$$H_n = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2t)^{n-2k}}{k!\,(n-2k)!}, \quad n \in \mathbb{N}_0,$$

form a Hilbert basis of $L^2(\mathbb{R})$.

Laguerre's polynomials

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} (-1) \binom{n+\alpha}{n-k} \frac{t^k}{k!}, \quad \alpha > -1, \; n \in \mathbb{N}_0,$$

form a basis of $L^2_v(\mathbb{R}^+)$, $v(t) = t e^{-t}$.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dt \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dt \right) \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx\, dy$$

$$= \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r \, d\theta \, dr$$

$$= \pi \int_0^{\infty} 2r e^{-r^2} dr$$

$$= \pi \left. e^{-r^2} \right|_{\infty}^{0} \Rightarrow I = \sqrt{\pi}$$

$$= \pi$$

---

2. Fourier series

Fourier (1822) proposed that every 2π-periodic function can be represented in the form

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + \dots$$

- Does the sum converge to f and in what sense?
- Does the series determine f uniquely and how the coefficients reflect the properties of f?

In this section we see what can be done by using Hilbert space methods, Set $L^2 = L^2([0,2\pi])$ and $f \in L^2$. If $n \in \mathbb{Z}$, then the most convenient way to define the n:th Fourier coefficient is

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx.$$

For example, if f is a trigonometric polynomial, that is,

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} c_n(\cos(nx) + i\sin(nx)),$$

then
$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\sum_{k=-N}^{N} c_n \int_0^{2\pi} e^{-i(k-n)x}\,dx = c_n$$
for $n \in \{-N, \dots, N\}$ and $\hat{f}(n) = 0$ for otherwise.

The expression

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx$$

defines an inner product in L^2. The Fourier coefficients are related to the orthonormal set

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z}, \quad x \in [0,2\pi],$$

see Section 1. Namely,

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{e^{inx}}\,dx = \frac{1}{\sqrt{2\pi}}\langle f, e_n\rangle.$$

Thus $\hat{f}(n)$ differs from the general definition in Section 1 by the constant factor $(2\pi)^{-\frac{1}{2}}$.

If $f \in L^2$ (or if $f \in L^p$, $1 \le p < \infty$), then the n:th partial sum of its Fourier series is

$$s_n(f,x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx}, \quad x \in [0,2\pi], \quad n \in \mathbb{N}_0.$$

The function $s_n(f,x)$ is pointwise defined continuous function, because e^{ikx} is continuous for all $k \in \mathbb{Z}$.

Lemma 2.1 Let $f \in L^1$. Then
$$s_n(f,x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) D_n(x-t)\,dt,$$
where
$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)}, \quad n \in \mathbb{N}_0,$$
is the n:th Dirichlet kernel.

Proof Clearly,
$$\begin{aligned} s_n(f,x) &= \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} \\ &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\,dt \right) e^{ikx} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{k=-n}^{n} e^{ik(x-t)} \right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) D_n(x-t)\,dt, \quad \text{where} \end{aligned}$$
$$\begin{aligned} D_n(x) &= \sum_{k=-n}^{n} e^{ikx} = e^{-ikx}\left(1 + e^{ix} + e^{i2x} + \dots + e^{i2nx}\right) \\ &= e^{-inx} \sum_{k=0}^{2n} (e^{ix})^k = \frac{1 - e^{i(2n+1)x}}{1 - e^{ix}} \\ &= \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}}. \end{aligned}$$
By multiplying this identity by
$e^{-ix/2}(e^{ix} - 1) = e^{ix/2} - e^{-ix/2}$ we deduce
$$(e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}) D_n(x) = e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x},$$
from which Euler's formula yields
$$2i \sin\frac{x}{2}\, D_n(x) = 2i \sin((n+\tfrac{1}{2})x).$$
$\square$

In the L^2-theory we will need the arithmetic means of Dirichlet kernels. These are called Fejér-kernels:
$$K_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(x), \quad x \in [0, 2\pi], \quad n \in \mathbb{N}_0.$$

Lemma 2.2 (i) $\frac{1}{2\pi}\int_0^{2\pi} K_n(x)\,dx = 1 = \frac{1}{2\pi}\int_0^{2\pi} D_n(x)\,dx$, $n\in\mathbb{N}_0$,

(ii) $K_n(x)\geq 0$ for all $x\in[0,2\pi]$ and

$$K_n(x) \leq \frac{2}{(n+1)(1-\cos\delta)}, \quad 0<\delta\leq x\leq 2\pi-\delta, \; n\in\mathbb{N}_0$$

for any fixed $\delta\in(0,\pi)$.

Remark By (ii) the Fejér-kernels K_n are non-negative and $K_n\to 0$ uniformly, as $n\to\infty$, in each compact subset of $(0,2\pi)$. The Dirichlet-kernels D_n do not have these properties, and hence the theory of pointwise convergence of Fourier series is difficult.

Proof (i) Since $\frac{1}{2\pi}\int_0^{2\pi} e^{int}\,dt = \delta_{n0}$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} D_n(x)\,dx = \sum_{k=-n}^{n}\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\,dx = 1, \quad n\in\mathbb{N}_0,$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi} K_n(x)\,dx = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_0^{2\pi} D_k(x)\,dx = 1.$$

(ii) By the proof of Lemma 2.1,

$(e^{ix}-1)\,D_n(x) = e^{i(n+1)x} - e^{-inx}$. Hence

$$\begin{aligned}
&(n+1)K_n(x)(e^{ix}-1)(e^{-ix}-1)\\
&= (e^{-ix}-1)\sum_{k=0}^{n}\left(e^{i(k+1)x}-e^{-ikx}\right)\\
&= (e^{-ix}-1)\left(e^{ix}\sum_{k=0}^{n}(e^{ix})^k - \sum_{k=0}^{n}(e^{-ix})^k\right)\\
&= \cancel{e^{ix}(e^{-ix}-1)}\,\frac{1-e^{i(n+1)x}}{\cancel{1-e^{ix}}} + \cancel{(e^{-ix}-1)}\,\frac{1-e^{-ix(n+1)}}{\cancel{1-e^{-ix}}}\\
&= 2-e^{i(n+1)x}-e^{-i(n+1)x}\\
&= 2-2\cos((n+1)x),
\end{aligned}$$

and it follows that

$$K_n(x) = \frac{2\,(1-\cos((n+1)x))}{(n+1)\,(1-e^{ix}-e^{-ix}+1)}$$
$$= \frac{2\,(1-\cos((n+1)x))}{(n+1)\,2\,(1-\cos x)}$$
$$= \frac{1-\cos((n+1)x)}{(n+1)\,(1-\cos x)} \geq 0,$$

Moreover,
$$K_n(x) \leq \frac{1}{(n+1)} \frac{2}{(1-\cos\delta)}$$
for $0 < \delta \leq x \leq 2\pi - \delta < 2\pi$. $\square$

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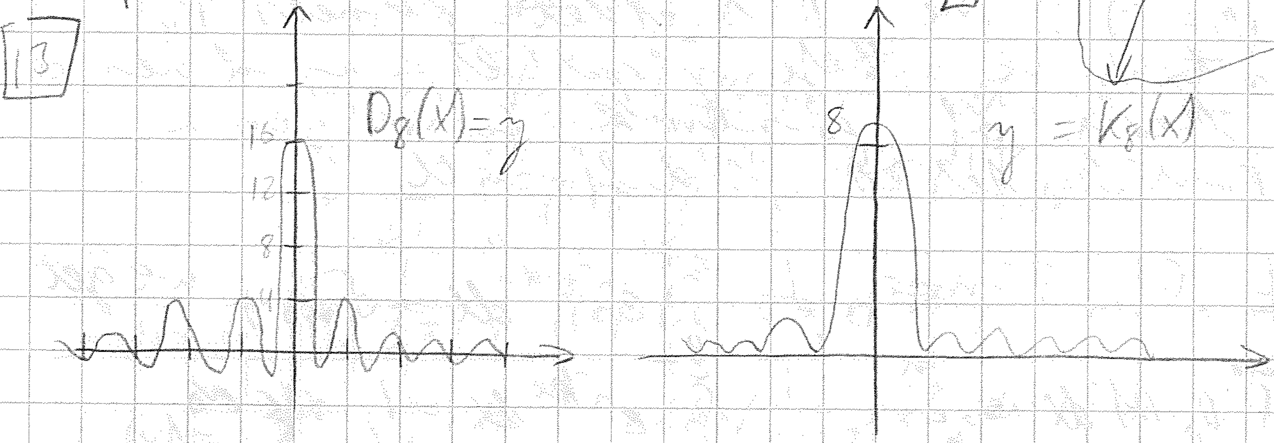
$\boxed{13}$



We next study how the Fejér kernel behaves in convolution.

<u>Theorem 2.3 (Fejér's theorem)</u> Let $f : [0, 2\pi] \to \mathbb{C}$ be continuous, $f(0) = f(2\pi)$ and denote
$$(K_n * f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, K_n(x-t)\, dt, \quad x \in [0, 2\pi],\ n \in \mathbb{N}.$$
Then $\| (K_n * f) - f \|_{L^\infty} = \sup_{x \in [0,2\pi]} |(K_n * f) - f| \to 0$, $n \to \infty$, that is, $K_n * f \to f$ uniformly on $[0, 2\pi]$, as $n \to \infty$.

<u>Proof</u> By the hypothesis f is uniformly continuous on $[0, 2\pi]$ and it can be defined as a continuous 2π-periodic function in the whole $\mathbb{R}$. Also K_n is 2π-periodic and continuous, so the change of variable $s = x - t$ yields

$$(K_n * f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, K_n(x-t)\,dt$$
$$= -\frac{1}{2\pi}\int_x^{x-2\pi} f(x-u)\, K_n(u)\, du \qquad (*)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(x-u)\, K_n(u)\, du = (f * K_n)(x).$$

Set $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta = \delta(\varepsilon) > 0$ s.th.

$$|f(x-u) - f(x)| < \frac{\varepsilon}{2},$$

whenever $|u| < \delta$ and $x \in [0, 2\pi]$. Set

$$M = \|f\|_{L^\infty} = \max_{u \in [0,2\pi]} |f(u)| < \infty.$$

By Lemma 2.2 (iii), there exists $n_0 = n_0(\varepsilon, M) \in \mathbb{N}$ such that

$$0 \le K_n(u) < \frac{\varepsilon}{4M}$$

provided $\delta \le u \le 2\pi - \delta$ and $n \ge n_0$. Then Lemma 2.2. and $(*)$ yield

$$|(K_n * f)(x) - f(x)| = \left|\frac{1}{2\pi}\int_0^{2\pi} \big(f(x-u) - f(x)\big) K_n(u)\,du\right|$$
$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(x-u) - f(x)|\, K_n(u)\, du$$
$$= \frac{1}{2\pi}\Bigg[\underbrace{\left(\int_0^{\delta} + \int_{2\pi-\delta}^{2\pi}\right)}_{=J_1} + \underbrace{\int_{\delta}^{2\pi-\delta}}_{=J_2}\Bigg] = J_1 + J_2.$$

Now, by the 2π-periodicity and Lemma 2.2(i)

$$J_1 = \frac{1}{2\pi}\left(\int_0^{\delta} + \int_{2\pi-\delta}^{2\pi}\right) |f(x-u) - f(x)|\, K_n(u)\, du$$
$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-u) - f(x)|\, K_n(u)\, du$$
$$\le \frac{\varepsilon}{2}\, \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(u)\, du \le \frac{\varepsilon}{2}.$$

Further,

$$J_2 = \frac{1}{2\pi} \int_\delta^{2\pi-\delta} \underbrace{|f(x-u) - f(x)|}_{\leq |f(x-u)| + |f(x)| \leq 2\|f\|_{L^\infty}} K_n(u)\,du$$

$$\leq \frac{1}{2\pi} \cdot 2\|f\|_{L^\infty} \sup_{u\in[0,2\pi]} K_n(u) \cdot 2\pi$$

$$< 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}, \quad n \geq n_0,$$

and that

$$|(K_n * f)(x) - f(x)| \leq J_1 + J_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad n \geq n_0,$$

and the assertion follows. $\square$

By Lemma 2.1,

$$(K_n * f)(x) = \left(\left(\frac{1}{n+1} \sum_{k=0}^{n} D_k\right) * f\right)(x)$$

$$(+) \quad = \frac{1}{n+1} \sum_{k=0}^{n} (D_k * f)(x)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2\pi} \int_0^{2\pi} f(t) D_k(x-t)\,dt$$

$$\overset{L.2.1.}{=} \frac{1}{n+1} \sum_{k=0}^{n} S_k(f,x), \quad n \in \mathbb{N}.$$

Therefore Theorem 2.3 shows that the arithmetic mean of the partial Fourier sums converges uniformly to f on $[0,2\pi]$. In particular

$$\frac{1}{n+1} \sum_{k=0}^{n} S_k(f,x) \to f(x), \quad n \to \infty,$$

for all $x \in [0,2\pi]$. This yields:

<u>Corollary 2.4</u> Let $f \in C([0,2\pi])$ be 2π-periodic ($f(0) = f(2\pi)$). If $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ then $f \equiv 0$.

Proof If $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$, then

$$S_n(f;x) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} = 0$$

for all $n \in \mathbb{N}$ and $x \in [0,2\pi]$. Hence

$$f(x) = \lim_{n\to\infty} (K_n * f)(x) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} S_k(f;x)$$
$$= \lim_{n\to\infty} 0 = 0, \quad x \in [0,2\pi],$$

and thus $f \equiv 0$. $\square$

We know by now that the Fourier coefficients of a continuous function determine the function uniquely, that is, if f and g are continuous 2π-periodic functions and $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$, then $f \equiv g$.

We proceed to generalize this uniqueness result for functions in L^2. To do this we will need the following result concerning approximation of L^p-functions. It says that smooth functions form a dense subset of $L^p([0,2\pi])$ when $p \neq \infty$.

Holder $\to p \in (0,1)$

Theorem 2.5 Let $1 \le p < \infty$ and $f \in L^p([0,2\pi])$. Then, for each $\varepsilon > 0$, there exists a 2π-periodic C^∞-function $g = g(f,\varepsilon)$ such that

(A) $$\|f-g\|_{L^p} = \left(\int_0^{2\pi} |f(x) - g(x)|^p dx \right)^{\frac{1}{p}} < \varepsilon.$$

Proof Note first that $K_n * g$ is a C^∞-function as a finite sum of C^∞-functions (trigonometric functions) for each continuous g, see the discussion after the proof of Theorem 2.3. Therefore, by Theorem 2.3, it suffices to find a continuous 2π-periodic g such that (A) holds, because in that case

$$\|f-(K_n * g)\|_{L^p} \le \|f-g\|_{L^p} + \|g-(K_n * g)\|$$

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where

$$\|g-(K_n * g)\|_{L^p} = \left(\int_0^{2\pi} |g(x)-(K_n * g)(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$\le (2\pi)^{\frac{1}{p}} \|g-(K * g)\|_{L^\infty} \to 0, \quad n \to \infty.$$

We search for g "step-by-step".

(1) Let f be the characteristic function of a <u>closed</u> set $F \subset [0,2\pi]$. Set

$$g_n(x) = \frac{1}{1+n\,\mathrm{dist}(x,F)}, \quad x \in [0,2\pi], \; n \in \mathbb{N}$$

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The function $x \mapsto \mathrm{dist}(x,F)$ is continuous, and hence $g_n : [0,2\pi] \to \mathbb{R}$ is continuous for all $n \in \mathbb{N}$. Moreover $g_n(x) = 1$ for $x \in F$ and $n \in \mathbb{N}$, and $g_n(x) \to 0$ for $x \notin F$, as $n \to \infty$ because in this case $\mathrm{dist}(x,F) > 0$. Thus $g_n(x) \to \chi_F(x)$ for all $x \in [0,2\pi]$, as $n \to \infty$.
Since $0 \le g_n(x) \le 1$ for all $x \in [0,2\pi]$ and $n \in \mathbb{N}$, Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \|g_n - \chi_F\|_{L^p}^p = \lim_{n\to\infty} \int_0^{2\pi} |g_n(x) - \chi_F(x)|^p \, dx$$

$$= \int_0^{2\pi} \lim_{n\to\infty} |g_n(x) - \chi_F(x)|^p \, dx = 0$$

The last thing to note in this case is that g_n is not necessarily 2π-periodic, but we can deal with this problem by re-defining g on $[0, \frac{1}{n}]$ in a suitable manner. The error caused by this re-definition is at most $1/n$.

(2) Let now f be the characteristic function of an open set $A \subset [0,2\pi]$. Then the assertion follows by (1), because $A^c = [0,2\pi] \setminus A$ is closed and $\chi_A = 1 - \chi_{A^c}$.

(3) Let $f = \chi_A$, where $A \subset [0, 2\pi]$ is Lebesgue measurable. Then, by the definition of Lebesgue measure, there exists a sequence of open sets $G_n \subset [0, 2\pi]$ such that

$$A \subset G_n, \quad n \in \mathbb{N}, \quad \text{and} \quad \lim_{n\to\infty} \mu(G_n \setminus A) = 0,$$

where μ denotes the Lebesgue measure. Let $\varepsilon > 0$, and choose $N = N(\varepsilon) \in \mathbb{N}$ such that $\mu(G_n \setminus A) < \left(\frac{\varepsilon}{2}\right)^p$ for all $n \geq N$. By (2) we can find a 2π-periodic function g for which $\|g - \chi_{G_n}\|_{L^p} < \frac{\varepsilon}{2}$, g cont. Hence

$$\|g - \chi_A\|_{L^p} \leq \|g - \chi_{G_n}\|_{L^p} + \|\chi_{G_n} - \chi_A\|_{L^p}$$

$$< \frac{\varepsilon}{2} + \mu(G_n \setminus A)^{\frac{1}{p}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for $n \geq N$ fixed.

(4) Let $f \in L^p$ be arbitrary. By the definition of the Lebesgue integral

$\int_0^{2\pi} |f(t)|^p \, dt$, we can find a simple function

$$g = \sum_{j=1}^{n} a_j \chi_{A_j}, \quad \text{where } A_1, \dots, A_n \subset [0, 2\pi]$$

are measurable, such that $\|f - g\|_{L^p} < \varepsilon/2$. By (3) we can find continuous 2π-periodic functions g_j such that

$$\|\chi_{A_j} - g_j\|_{L^p} < \frac{\varepsilon}{2nM},$$

where $M = \max\{|a_1|, \dots, |a_n|\}$. Therefore, by triangle inequality

$$\left\|f - \sum_{j=1}^{n} a_j g_j\right\| \leq \|f - g\|_{L^p} + \left\|g - \sum_{j=1}^{n} a_j g_j\right\|_{L^p}$$

$$= \|f-g\|_{L^p} + \Big\| \sum_{j=1}^{m} a_j \chi_{A_j} - \sum_{j=1}^{m} a_j g_j \Big\|$$

$$\le \|f-g\|_{L^p} + \sum_{j=1}^{m} |a_j| \, \|\chi_{A_j} - g_j\|_{L^p}$$

$$< \frac{\varepsilon}{2} + \sum_{j=1}^{m} |a_j| \frac{\varepsilon}{2mM} \le \varepsilon,$$

Since $\sum_{j=1}^{m} a_j g_j$ is continuous and 2π-periodic, we are done. $\square$

$\boxed{14}$ $f \in L^p([0,2\pi])$, $0<p<\infty$, $\exists\, g = g(f,\varepsilon) \in C^\infty$ s.th.

(A) $\|f-g\|_{L^p} < \varepsilon$

Remarks (1) The assertion in Theorem 2.5. fails for $p=\infty$, because even $C([0,2\pi])$ is not dense in $(L^\infty([0,2\pi]), \|\cdot\|_{L^\infty})$.

(2) It is obvious from Theorem 2.5 that C^∞ is dense in $L^p([a,b])$, when $a<b$ and $0<p<\infty$ (re-scaling).

(3) If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ then for each $\varepsilon>0$ there exists $M = M(\varepsilon,f) \in [0,\infty)$ such that

$$\int_{J_m} |f(x)|^p \, dx < \varepsilon, \quad J_m = \{x \in \mathbb{R} : |x| > m\},$$

for all $m \ge M$. This follows by Lebesgue's dominated convergence theorem when one considers

$f_m = \chi_m f$ and notes that $|f_m(x)| \le |f(x)|$,

where $f \in L^p(\mathbb{R})$, and $\lim_{m\to\infty} f_m(x) = 0$ for all $x \in \mathbb{R}$

One can use this fact to deduce that $C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$

If $\Omega \subseteq \mathbb{R}$ is measurable, $\mu(\Omega) > 0$ and $f \in L^p(\Omega)$, then

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \in \mathbb{R} \setminus \Omega \end{cases}$$

belongs to $L^p(\mathbb{R})$, we can deduce that $C^\infty|_\Omega \cap L^p(\Omega)$ is dense in $L^p(\Omega)$.

Uniqueness of the Fourier-series of L^2-functions follows:

Corollary 2.6 If $f \in L^2([0,2\pi])$ and

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\, dt = 0, \quad k \in \mathbb{Z},$$

then $f \equiv 0$. Therefore, if $f, g \in L^2$ and $\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$ then $f \equiv g$, that is, $f = g$ almost everywhere. In particular, the functions

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad x \in [0, 2\pi], \quad n \in \mathbb{Z},$$

form a Hilbert basis of L^2.

Proof Set $\varepsilon > 0$. By Theorem 2.5 there exist a continuous 2π-periodic function g such that $\|f - g\|_{L^2} < \varepsilon$. Moreover, by Theorem 2.3 (Fejér), $K_n * g \to g$ uniformly on $[0, 2\pi]$, as $n \to \infty$, so

$$\|g - (K_n * g)\|_{L^2} \leq \sqrt{2\pi}\, \|g - (K_n * g)\|_{L^\infty} < \varepsilon,$$

when $n \in \mathbb{N}$ is sufficiently large. By (T),

$$(K_n * g)(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(f, x)$$

$$= \sum_{k=-n}^{n} a_k e_k(x),$$

where $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$.

Since $\{e_k\}$ is orthonormal, the corresponding orthogonal projection minimizes the distance to the subspace $\text{span}(\{e_k : -n \le k \le n\})$, that is, ORTOPROJEKTIO

$$\| f - \sum_{k=-n}^{n} \langle f, e_k \rangle e_k \|_{L^2} \le \| f - \sum_{k=-n}^{n} a_k e_k \|_{L^2},$$

see the discussion before Theorem 1.30. On the other hand, by the hypothesis, $\langle f, e_k \rangle = \sqrt{2\pi}\, \hat{f}(k) = 0$ for all $k \in \mathbb{Z}$, so the triangle inequality yields

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$$\|f\|_{L^2} = \| f - \sum_{k=-n}^{n} \langle f, e_k \rangle e_k \|_{L^2}$$

$$\le \| f - (K_n * g) \|_{L^2} \le \| f - g \|_{L^2} + \| g - (K_n * g) \|_{L^2} < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce $f = 0$ for almost everywhere on $[0, 2\pi]$. Thus $f \equiv 0$ (as a member of L^2).

Theorem 1.32 (b) shows that $\{e_n\}$ is a Hilbert basis of $L^2([0, 2\pi])$. $\square$

? $\Rightarrow$ Resumen on L^2-theory of Fourier-series

① $\{e_n\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$ is orthonormal in L^2.

② If $f \in L^2$ and $\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \langle f, e_n \rangle_{L^2} = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$. Hence $\{e_n\}$ is a Hilbert basis of L^2.

③ If $f \in L^2$, then

$$f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle_{L^2} e_n = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx},$$

where the series converges in L^2, that is,

$$\lim_{n \to \infty} \int_0^{2\pi} \left| f(x) - \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} \right|^2 dx = 0.$$

④ By Parseval's identity

$$\frac{1}{2\pi}\|f\|^2_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2, \quad f \in L^2,$$

because $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \langle f, e_k \rangle_{L^2}$ for all $k \in \mathbb{Z}$.

(5) Conversely, if $\{\lambda_k\} \in \ell^2$, then there exists $f \in L^2$ for which $\hat{f}(k) = \lambda_k$ for all $k \in \mathbb{Z}$, see Riesz-Fischer. The function f here is

$$f(t) = \sum_{k=-\infty}^{\infty} \lambda_k e^{ikt}.$$

(6) Plancherel's formula yields

$$\frac{1}{2\pi}\langle f, g\rangle_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} f(s)\overline{g(s)}\, ds$$
$$= \sum_{k=-\infty}^{\infty} \hat{f}(k)\overline{\hat{g}(k)}, \quad f, g \in L^2.$$

From the theory of pointwise convergence we will only prove the following result

Theorem 2.7 Let $f: \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that
$|f(x) - f(y)| \le L|x-y|$ for all $x, y \in \mathbb{R}$ (Lipschitz condition). Then the partial sums of Fourier series at x converge to f at x, that is

$$s_n(f, x) = \sum_{k=-n}^{n} \hat{f}(k)\underbrace{e^{ikx}}_{=g_k(x)} \to f(x), \quad n \to \infty,$$

for all $x \in \mathbb{R}$.

Proof If $h \in L^2$, then Bessel's inequality gives

(#) $\int_0^{2\pi} h(t)\cos(nt)\, dt = \frac{1}{2}(\langle h, g_n\rangle_{L^2} + \langle h, \overline{g_n}\rangle_{L^2}), g$

$$= \frac{\sqrt{2\pi}}{2}\left(\langle h, e_n\rangle_{L^2}^{\nearrow 0} + \langle h, e_{-n}\rangle_{L^2}^{\nearrow 0}\right) \to 0, \quad \begin{matrix}\{e_n\} \text{ ON} \\ \{e_{-n}\} \text{ ON}\end{matrix}$$

$n \to \infty$, and similarly

$$(\&) \quad \lim_{n\to\infty} \int_0^{2\pi} h(t) \sin(nt)\, dt = 0$$

Choose $h_x(t) = \left\{ \dfrac{f(x-t) - f(x)}{\sin \frac{t}{2}} \right.$.

Then, by the Lipschitz condition, $h_x \in L^\infty([0,2\pi]) \subset L^2([0,2\pi])$ because

$$|h_x(t)| \le L \frac{|t|}{|\sin \frac{t}{2}|} \to 2L, \quad t \to 0^+.$$

Lemmas 2.1 and 2.2 imply

$$s_n(f,x) - f(x) \overset{L\,2.1}{=} \frac{1}{2\pi}\int_0^{2\pi} D_n(x-t) f(t)\, dt - f(x)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} D_n(t) f(x-t)\, dt - f(x)$$

$$\overset{L\,2.2}{=} \frac{1}{2\pi}\int_0^{2\pi} D_n(t)\left(f(x-t) - f(x)\right) dt,$$

Since $D_n(t) = \dfrac{\sin((n+\frac{1}{2})t)}{\sin \frac{t}{2}} = \cos(nt) + \dfrac{\sin(nt)\cos\frac{t}{2}}{\sin\frac{t}{2}}$

we deduce

$$s_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(nt) \overbrace{\left(f(x-t) - f(x)\right)}^{\in L^2, \text{ as a function of } t} dt$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \underbrace{h_x(t) \cos\frac{t}{2}}_{\in L^\infty \subset L^2, \text{ as a function of } t} \sin(nt)\, dt$$

$\to 0$, $n \to \infty$, by (#) and (&). $\square$