) = -f(x)}$$

EVEN









ODD ?
EVEN ?

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \overset{n=0}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\underbrace{\cos(n\cdot 0)}_{=\cos(0)=1}\,dx = a_0$$

JUST NEED TO CALCULATE b_n'S

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \overbrace{f(x)}^{=1}\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$\int_{-\pi}^{0} (\ )\,dx \qquad \int_0^{\pi} (\ )\,dx$$

EQUAL

LET'S CALCULATE WITHOUT THE TRICK

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \underbrace{\frac{1}{\pi}\int_{-\pi}^{0} \underbrace{f(x)}_{=-1}\sin(nx)\,dx}_{=B} + \underbrace{\frac{1}{\pi}\int_0^{\pi} \underbrace{f(x)}_{=1}\sin(nx)\,dx}_{=A}$$

$$A = \int_0^{\pi}\sin(nx)\,dx = \left[\frac{-\cos(nx)}{n}\right]_{x=0}^{x=\pi}$$

$$\left[F(x)\right]_{x=a}^{x=b} = F(b) - F(a)$$

$$= \frac{-\overbrace{\cos(n\pi)}^{(-1)^n}}{n} - \left(\frac{-\overbrace{\cos(n\cdot 0)}^{=\cos(0)=1}}{n}\right)$$

$\cos(\pi) = -1$
$\cos(2\pi) = 1$
CALCULATOR

$$= \frac{-(-1)^n}{n} + \frac{1}{n}$$

$$= \frac{1+(-1)^{n+1}}{n}$$

$$B = -\int_{-\pi}^{0}\sin(nx)\,dx$$

$$= +\left[\frac{\cos(nx)}{}\right]^{x=0} = \frac{\cos(0)}{} - \frac{\cos(-n\pi)}{}$$

$$= -\left[\frac{\cos(nx)}{n}\right]_{x=-\pi}^{x=0} = \frac{\cos(0)}{n} - \frac{\cos(-n\pi)}{n}$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} = \frac{1+(-1)^{n+1}}{n}$$

$$\Rightarrow b_n = \frac{2}{\pi}\,\frac{1+(-1)^{n+1}}{n}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{1+(-1)^{n+1}}{n}\sin(nx)$$

5. Let $f(x) = \frac{1}{4}x^2$ for $-\pi \le x \le \pi$ and let $f(x)$ be periodic with period 2π. It's Fourier series is

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos(nx) \qquad (1)$$

(a) Find the Fourier series of the 2π periodic function $g(x)$ for which

D

$$g(x) = \frac{x}{2}, \quad \text{when} \quad -\pi \le x \le \pi.$$

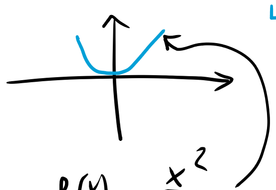
D

(b) Is $f(x)$ or $g(x)$ odd?
(c) Is $f(x)$ or $g(x)$ even?

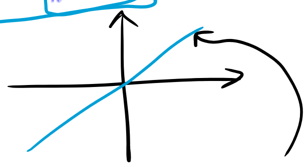
Solution. We have $Df(x) = g(x)$. Therefore, differentiate equation (1) on both sides to obtain

(a)

$$g(x) = 0 - \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}\sin(nx).$$



$y = f(x) = \frac{x^2}{4}$



$y = g(x) = \frac{x}{2}$ LINE

(b) ODD. g IS ODD. f IS NOT ODD.

(c) EVEN. f IS EVEN. g IS NOT EVEN.

6. Find the discrete Fourier transform of [2, 3]. In other words, calculate by hand fft([2, 3]). 7

Discrete Fourier transform / FFT

Transform and inverse transform

FFT

$$\begin{cases} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{cases}, \qquad \begin{cases} y_0 &= \frac{1}{2}(x_0 + x_1) \\ y_1 &= \frac{1}{2}(x_0 - x_1) \end{cases}$$

IFFT

SOLUTION.

$$\begin{cases} x_0 = 2 \\ x_1 = 3 \end{cases} \qquad y_0 = 2+3 = 5, \quad y_1 = 2-3 = -1$$

ANSWER: fft([2,3]) = [5, -1]