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LINEAR DIFFERENTIAL EQUATIONS WITH SLOWLY GROWING SOLUTIONS

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ABSTRACT. This research concerns linear differential equations in the unit disc of the complex plane. In the higher order case the separation of zeros (of maximal multiplicity) of solutions is considered, while in the second order case slowly growing solutions in H^∞, BMOA and the Bloch space are discussed. A counterpart of the Hardy-Stein-Spencer formula for higher derivatives is proved, and then applied to study solutions in the Hardy spaces.

1. INTRODUCTION

A fundamental objective in the study of complex linear differential equations with analytic coefficients in a complex domain is to relate the growth of coefficients to the growth of solutions and to the distribution of their zeros. In the case of fast growing solutions, Nevanlinna and Wiman-Valiron theories have turned out to be very useful both in the unit disc [10, 24] and in the complex plane [23, 24].

We restrict ourselves to the case of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In addition to methods above, theory of conformal maps has been used to establish interrelationships between the growth of coefficients and the geometric distribution (and separation) of zeros of solutions. This connection was one of the highlights in Nehari's seminal paper [25], according to which a sufficient condition for the injectivity of a locally univalent meromorphic function can be given in terms of its Schwarzian derivative. In the setting of differential equations, Nehari's theorem [25, Theorem I] admits the following (equivalent) formulation: if A is analytic in $\mathbb{D}$ and

$$\sup_{z\in\mathbb{D}} |A(z)|(1-|z|^2)^2 \tag{1.1}$$

is at most one, then each non-trivial solution ($f \not\equiv 0$) of

$$f'' + Af = 0 \tag{1.2}$$

has at most one zero in $\mathbb{D}$. A few years later, in 1955, Schwarz showed [36, Theorems 3–4] that if A is analytic in $\mathbb{D}$ then zero-sequences of all non-trivial solutions of (1.2) are separated in the hyperbolic metric if and only if (1.1) is finite. The necessary condition, corresponding to Nehari's theorem, was given by Kraus [22]. For recent developments based on localization of the classical results, see [5]. In the case of higher order linear differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0, \quad k \in \mathbb{N}, \tag{1.3}$$

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with analytic coefficients $A_0, \dots, A_{k-1}$, this line of reasoning has not given complete results. Some progress on the subject was obtained in the seventies and eighties by Kim and Lavie, among many other authors.

Nevanlinna and Wiman-Valiron theories, in the form they are known today, do not seem to be sufficiently delicate tools to study slowly growing solutions of (1.2), and hence different approach must be employed. An important breakthrough in this regard was [33], where Pommerenke obtained a sharp sufficient condition for the analytic coefficient A which places all solutions f of (1.2) to the classical Hardy space H^2. Pommerenke's idea was to use Green's formula twice to write the H^2-norm of f in terms of f'', employ the differential equation (1.2), and then apply Carleson's theorem for the Hardy spaces [8, Theorem 9.3]. Consequently, the coefficient condition was given in terms of Carleson measures. The leading idea of this (operator theoretic) approach has been extended to study, for example, solutions in the Hardy and Bergman spaces [28, 35], Dirichlet type spaces [19] and growth spaces [16, 21], to name a few instances.

Our intention is to establish sufficient conditions for the coefficient of (1.2) which place all solutions to H^∞, BMOA or to the Bloch space. In principle, Pommerenke's original idea could be modified to cover these cases, but in practice, this approach falls short since either it is difficult to find a useful expression for the norm in terms of the second derivative (in the case of H^∞) or the characterization of Carleson measures is not known (in the cases of BMOA and Bloch). Concerning Carleson measures for the Bloch space, see [13]. Curiously enough, the best known coefficient condition placing all solutions of (1.2) to the Bloch space is obtained by straightforward integration [21]. Our approach takes advantage of the reproducing formulae, and is different to ones in the literature.

2. Main results

Let $\mathcal{H}(\mathbb{D})$ denote the collection of functions analytic in $\mathbb{D}$, and let m be the Lebesgue area measure, normalized so that $m(\mathbb{D}) = 1$. By postponing the rigorous definitions to the forthcoming sections, we proceed to outline our results. We begin with the zero distribution of non-trivial solutions of the linear differential equation

$$f''' + A_2 f'' + A_1 f' + A_0 f = 0 \tag{2.1}$$

with analytic coefficients. Note that zeros of non-trivial solutions of (2.1) are at most two-fold. Let $\varphi_a(z) = (a-z)/(1-\overline{a}z)$, for $a, z \in \mathbb{D}$, denote a conformal automorphism of $\mathbb{D}$ which coincides with its own inverse.

Theorem 1. *Let f be a non-trivial solution of* (2.1) *where $A_0, A_1, A_2 \in \mathcal{H}(\mathbb{D})$.*

(i) *If*

$$\sup_{z\in\mathbb{D}} |A_j(z)|(1-|z|^2)^{3-j} < \infty, \quad j = 0, 1, 2, \tag{2.2}$$

then the sequence of two-fold zeros of f is a finite union of separated sequences.

(ii) *If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A_j(z)|(1-|z|^2)^{1-j}\big(1-|\varphi_a(z)|^2\big)\, dm(z) < \infty, \quad j = 0, 1, 2, \tag{2.3}$$

then the sequence of two-fold zeros of f is a finite union of uniformly separated sequences.

Theorem 1(i) should be compared to the second order case [36, Theorem 3], which was already mentioned in the introduction. For the second order counterpart of Theorem 1(ii), see [14, Theorem 1]. By a standard transformation as in [23, p. 74], both [36, Theorem 3] and [14, Theorem 1] admit immediate generalizations to second order differential equations (1.3) with an intermediate coefficient A_1. The proof of Theorem 1 is presented in Section 3, and it is based on a conformal transformation of (2.1), Jensen's formula, and on a sharp growth estimate for solutions of (2.1). Theorem 1 extends to the case of higher order differential equations (1.3), but we leave details for the interested reader.

The following results concern slowly growing solutions of the second order differential equation (1.2), however, our methods could also be applied in more general situations. A sufficient condition for the analytic coefficient A, which forces all solutions of (1.2) to be bounded, is given in terms of Cauchy transforms. The space $\mathcal{K}$ of Cauchy transforms consists of functions in $\mathcal{H}(\mathbb{D})$ that take the form $\int_{\mathbb{T}}(1-\overline{\zeta}z)^{-1}\,d\mu(\zeta)$, where μ is a finite, complex, Borel measure on the unit circle $\mathbb{T}=\partial\mathbb{D}$. For more details we refer to Section 5, where the following theorem is proved.

Theorem 2. *Let $A\in\mathcal{H}(\mathbb{D})$.*

(i) *If* $\limsup_{r\to1^-}\sup_{z\in\mathbb{D}}\|A_{r,z}\|_{\mathcal{K}}<1$ *for*

$$A_{r,z}(u)=\overline{\int_0^z\int_0^\zeta\frac{A(rw)}{1-\overline{u}w}\,dw\,d\zeta},\qquad u\in\mathbb{D},$$

then all solutions of (1.2) *are bounded.*

(ii) *If a primitive of A belongs to the Hardy space H^1, then all solutions of* (1.2) *have their first derivative in H^1.*

For $f\in\mathcal{H}(\mathbb{D})$, $f'\in H^1$ if and only if f admits a continuous extension to $\overline{\mathbb{D}}$ and is absolutely continuous on $\mathbb{T}$ [8, Theorem 3.11]. Therefore, as a consequence of Theorem 2(ii), we obtain a coefficient condition which places all solutions of (1.2) to the disc algebra.

The question converse to Theorem 2(i) is open and appears to be difficult. The boundedness of one non-trivial solution of (1.2) is not enough to guarantee that (1.1) is finite, which can be easily seen by considering the solution $f(z)=\exp(-(1+z)/(1-z))$ of (1.2) for $A(z)=-4z/(1-z)^4$, $z\in\mathbb{D}$. However, if (1.2) admits linearly independent solutions $f_1,f_2\in H^\infty$ such that $\inf_{z\in\mathbb{D}}\big(|f_1(z)|+|f_2(z)|\big)>0$, then (1.1) is finite. This is a consequence of the Corona theorem [8, Theorem 12.1], according to which there exist $g_1,g_2\in H^\infty$ such that $f_1g_1+f_2g_2\equiv1$, and consequently $A=A+(f_1g_1+f_2g_2)''=2(f_1'g_1'+f_2'g_2')+f_1g_1''+f_2g_2''$.

We proceed to consider BMOA, which consists of those functions in the Hardy space H^2 whose boundary values are of bounded mean oscillation. The following result should be compared to [33, Theorem 2] as BMOA is a conformally invariant subspace of H^2.

Theorem 3. *Let $A\in\mathcal{H}(\mathbb{D})$. If*

$$\sup_{a\in\mathbb{D}}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\tag{2.4}$$

is sufficiently small, then all solutions of (1.2) *belong to* BMOA.

To the best of our knowledge BMOA solutions of (1.2) have not been discussed in the literature before. The coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded, see Example 2 in Section 6. By [28, Lemma 5.3] or [40, Theorem 1], (2.4) is comparable to

$$\sup_{a\in\mathbb{D}}\frac{\left(\log\frac{e}{1-|a|}\right)^2}{1-|a|}\int_{S_a}|A(z)|^2(1-|z|^2)^3\,dm(z),\tag{2.5}$$

where $S_a=\{re^{i\theta}:|a|<r<1,\,|\theta-\arg(a)|\le(1-|a|)/2\}$ denotes the Carleson square with respect to $a\in\mathbb{D}\setminus\{0\}$ and $S_0=\mathbb{D}$. See also [37, Lemma 3.4]. Solutions in VMOA, the closure of polynomials in BMOA, are discussed in Section 6 in which Theorem 3 is proved.

The case of the Bloch space $\mathcal{B}$ is especially interesting. For $0<\alpha<\infty$, let $\mathcal{L}^\alpha$ denote the collection of those $A\in\mathcal{H}(\mathbb{D})$ for which

$$\|A\|_{\mathcal{L}^\alpha}=\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2\left(\log\frac{e}{1-|z|}\right)^\alpha<\infty.$$

The comparison between H_2^∞, $\mathcal{L}^\alpha$ and the functions for which (2.4) is finite is presented in Section 4. It is known that, if $\|A\|_{\mathcal{L}^1}$ is sufficiently small, then all solutions of (1.2) belong to $\mathcal{B}$. This result was recently discovered with the best possible upper bound for $\|A\|_{\mathcal{L}^1}$ in [21, Corollary 4(b) and Example 5(b)]. Moreover, if $A \in \mathcal{L}^1$ then all solutions of (1.2) are in H^2 by [33, Corollary 1]. We point out that, if $A \in \mathcal{L}^\alpha$ for any $1 < \alpha < \infty$, then all solutions of (1.2) are bounded by [18, Theorem G(a)]. Solutions in the little Bloch space $\mathcal{B}_0$, the closure of polynomials in $\mathcal{B}$, are discussed in Section 7, among other results involving the Bloch space.

The proof of Theorem 2(i) is based on an application of the reproducing formula for H^1 functions, and it is natural to ask whether this method extends to the cases of $\mathcal{B}$ and BMOA. In the case of $\mathcal{B}$, by using the reproducing formula for weighted Bergman spaces, we prove a result (namely, Theorem 10) offering a family of coefficient conditions, which are given in terms of Bergman spaces induced by doubling weights. The case of BMOA, with the reproducing formula for H^1, is further considered in Section 8.

A careful reader observes that the results above are closely related to operator theory. If f is a solution of (1.2), then

$$f(z) = -\int_0^z \left(\int_0^\zeta f(w)A(w)\,dw \right) d\zeta + f'(0)z + f(0), \quad z \in \mathbb{D}. \tag{2.6}$$

By denoting

$$S_A(f)(z) = \int_0^z \left(\int_0^\zeta f(w)A(w)\,dw \right) d\zeta, \quad z \in \mathbb{D},$$

we obtain an integral operator, induced by the symbol $A \in \mathcal{H}(\mathbb{D})$, that sends $\mathcal{H}(\mathbb{D})$ into itself. With this approach, the search of sufficient coefficient conditions boils down to finding sufficient conditions for the boundedness of S_A. Therefore, it is not a surprise that many results on slowly growing solutions are inspired by study of the classical integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

see [2, 3, 7, 32, 38]. The strength of the operator theoretic approach is demonstrated by proving that the coefficient conditions arising from Theorem 10 are essentially interchangeable with $A \in \mathcal{L}^1$, see Theorem 11.

Deep duality relations are implicit in the proofs of Theorems 2(i), 10 and 14. The dual of H^1 is isomorphic to BMOA with the Cauchy pairing by Fefferman's theorem [12, Theorem 7.1], the dual of the disc algebra is isomorphic to the space of Cauchy transforms with the dual pairing $\langle f, K\mu \rangle = \int f\,\overline{d\mu}$ [6, Theorem 4.2.2], and the dual of A^1_ω is isomorphic to the Bloch space with the dual pairing $\langle f, g \rangle_{A^2_\omega} = \int_{\mathbb{D}} f\overline{g}\,\omega\,dm$ [30, Corollary 7].

Finally, we turn to consider coefficient conditions which place solutions of (1.2) in the Hardy spaces. Our results are inspired by an open question, which is closely related to the Hardy-Stein-Spencer formula

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|}\,dm(z), \tag{2.7}$$

that holds for $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$. For $p = 2$, (2.7) is the well-known Littlewood-Paley identity, while the general case follows from [17, Theorem 3.1] by integration.

Question 1. Let $0 < p < \infty$. Is it true that

$$\|f\|_{H^p}^p \le C(p) \int_{\mathbb{D}} |f(z)|^{p-2} |f''(z)|^2 (1 - |z|^2)^3\,dm(z) + |f(0)|^p + |f'(0)|^p \tag{2.8}$$

for any $f \in \mathcal{H}(\mathbb{D})$, where $C(p)$ is a positive constant such that $C(p) \to 0^+$ as $p \to 0^+$?

Affirmative answer to this question would have an immediate application to differential equations, see Section 9.2. In the context of second order differential equation (1.2), it suffices to consider Question 1 under the additional assumptions that all zeros of f are simple and f'' vanishes at zeros of f. The estimate in Question 1 is valid for a non-trivial subclass of $\mathcal{H}(\mathbb{D})$, see Section 9.1.

Function $f \in \mathcal{H}(\mathbb{D})$ is uniformly locally univalent if there is a constant $0 < \delta \le 1$ such that f is univalent in each pseudo-hyperbolic disc $\Delta(z,\delta) = \{w \in \mathbb{D} : |\varphi_z(w)| < \delta\}$ for $z \in \mathbb{D}$. A partial solution to Question 1 is given by Theorem 4. Here $a \lesssim b$ means that there exists $C > 0$ such that $a \le Cb$. Moreover, $a \asymp b$ if and only if $a \lesssim b$ and $a \gtrsim b$.

Theorem 4. *Let $f \in \mathcal{H}(\mathbb{D})$, and $k \in \mathbb{N}$.*

(i) *If $0 < p \le 2$, then*

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p. \tag{2.9}$$

(ii) *If $2 \le p < \infty$, then*

$$\int_{\mathbb{D}} |f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \lesssim \|f\|_{H^p}^p. \tag{2.10}$$

(iii) *If $0 < p < \infty$ and f is uniformly locally univalent, then* (2.10) *holds.*

The comparison constants are independent of f; in (i) *and* (ii) *they depend on p, and in* (iii) *it depends on δ (the constant of uniform local univalence) and p.*

The proof of Theorem 4 is presented in Section 9, and it takes advantage of a norm in H^p, given in terms of higher derivatives and area functions, and an estimate of the non-tangential maximal function.

3. Zero distribution of solutions

For $0 \le p < \infty$, the growth space H_p^∞ consists of those $g \in \mathcal{H}(\mathbb{D})$ for which

$$\|g\|_{H_p^\infty} = \sup_{z\in\mathbb{D}} |g(z)|(1-|z|^2)^p < \infty.$$

We write $H^\infty = H_0^\infty$, for short. The sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ is called uniformly separated if

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \left| \frac{z_n - z_k}{1-\overline{z}_n z_k} \right| > 0,$$

while $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ is said to be separated in the hyperbolic metric if there exists a constant $\delta > 0$ such that $|z_n - z_k|/|1-\overline{z}_n z_k| > \delta$ for any $n \ne k$. After the proof of Theorem 1, we present an auxiliary result which provides an estimate for the number of sequences in the finite union appearing in the claim.

Proof of Theorem 1. (i) If f is a non-trivial solution of (2.1), then $g = f \circ \varphi_a$ solves

$$g''' + B_2 g'' + B_1 g' + B_0 g = 0, \tag{3.1}$$

where

$$\begin{aligned} B_0 &= (A_0 \circ \varphi_a)(\varphi_a')^3, \qquad B_2 = (A_2\circ\varphi_a)\varphi_a' - 3\,\frac{\varphi_a''}{\varphi_a'},\\ B_1 &= (A_1\circ\varphi_a)(\varphi_a')^2 - (A_2\circ\varphi_a)\varphi_a'' + 3\left(\frac{\varphi_a''}{\varphi_a'}\right)^2 - \frac{\varphi_a'''}{\varphi_a'}. \end{aligned} \tag{3.2}$$

By a conformal change of variable, we deduce $\|B_0\|_{H_3^\infty} = \|A_0\|_{H_3^\infty}$,

$$
\begin{aligned}
\|B_2\|_{H_1^\infty} &\leq \sup_{z\in\mathbb{D}} |A_2(z)|\,(1-|z|^2) + \sup_{z\in\mathbb{D}} \frac{6|a|}{|1-\overline{a}z|}\,(1-|z|^2) \leq \|A_2\|_{H_1^\infty} + 12,\\
\|B_1\|_{H_2^\infty} &\leq \sup_{z\in\mathbb{D}} |A_1(z)|\,(1-|z|^2)^2 + \sup_{w\in\mathbb{D}} |A_2(w)|\,(1-|w|^2)\left|\frac{\varphi_a''(\varphi_a(w))}{\varphi_a'(\varphi_a(w))}\right|(1-|\varphi_a(w)|^2)\\
&\qquad + \sup_{z\in\mathbb{D}} \frac{12|a|^2}{|1-\overline{a}z|^2}\,(1-|z|^2)^2 + \sup_{z\in\mathbb{D}} \frac{6|a|^2}{|1-\overline{a}z|^2}\,(1-|z|^2)^2\\
&\leq \|A_1\|_{H_2^\infty} + 4\|A_2\|_{H_1^\infty} + 72.
\end{aligned}
$$

Let $\mathcal{Z} = \mathcal{Z}(f)$ be the sequence of two-fold zeros of f, and let $a \in \mathcal{Z}$; we may assume that $\mathcal{Z}$ is not empty, for otherwise there is nothing to prove. Then, the zero of $g = f \circ \varphi_a$ at the origin is two-fold. By applying Jensen's formula to $z \mapsto g(z)/z^2$ we obtain

$$
\sum_{\substack{z_k\in\mathcal{Z}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{g(re^{i\theta})}{g''(0)}\right| d\theta + \log\frac{2}{r^2}, \quad 0<r<1, \tag{3.3}
$$

where $\log^+ x = \max\{0, \log x\}$ for $0 \leq x < \infty$. Since

$$
\begin{aligned}
\int_0^1 \left(\sum_{\substack{z_k\in\mathcal{Z}\\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|}\right) r\,dr &= \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \int_{|\varphi_a(z_k)|}^1 r\log\frac{r}{|\varphi_a(z_k)|}\,dr\\
&\geq \frac{1}{8}\sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2,
\end{aligned}
$$

the estimate (3.3) implies

$$
\sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2 \leq 4\int_{\mathbb{D}} \log^+\left|\frac{g(z)}{g''(0)}\right| dm(z) + 4\log 2 + 4.
$$

Consider the normalized solution $h(z) = g(z)/g''(0)$ of (3.1), which has the initial values $h(0) = h'(0) = 0$ and $h''(0) = 1$. By the proofs of the growth estimates [18, Theorems 3.1 and 4.1, and Corollary 4.2], there exists an absolute constant $C_1 > 0$ such that

$$
\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|h(re^{i\theta})\right| d\theta \leq C_1 \sum_{j=0}^{2}\sum_{n=0}^{j} \int_0^{2\pi}\int_0^r |B_j^{(n)}(se^{i\theta})|(1-s)^{3-j+n-1}\,ds\,d\theta.
$$

By Cauchy's integral formula and the estimates above, there exists a positive constant $C_2 = C_2(\|A_0\|_{H_3^\infty}, \|A_1\|_{H_2^\infty}, \|A_2\|_{H_1^\infty})$, independent of $a \in \mathbb{D}$, such that

$$
\|B_j^{(n)}\|_{H_{3-j+n}^\infty} \leq C_2, \quad j=0,1,2, \quad n=0,\dots,j.
$$

Let $M_\infty(s, B_j^{(n)})$ denote the maximum modulus of $B_j^{(n)}$ on the circle of radius s. Now

$$
\begin{aligned}
&\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2\\
&\qquad \leq 4\log 2 + 4 + 16\pi\, C_1 \sup_{a\in\mathcal{Z}} \sum_{j=0}^{2}\sum_{n=0}^{j}\int_0^1\int_0^r M_\infty(s, B_j^{(n)})(1-s)^{2-j+n}\,ds\,dr\\
&\qquad \leq 4\log 2 + 4 + 16\pi\, C_1C_2 \sum_{j=0}^{2}\sum_{n=0}^{j}\int_0^1\int_0^r \frac{ds}{1-s^2}\,dr < \infty.
\end{aligned}
$$

The assertion of Theorem 1(i) follows from Lemma 5(i) below.

(ii) As in the proof of (i), we conclude that $g = f \circ \varphi_a$ is a solution of (3.1), where the coefficients B_0, B_1, B_2 depend on $a \in \mathbb{D}$. By (2.3),

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |B_j^{(n)}(z)|(1-|z|^2)^{2-j+n}\,dm(z) < \infty, \quad j = 0,\dots,2, \quad n = 0,\dots,j. \tag{3.4}$$

In order to conclude (3.4), first get rid of the derivatives by standard estimates, and then integrate the coefficients (3.2) term-by-term.

Let $\mathcal{Z}$ be the sequence of two-fold zeros of f. As above, there exists an absolute constant $C_3 > 0$ such that

$$\sup_{a\in\mathcal{Z}} \sum_{\substack{z_k\in\mathcal{Z} \\ 0<|\varphi_a(z_k)|<r}} \log\frac{r}{|\varphi_a(z_k)|} \le \log\frac{2}{r^2} + C_3 \sup_{a\in\mathcal{Z}} \sum_{j=0}^{2}\sum_{n=0}^{j} \int_{\mathbb{D}} |B_j^{(n)}(z)|(1-|z|^2)^{2-j+n}\,dm(z)$$

for $0 < r < 1$. By letting $r \to 1^-$, we obtain

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right) < \infty.$$

This implies the assertion of Theorem 1(ii) by Lemma 5(ii) below. □

The following lemma gives an estimate for the number of sequences in the finite union appearing in the statement of Theorem 1. For more details, we refer to [9, Chapter 2.11].

Lemma 5. *Let $\mathcal{Z} = \{z_k\}$ be a sequence of points in $\mathbb{D}$ such that the multiplicity of each point is at most $p \in \mathbb{N}$, and let $M \in \mathbb{N}$.*

(i) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right)^2 \le M < \infty,$$

then $\{z_k\}$ can be expressed as a finite union of at most $M+p$ separated sequences.

(ii) *If*

$$\sup_{a\in\mathcal{Z}} \sum_{z_k\in\mathcal{Z}\setminus\{a\}} \left(1-|\varphi_a(z_k)|^2\right) \le M < \infty, \tag{3.5}$$

then $\{z_k\}$ can be expressed as a finite union of at most $M+p$ uniformly separated sequences.

Proof. (i) Assume on contrary to the claim, that every partition of $\mathcal{Z}$ into separated subsequences is a union of at least $M+p+1$ sequences. Then, for each $n \in \mathbb{N}$, there exists a point $z_n \in \mathcal{Z}$ such that

$$\#\left\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \le 2^{-n}\right\} \ge M+p+1.$$

Now

$$\begin{aligned} p + M &\ge p + \sum_{z_k\in\mathcal{Z}\setminus\{z_n\}} \left(1-|\varphi_{z_n}(z_k)|^2\right)^2 \ge \sum_{z_k\in\mathcal{Z}} \left(1-|\varphi_{z_n}(z_k)|^2\right)^2 \\ &\ge \#\left\{z_k \in \mathcal{Z} : |\varphi_{z_n}(z_k)| \le 2^{-n}\right\} \cdot (1-4^{-n})^2 \ge (M+p+1)(1-4^{-n})^2. \end{aligned}$$

By letting $n \to \infty$ we arrive to a contradiction. Hence $\mathcal{Z}$ can be expressed as a union of at most $M+p$ separated sequences.

(ii) By part (i), $\mathcal{Z}$ can be expressed as a union of at most $M+p$ separated sequences, and each of these separated sequences is uniformly separated by (3.5). □

Example 1. If $\{f, g\}$ is a solution base of (1.2), then $\{f^2, g^2, fg\}$ is a solution base of

$$h''' + 4Ah' + 2A'h = 0. \tag{3.6}$$

Let us apply this property to a classical example [36, p. 162] originally due to Hille [20, p. 552]. For $\gamma > 0$, the differential equation (1.2) with $A(z) = (1+4\gamma^2)/(1-z^2)^2$, $z \in \mathbb{D}$, admits the solution

$$f(z) = \sqrt{1-z^2}\,\sin\left(\gamma \log \frac{1+z}{1-z}\right), \quad z \in \mathbb{D}.$$

The zeros of f are simple and real, and moreover, the hyperbolic distance between two consecutive zeros is precisely $\pi/(2\gamma)$. Consequently, (3.6) admits the solution $h = f^2$ whose zero-sequence is a union of two separated sequences. This sequence is a union of two uniformly separated sequences (in fact, a union of two exponential sequences), since all zeros are real [8, Theorem 9.2]. In this case the coefficients of (3.6) satisfy both conditions (2.2) and (2.3). ◇

4. Inclusion relations between function spaces

The following result can be used to compare the coefficient conditions. In particular, Lemma 6 shows that the coefficient condition in Theorem 3 (which implies that all solutions of (1.2) are in BMOA) is strictly stronger than $A \in \mathcal{L}^1$ with sufficiently small norm (which places all solutions in $\mathcal{B} \cap H^2$). And further, Lemma 6 proves that $A \in \mathcal{L}^1$ with sufficiently small norm is strictly stronger than the coefficient condition in Theorem A below (which forces solutions to be in Hardy spaces). The reader is invited to compare Lemma 6 to the results in [4, Section 5].

If $A \in \mathcal{H}(\mathbb{D})$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) \tag{4.1}$$

is finite, then we write $A \in \mathrm{BMOA}''$. Note that $A \in \mathrm{BMOA}''$ if and only if there exists a function $g = g(A) \in \mathrm{BMOA}$ such that $A = g''$, which follows from standard estimates. Correspondingly, if $A \in \mathcal{H}(\mathbb{D})$ and

$$\|A\|^2_{\mathrm{LMOA}''} = \sup_{a\in\mathbb{D}} \left(\log \frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z) < \infty,$$

then $A \in \mathrm{LMOA}''$. As expected, LMOA'' consists of those functions in $\mathcal{H}(\mathbb{D})$ which can be represented as the second derivative of a function in LMOA. For more details on LMOA, see [4, 37]. Finally, part (iv) of Lemma 6 gives a sufficient condition for a lacunary series to be in LMOA''.

Lemma 6. *The following assertions hold:*

(i) $\mathcal{L}^{\alpha_1} \subsetneq \mathcal{L}^{\alpha_2} \subsetneq H_2^\infty$ *for any* $0 < \alpha_2 < \alpha_1 < \infty$;
(ii) $\mathrm{LMOA}'' \subsetneq \mathcal{L}^1 \subsetneq \mathcal{L}^\alpha \subsetneq \mathrm{BMOA}'' \subsetneq H_2^\infty$ *for any* $1/2 < \alpha < 1$;
(iii) $\mathcal{L}^{3/2} \subsetneq \mathrm{LMOA}''$, *and* $\mathrm{LMOA}'' \setminus \bigcup_{1<\alpha<\infty} \mathcal{L}^\alpha$ *is non-empty;*
(iv) *if* $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ *and* $\{a_k\}_{k=1}^\infty \subset \mathbb{C}$ *satisfy the conditions* $\inf_{k\in\mathbb{N}} n_{k+1}/n_k > 1$ *and* $\sum_{k=1}^\infty |a_k|^2(\log n_k)^3/n_k^4 < \infty$, *then* $\left(\sum_{k=1}^\infty a_k z^{n_k}\right) \in \mathrm{LMOA}''$.

Proof. As (i) is an immediate consequence of the definitions, we proceed to prove (ii). Let $A \in \mathrm{LMOA}''$. Since (2.5) is finite and $|A|^2$ is subharmonic, we deduce $\|A\|^2_{\mathcal{L}^1} \lesssim \|A\|^2_{\mathrm{LMOA}''}$. Assume on contrary to the assertion that $\mathrm{LMOA}'' = \mathcal{L}^1$. By [15, Theorem 1], there exist $A_0, A_1 \in \mathcal{H}(\mathbb{D})$ satisfying

$$|A_0(z)| + |A_1(z)| \asymp \frac{1}{(1-|z|^2)^2 \log \frac{e}{1-|z|}}, \quad z \in \mathbb{D}.$$

Since $A_0, A_1 \in \mathrm{LMOA}''$, we deduce

$$\int_{S_a} \frac{dm(z)}{(1-|z|^2)\big(\log\frac{e}{1-|z|}\big)^2} \lesssim \int_{S_a} \big(|A_0(z)|+|A_1(z)|\big)^2(1-|z|^2)^3\,dm(z) \lesssim \frac{1-|a|}{\big(\log\frac{e}{1-|a|}\big)^2}$$

as $|a|\to 1^-$. This contradicts the fact

$$\int_{S_a}\frac{dm(z)}{(1-|z|^2)\left(\log\frac{e}{1-|z|}\right)^2}\asymp\frac{1-|a|}{\log\frac{e}{1-|a|}},\quad |a|\to 1^-,$$

and hence $\mathrm{LMOA}''\neq\mathcal{L}^1$. The remaining part of (ii) is a straightforward computation. Note that the inclusion $\mathcal{L}^\alpha\subsetneq\mathrm{BMOA}''$, for any $1/2<\alpha<\infty$, is strict by $A(z)=(1-z)^{-2}$.

To prove (iii) it suffices to prove the latter assertion, as $\mathcal{L}^{3/2}\subset\mathrm{LMOA}''$ follows directly from (2.5). If $A(z)=(1-z)^{-2}\left(\log\frac{e}{1-z}\right)^{-1}$ for $z\in\mathbb{D}$, then $A\notin\bigcup_{1<\alpha<\infty}\mathcal{L}^\alpha$. To show that $A\in\mathrm{LMOA}''$, it is enough to verify (2.5) for $0<a<1$. Since

$$\left|\log\frac{e}{1-z}\right|\geq\log\frac{e}{|1-z|}\geq\log\frac{e}{2(1-a)},\quad z\in S_a, \tag{4.2}$$

we conclude

$$\begin{aligned}&\sup_{0<a<1}\frac{\left(\log\frac{e}{1-a}\right)^2}{1-a}\int_{S_a}|A(z)|^2(1-|z|^2)^3\,dm(z)\\&\quad\lesssim\sup_{0<a<1}\frac{1}{1-a}\int_a^1\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^4}(1-r^2)^3\,r\,dr<\infty.\end{aligned} \tag{4.3}$$

In order to prove (iv), let $A(z)=\sum_{k=1}^\infty a_kz^{n_k}$ for $z\in\mathbb{D}$. If $h(z)=\sum_{k=1}^\infty z^{n_k}$ for $z\in\mathbb{D}$, then $h\in\mathcal{B}$ with $M_\infty(r,h)=\sum_{k=1}^\infty r^{n_k}\lesssim\log\frac{e}{1-r}$ for $0<r<1$. By the Cauchy-Schwarz inequality,

$$M_\infty(r,A)\lesssim\left(\sum_{k=1}^\infty|a_k|^2r^{n_k}\right)^{1/2}\left(\log\frac{e}{1-r}\right)^{1/2},\quad 0<r<1.$$

It follows that

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\frac{\left(\log\frac{e}{1-|a|}\right)^2}{1-|a|}\int_{S_a}|A(z)|^2(1-|z|^2)^3\,dm(z)\\&\quad\lesssim\int_0^1M_\infty(r,A)^2(1-r)^3\left(\log\frac{e}{1-r}\right)^2dr\\&\quad\lesssim\sum_{k=1}^\infty|a_k|^2\int_0^1r^{n_k}(1-r)^3\left(\log\frac{e}{1-r}\right)^3dr\asymp\sum_{k=1}^\infty|a_k|^2\frac{(\log n_k)^3}{n_k^4},\end{aligned}$$

where the asymptotic equality follows from [28, Lemma 1.3]. This completes the proof of Lemma 6. □

5. Bounded solutions

We consider bounded solutions of (1.2). As usual, the space H^∞ consists of $f\in\mathcal{H}(\mathbb{D})$ for which $\|f\|_{H^\infty}=\sup_{z\in\mathbb{D}}|f(z)|<\infty$. The proof of Theorem 2(i) takes advantage of the well-known representation formula

$$g(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{g(e^{it})}{1-e^{-it}\zeta}\,dt,\quad\zeta\in\mathbb{D}, \tag{5.1}$$

which holds for any $g\in H^1$ [8, Theorem 3.6].

Let M be the collection of all (finite) complex Borel measures on $\mathbb{T}$. For $\mu\in M$, the total variation measure $|\mu|$ is defined as a set function

$$|\mu|(E)=\sup\sum_j|\mu(E_j)|,$$

where the supremum is taken over all countable (Borel) partitions $\{E_j\}$ of $E \subset \mathbb{T}$. Moreover, $\|\mu\| = |\mu|(\mathbb{T})$ is the total variation of μ [34, Chapter 6]. Let $\mathcal{K}$ be the space of Cauchy transforms, which consists of analytic functions in $\mathbb{D}$ of the form

$$(K\mu)(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}, \quad z \in \mathbb{D},$$

for some $\mu \in M$. For each $f \in \mathcal{K}$ there is a set $M_f = \{\mu \in M : f = K\mu\}$ of measures that represent f, and produce the norm

$$\|f\|_{\mathcal{K}} = \inf\{\|\mu\| : \mu \in M_f\}.$$

We refer to [6] for more details.

Proof of Theorem 2(i). Let f be any solution of (1.2), and write $f_r(z) = f(rz)$ for $0 \le r < 1$. Then f_r is analytic in $\overline{\mathbb{D}}$ and satisfies $f_r''(w) + r^2 A(rw) f_r(w) = 0$ for $w \in \mathbb{D}$. By (2.6), (5.1) for $g = f_r$, and Fubini's theorem, we conclude

$$f_r(z) = -\frac{1}{2\pi}\int_0^{2\pi} f_r(e^{it}) \int_0^z \int_0^\zeta \frac{r^2 A(rw)}{1-e^{-it}w}\, dw\, d\zeta\, dt + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}.$$

For all $0 < r < 1$ sufficiently large, and $z \in \mathbb{D}$, there exists $\mu_{r,z} \in M$ such that

$$A_{r,z}(u) = (K\mu_{r,z})(u), \quad u \in \mathbb{D}, \tag{5.2}$$

and $\|\mu_{r,z}\| < \delta$ for some absolute constant $0 < \delta < 1$. Hence, by [6, Theorem 4.2.2],

$$\begin{aligned} f_r(z) &= -\frac{r^2}{2\pi}\int_0^{2\pi} f_r(e^{it})\overline{(K\mu_{r,z})(e^{it})}\, dt + f_r'(0)z + f_r(0) \\ &= -r^2 \int_{\mathbb{T}} f_r(x)\overline{d\mu_{r,z}(x)} + f_r'(0)z + f_r(0), \quad z \in \mathbb{D}. \end{aligned}$$

By [34, Theorem 6.12], there exist measurable functions $h_{r,z}$ such that $|h_{r,z}(\zeta)| = 1$ for all $\zeta \in \mathbb{T}$ and the polar decompositions $d\mu_{r,z} = h_{r,z}\, d|\mu_{r,z}|$ hold. Therefore

$$\begin{aligned} |f_r(z)| &\le \left| \int_{\mathbb{T}} f_r(x)\overline{h_{r,z}(x)}\, d|\mu_{r,z}|(x) \right| + |f_r'(0)| + |f_r(0)| \\ &\le \|f_r\|_{H^\infty} \int_{\mathbb{T}} d|\mu_{r,z}| + |f_r'(0)| + |f_r(0)| \\ &\le \|f_r\|_{H^\infty}\|\mu_{r,z}\| + |f'(0)| + |f(0)|, \quad z \in \mathbb{D}. \end{aligned}$$

This implies $\|f\|_{H^\infty} \le (|f(0)| + |f'(0)|)/(1-\delta)$, and hence completes the proof of Theorem 2(i). □

Let $0 < p < \infty$, $n \in \mathbb{N}$ and $f \in \mathcal{H}(\mathbb{D})$. The proof of Theorem 2(ii) relies on a classical representation

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2}\, dm(z) \right)^{p/2} |d\zeta| + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \tag{5.3}$$

which involves non-tangential approach regions; see [1, p. 125], for example. Hardy spaces H^p are further considered in Section 9. For a fixed $1 < \alpha < \infty$, the non-tangential approach region of aperture $2\arctan\sqrt{\alpha^2-1}$, with vertex at $\zeta \in \mathbb{T}$, is given by $\Gamma(\zeta) = \{z \in \mathbb{D} : |z-\zeta| \le \alpha(1-|z|)\}$. The corresponding non-tangential maximal function is

$$f^\star(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{T}. \tag{5.4}$$

Proof of Theorem 2(ii). Let $A(z) = \sum_{n=0}^\infty a_n z^n$ for $z \in \mathbb{D}$. By the assumption, $\mathcal{A}(z) = \int_0^z A(\zeta)\, d\zeta$ satisfies $\mathcal{A} \in H^1$. We compute

$$\int_0^1 M_\infty(r, A)(1-r)\, dr \le \int_0^1 \left(\sum_{n=0}^\infty |a_n| r^n \right)(1-r)\, dr = \sum_{n=0}^\infty \frac{|a_n|}{(n+1)(n+2)} \le \pi\, \|\mathcal{A}\|_{H^1},$$

where the last estimate follows from Hardy's inequality [8, p. 48]. By [19, Corollary 3.16], we conclude that all solutions of (1.2) are bounded.

Let f be a solution of (1.2). Then

$$f'(z) = -\int_0^z f(\zeta)A(\zeta)\,d\zeta + f'(0), \quad z \in \mathbb{D},$$

and hence by (5.3), we deduce

$$\begin{aligned}\|f'\|_{H^1} &\le \left\| \int_0^z f(\zeta)A(\zeta)\,d\zeta \right\|_{H^1} + |f'(0)| \\ &\asymp \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} |f(z)|^2 |A(z)|^2\,dm(z) \right)^{1/2} |d\zeta| + |f'(0)| + |f''(0)| \\ &\le \|f\|_{H^\infty} \|\mathcal{A}\|_{H^1} + |f'(0)| + |f''(0)|.\end{aligned}$$

The assertion $f' \in H^1$ follows. □

Remark 1. For each $0 < r < 1$ and $z \in \mathbb{D}$, it is easy to see that

$$d\widetilde{\mu}_{r,z}(x) = \overline{\left(\int_0^z \int_0^\zeta \frac{A(rw)}{x-w}\,dw\,d\zeta \right) \frac{dx}{2\pi i}}, \quad x \in \mathbb{T},$$

is one of the representing measures for which (5.2) holds, and hence $\|A_{r,z}\|_{\mathcal{K}} \le \|\widetilde{\mu}_{r,z}\|$. Moreover, the behavior of the second primitive of A is controlled by this measure in the sense that

$$\int_0^z \int_0^\zeta A(rw)\,dw d\zeta = \int_0^z \int_0^\zeta \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{dx}{x-w} \right) A(rw)\,dw d\zeta = \int_{\mathbb{T}} \overline{d\widetilde{\mu}_{r,z}(x)},$$

which follows from Cauchy's integral formula and Fubini's theorem.

6. Solutions of bounded and vanishing mean oscillation

The space BMOA consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^2_{\mathrm{BMOA}} = \sup_{a\in\mathbb{D}} \|f_a\|^2_{H^2} < \infty, \tag{6.1}$$

where $f_a(z) = f(\varphi_a(z)) - f(a)$ for $a, z \in \mathbb{D}$. By the Littlewood-Paley identity,

$$\|f\|^2_{\mathrm{BMOA}} \le 4 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)\,dm(z) \le 4\,\|f\|^2_{\mathrm{BMOA}}, \tag{6.2}$$

see [11, pp. 228–230]. Clearly, BMOA is a subspace of the Bloch space $\mathcal{B}$.

A positive Borel measure μ on $\mathbb{D}$ is called a Carleson measure, if

$$\|\mu\|_{\mathrm{Carleson}} = \sup_{a\in\mathbb{D}} \frac{\mu(S_a)}{1-|a|} < \infty.$$

There exists a constant $1 \le \alpha < \infty$ such that

$$\frac{1}{1-|a|} \le \alpha\,\frac{1-|a|^2}{|1-\overline{a}z|^2} = \alpha\,|\varphi_a'(z)|, \quad z \in S_a, \quad a \in \mathbb{D},$$

since $|1-\overline{a}z| \le |1-|a|^2| + ||a|^2 - \overline{a}z| \lesssim 1 - |a|$. Consequently,

$$\|\mu\|_{\mathrm{Carleson}} = \sup_{a\in\mathbb{D}} \int_{S_a} \frac{1}{1-|a|}\,d\mu(z) \le \alpha \cdot \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|\,d\mu(z). \tag{6.3}$$

We prove Theorem 3 and consider its counterpart for VMOA. Theorem 3 is inspired by [37, Theorem 3.1]. We return to consider BMOA and VMOA solutions in Section 8, where parallel results are obtained by using the representation formula for H^1 functions.

Proof of Theorem 3. The proof consists of two steps. First, we show that

$$\sup_{1/2<r<1}\sup_{a\in\mathbb{D}}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\lesssim\|A\|^2_{\mathrm{LMOA}''}. \tag{6.4}$$

Denote

$$I(a,r)=\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z),\quad 0<r<1,\quad a\in\mathbb{D},$$

for short. For $|a|\le 1/2$ the estimate (6.4) is trivial. Let $1/2<|a|<1/(2-r)$. Since $|1-\overline{a}z|\le 2\,|1-\overline{a}z/r|$ for $|z|\le r$, we deduce

$$\begin{aligned}I(a,r)&=\int_{D(0,r)}|A(z)|^2\big(1-\big|\tfrac{z}{r}\big|^2\big)^3\frac{1-|a|^2}{\big|1-\overline{a}\,\frac{z}{r}\big|^2}\frac{dm(z)}{r^2}\\&\le\frac{4}{r^2}\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\le 16\,\|A\|^2_{\mathrm{LMOA}''}\left(\log\frac{e}{1-|a|}\right)^{-2}.\end{aligned}$$

for any $1/2<r<1$. Let $1/(2-r)\le|a|<1$. Now

$$\begin{aligned}I(a,r)&\le\|A\|^2_{\mathcal{L}^1}\int_{\mathbb{D}}\frac{\big(1-|z|^2\big)^2\big(1-|\varphi_a(z)|^2\big)}{\big(1-|rz|^2\big)^4\big(\log\frac{e}{1-|rz|}\big)^2}\,dm(z)\\&\lesssim\|A\|^2_{\mathcal{L}^1}\int_0^1\frac{(1-s)^3(1-|a|)}{(1-rs)^4\big(\log\frac{e}{1-rs}\big)^2(1-|a|s)}\,ds.\end{aligned}$$

As $t\mapsto(1-t)^2\big(\log\frac{e}{1-t}\big)$ is decreasing for $0<t<1$, we apply $r\le 2-1/|a|$ to obtain

$$\begin{aligned}I(a,r)&\lesssim\|A\|^2_{\mathcal{L}^1}(1-|a|)\int_0^{|a|}\frac{ds}{(1-s)^2\big(\log\frac{e}{1-s}\big)^2}+\frac{\|A\|^2_{\mathcal{L}^1}}{(1-|a|)^4\big(\log\frac{e}{1-|a|}\big)^2}\int_{|a|}^1(1-s)^3\,ds\\&\lesssim\|A\|^2_{\mathcal{L}^1}\left(\log\frac{e}{1-|a|}\right)^{-2}.\end{aligned}$$

Since $\|A\|^2_{\mathcal{L}^1}\lesssim\|A\|^2_{\mathrm{LMOA}''}$ by the proof of Lemma 6(ii), this completes the proof of (6.4).

Second, we proceed to consider the differential equation (1.2). Let f be a non-trivial solution of (1.2). By Lemma 6(ii) and [21, Corollary 4(b)], we may assume that $f\in\mathcal{B}$. Now, (1.2) and (6.2) yield

$$\begin{aligned}\|f_r\|^2_{\mathrm{BMOA}}&\lesssim\sup_{a\in\mathbb{D}}\left(|f'(ra)|^2(1-|a|^2)^2\,r^2+\int_{\mathbb{D}}r^4|f''(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\right)\\&\lesssim\|f_r\|^2_{\mathcal{B}}+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_r(z)-f_r(a)|^2\,|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\\&\qquad+\sup_{a\in\mathbb{D}}|f_r(a)|^2\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)\\&\lesssim\|f_r\|^2_{\mathcal{B}}+I_1+I_2\end{aligned}$$

with absolute comparison constants. By Carleson's theorem [8, Theorem 9.3], (6.1) and (6.3),

$$\begin{aligned}I_1&\lesssim\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f_r)_a(z)|^2\,\big|A(r\varphi_a(z))\big|^2\big(1-|\varphi_a(z)|^2\big)^3\,|\varphi_a'(z)|\,dm(z)\\&\lesssim\sup_{a\in\mathbb{D}}\left(\big\|(f_r)_a\big\|^2_{H^2}\cdot\sup_{b\in\mathbb{D}}\int_{\mathbb{D}}\big|A(r\varphi_a(z))\big|^2\big(1-|\varphi_a(z)|^2\big)^3|\varphi_a'(z)||\varphi_b'(z)|\,dm(z)\right)\\&\lesssim\|f_r\|^2_{\mathrm{BMOA}}\cdot\sup_{c\in\mathbb{D}}\int_{\mathbb{D}}|A(rz)|^2(1-|z|^2)^2\big(1-|\varphi_c(z)|^2\big)\,dm(z).\end{aligned}$$

Estimation of I_2 is easier. By [12, Corollary 5.3],

$$I_2 \lesssim \|f_r\|_{\mathrm{BMOA}}^2 \cdot \sup_{a\in\mathbb{D}} \left(\log\frac{e}{1-|a|}\right)^2 \int_{\mathbb{D}} |A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z).$$

If (2.4) is sufficiently small, then (6.4) implies that $\|f_r\|_{\mathrm{BMOA}}$ is uniformly bounded for $1/2<r<1$. By letting $r\to 1^-$, we conclude $f\in\mathrm{BMOA}$. □

The following example reveals that the coefficient condition in Theorem 3 allows solutions of (1.2) to be unbounded. Moreover, the same construction with $1<\alpha<\infty$ illustrates that the finiteness of (2.4) is not enough to guarantee that all solutions of (1.2) are in BMOA. The same construction is applied in [21, Example 5(b)].

Example 2. Let $0<\alpha\le 1$, and define

$$A(z)=\frac{-\alpha}{(1-z)^2}\left((\alpha-1)\left(\log\frac{e}{1-z}\right)^{-2}+\left(\log\frac{e}{1-z}\right)^{-1}\right),\quad z\in\mathbb{D}.$$

Then $A\in\mathcal{H}(\mathbb{D})$, and (1.2) admits two linearly independent solutions

$$f_1(z)=\left(\log\frac{e}{1-z}\right)^{\alpha},\quad f_2(z)=\left(\log\frac{e}{1-z}\right)^{\alpha}\int_0^z\left(\log\frac{e}{1-\zeta}\right)^{-2\alpha}d\zeta,\quad z\in\mathbb{D},$$

which are unbounded on positive real axis; see also [21, Example 5(b)]. We denote $A=-\alpha B_1-\alpha(\alpha-1)B_2$, where $B_j(z)=(1-z)^{-2}(\log(e/(1-z)))^{-j}$ for $z\in\mathbb{D}$ and $j=1,2$. Since $|B_2(z)|\le|B_1(z)|\,(\log(e/2))^{-1}$ for all $z\in\mathbb{D}$, and (4.2) holds for any $0<a<1$, we conclude (4.3). We point out that, for a sufficiently small α, the coefficient A satisfies the assumptions of Theorem 3 and hence all solutions of (1.2) are in BMOA.

The space VMOA consists of those $f\in H^2$ for which

$$\lim_{|a|\to 1^-}\|f_a\|_{H^2}^2=0,$$

where f_a is the auxiliary function in the beginning of Section 6. Clearly, VMOA is a subspace of the little Bloch space $\mathcal{B}_0$. As Theorem 3 is motivated by [37, Theorem 3.1], the counterpart of the following result is [37, Theorem 3.6].

Theorem 7. *Let $A\in\mathcal{H}(\mathbb{D})$. If* (2.4) *is sufficiently small and*

$$\lim_{|a|\to 1^-}\left(\log\frac{e}{1-|a|}\right)^2\int_{\mathbb{D}}|A(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\,dm(z)=0,$$

then all solutions f of (1.2) *satisfy* $f\in\mathrm{VMOA}$.

The proof of Theorem 7 is omitted, since it is similar to the proof of Theorem 3. Note that the coefficient condition in Theorem 7 implies (7.11), and hence forces all solutions of (1.2) to be in the little Bloch space $\mathcal{B}_0$. See the end of Section 7 for more details.

7. Solutions in the Bloch and the little Bloch spaces

An integrable function $\omega:\mathbb{D}\to[0,\infty)$ is called a weight. The weight ω is said to be radial if $\omega(u)=\omega(|u|)$ for all $u\in\mathbb{D}$. For $0<p<\infty$ and a weight ω, the weighted Bergman space A^p_ω consists of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A^p_\omega}^p=\int_{\mathbb{D}}|f(u)|^p\omega(u)\,dm(u)<\infty.$$

For a radial weight ω, we define $\widehat{\omega}(u)=\int_{|u|}^1\omega(r)\,dr$ for $u\in\mathbb{D}$. We denote $\omega\in\mathcal{D}$ whenever ω is radial and there exist constants $C=C(\omega)\ge 1$, $\alpha=\alpha(\omega)>0$ and $\beta=\beta(\omega)\ge\alpha$ such that

$$C^{-1}\left(\frac{1-r}{1-t}\right)^{\alpha}\widehat{\omega}(t)\le\widehat{\omega}(r)\le C\left(\frac{1-r}{1-t}\right)^{\beta}\widehat{\omega}(t) \tag{7.1}$$

for all $0 \le r \le t < 1$. The existence of constants $\beta = \beta(\omega) > 0$ and $C = C(\omega) > 0$ for which the right-hand side inequality of (7.1) is satisfied is equivalent to the existence of a constant $K = K(\omega) \ge 1$ such that the doubling property $\widehat{\omega}(r) \le K\,\widehat{\omega}((1+r)/2)$ holds for all $0 \le r < 1$ [29, Lemma 1]. Moreover, the left-hand side inequality of (7.1) is equivalent to the existence of constants $K = K(\omega) > 1$ and $L = L(\omega) > 1$ such that $\widehat{\omega}(r) \ge K\,\widehat{\omega}(1-(1-r)/L)$ for all $0 \le r < 1$, see [31] for more details.

Let $0 < p < \infty$ and ω be a radial weight. If $\widehat{\omega}(r) = 0$ for some $0 < r < 1$, then $A^p_\omega = \mathcal{H}(\mathbb{D})$. Let ω be a radial weight such that $\widehat{\omega}(r) > 0$ for all $0 \le r < 1$. By standard estimates,

$$\|f\|^p_{A^p_\omega} \gtrsim M_p\left(\frac{1+r}{2}, f\right)^p \widehat{\omega}\left(\frac{1+r}{2}\right) \gtrsim M_\infty(r,f)^p(1-r)\,\widehat{\omega}\left(\frac{1+r}{2}\right), \quad 0 < r < 1,$$

where $M_p(r,f)$ denotes the H^p mean of f, and hence

$$|f(z)| \lesssim \frac{\|f\|_{A^p_\omega}}{\widehat{\omega}\left(\frac{1+|z|}{2}\right)^{1/p}(1-|z|)^{1/p}}, \quad z \in \mathbb{D}. \tag{7.2}$$

We will concentrate on the case $p = 2$. By (7.2), the norm convergence in A^2_ω implies the uniform convergence on compact subsets of $\mathbb{D}$, and consequently each point evaluation $L_\zeta(f) = f(\zeta)$ is a bounded linear functional in the Hilbert space A^2_ω. Hence, there exist unique reproducing kernels $B^\omega_\zeta \in A^2_\omega$ with $\|L_\zeta\| = \|B^\omega_\zeta\|_{A^2_\omega}$ such that

$$f(\zeta) = \langle f, B^\omega_\zeta \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(u)\overline{B^\omega_\zeta(u)}\omega(u)\,dm(u), \quad f \in A^2_\omega. \tag{7.3}$$

Moreover, the normalized monomials $(2\,\omega_{2n+1})^{-1/2} z^n$, for $n \in \mathbb{N} \cup \{0\}$, form the standard orthonormal basis of A^2_ω, and hence

$$B^\omega_\zeta(u) = \sum_{n=0}^\infty \frac{(u\overline{\zeta})^n}{2\,\omega_{2n+1}}, \quad u, \zeta \in \mathbb{D}; \tag{7.4}$$

see [41, Theorem 4.19] for details in the classical case. Here $\omega_x = \int_0^1 r^x \omega(r)\,dr$ for $1 \le x < \infty$. Weight ω is called normalized if $\omega_1 = 1/2$, which implies that $\omega(\mathbb{D}) = \int_{\mathbb{D}} \omega(u)\,dm(u) = 2\omega_1 = 1$.

We begin with a lemma which shows that the derivative of B^ω_ζ is closely related to the reproducing kernel of another Bergman space with a suitably chosen weight. For example, $B^\omega_\zeta(u) = (1-u\overline{\zeta})^{-2-\alpha}$ is the reproducing kernel corresponding to the standard weight $\omega(u) = (\alpha+1)(1-|u|^2)^\alpha$, $\alpha > -1$, while $(B^\omega_\zeta)'(u) = (2+\alpha)\overline{\zeta}(1-u\overline{\zeta})^{-3-\alpha}$ is related to the reproducing kernel of the Bergman space with the weight $\widetilde{\omega}(u) = (1-|u|^2)^{\alpha+1}$. In general, we define

$$\widetilde{\omega}(u) = 2\int_{|u|}^1 \omega(r) r\,dr, \quad u \in \mathbb{D},$$

for any radial weight ω.

Lemma 8. *If ω is radial then $(B^\omega_\zeta)'(u) = \overline{\zeta}\,B^{\widetilde{\omega}}_\zeta(u)$ for $u, \zeta \in \mathbb{D}$.*

Proof. It is clear that representations (7.4) exist for both B^ω_ζ and $B^{\widetilde{\omega}}_\zeta$. By Fubini's theorem,

$$\widetilde{\omega}_{2n+1} = 2\int_0^1 \omega(s)s \int_0^s r^{2n+1}\,dr\,ds = \frac{\omega_{2n+3}}{n+1}, \quad n \in \mathbb{N} \cup \{0\},$$

and hence

$$(B^\omega_\zeta)'(u) = \overline{\zeta} \sum_{n=0}^\infty \frac{(n+1)(u\overline{\zeta})^n}{2\,\omega_{2n+3}} = \overline{\zeta}\,B^{\widetilde{\omega}}_\zeta(u), \quad u, \zeta \in \mathbb{D}.$$

This proves the assertion. □

The following auxiliary result is well-known to experts. For a radial weight ω, we define

$$\omega^\star(u)=\int_{|u|}^1\log\frac{r}{|u|}\,\omega(r)\,r\,dr,\quad u\in\mathbb{D}\setminus\{0\}.$$

Lemma 9. *If $f,g\in H^2$, then*

$$\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\overline{g(e^{it})}\,dt=2\int_{\mathbb{D}}f'(u)\overline{g'(u)}\log\frac{1}{|u|}\,dm(u)+f(0)\overline{g(0)}. \tag{7.5}$$

Moreover, if $f,g\in\mathcal{H}(\mathbb{D})$ and ω is a normalized radial weight, then

$$\langle f,g\rangle_{A^2_\omega}=4\,\langle f',g'\rangle_{A^2_{\omega^\star}}+f(0)\overline{g(0)}.$$

Proof. Identity (7.5) is a special case of [41, Theorem 9.9]. Let $f,g\in\mathcal{H}(\mathbb{D})$. By (7.5),

$$\frac{1}{\pi}\int_0^{2\pi}f(re^{it})\overline{g(re^{it})}\,dt=4\int_{D(0,r)}f'(u)\overline{g'(u)}\log\frac{r}{|u|}\,dm(u)+2f(0)\overline{g(0)}.$$

The assertion follows by integrating both sides with respect to the measure $\omega(r)r\,dr$ and using Fubini's theorem. □

Recall that the Bloch space $\mathcal{B}$ consists of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}}=\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)<\infty.$$

Theorem 10. *Let $\omega\in\mathcal{D}$ be normalized, and $A\in\mathcal{H}(\mathbb{D})$ such that*

$$\limsup_{r\to1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)<\frac14. \tag{7.6}$$

Then every solution f of (1.2) satisfies $f\in\mathcal{B}$, and

$$\|f\|_{\mathcal{B}}\le\frac{1}{1-4\,X_{\mathcal{B}}(A)}\left(|f(0)|\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_0^zA(\zeta)\,d\zeta\right|+|f'(0)|\right),$$

where

$$X_{\mathcal{B}}(A)=\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)<\frac14.$$

Proof. Observe that $\omega^\star(u)/(1-|u|^2)\asymp\widetilde{\omega}(u)$ as $|u|\to1^-$, since $\omega\in\mathcal{D}$ by the hypothesis. For fixed $z\in\mathbb{D}$, Fubini's theorem and Lemma 8 yield

$$\begin{aligned}
&\limsup_{r\to1^-}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\\
&\quad\gtrsim(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z\overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\widetilde{\omega}(u)\,dm(u)\\
&\quad\ge(1-|z|^2)\left|\int_0^z\langle1,B^{\widetilde{\omega}}_\zeta\rangle_{A^2_{\widetilde{\omega}}}A(\zeta)\zeta\,d\zeta\right|\ge(1-|z|^2)\left|\int_0^zA(\zeta)\zeta\,d\zeta\right|,
\end{aligned}\tag{7.7}$$

and it follows that $A\in H^\infty_2$. Note that the use of the reproducing formula could be avoided by a straightforward integration.

Let f be any solution of (1.2). Then

$$f_r'(z)=-\int_0^zf_r(\zeta)r^2A(r\zeta)\,d\zeta+f_r'(0),\quad z\in\mathbb{D}. \tag{7.8}$$

The reproducing formula (7.3) and Fubini's theorem imply

$$\begin{aligned}
f_r'(z)&=-\int_0^z\left(\int_{\mathbb{D}}f_r(u)\overline{B^\omega_\zeta(u)}\omega(u)\,dm(u)\right)r^2A(r\zeta)\,d\zeta+f_r'(0)\\
&=-\int_{\mathbb{D}}f_r(u)\left(\int_0^z\overline{B^\omega_\zeta(u)}r^2A(r\zeta)\,d\zeta\right)\omega(u)\,dm(u)+f_r'(0),\quad z\in\mathbb{D},
\end{aligned}$$

from which the second part of Lemma 9 yields

$$\begin{aligned} f_r'(z) &= -4\int_{\mathbb{D}} f_r'(u)\left(\int_0^z \overline{(B_\zeta^\omega)'(u)} r^2 A(r\zeta)\,d\zeta\right)\omega^\star(u)\,dm(u) \\ &\quad - f_r(0)\int_0^z r^2 A(r\zeta)\,d\zeta + f_r'(0), \quad z\in\mathbb{D}. \end{aligned}$$

It follows that

$$\begin{aligned} \|f_r\|_{\mathcal{B}} &\le 4\,\|f_r\|_{\mathcal{B}} \sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta)\,d\zeta\right| \frac{\omega^\star(u)}{1-|u|^2}\,dm(u) \\ &\quad + |f(0)|\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_0^z A(r\zeta)\,d\zeta\right| + |f'(0)|, \quad 0<r<1. \end{aligned}$$

We deduce $f\in\mathcal{B}$ by re-organizing the terms and letting $r\to 1^-$.

Since $f\in\mathcal{B}$, we know that $M_\infty(r,f)\lesssim \log(e/(1-r))$ for $0<r<1$. Hence, for any $0<p<\infty$,

$$\|f\|_{A^p_\omega}^p \lesssim \widehat{\omega}(0) + p\int_0^1\left(\log\frac{e}{1-r}\right)^{p-1}\frac{1}{(1-r)^{1-\alpha}}\,dr<\infty$$

by partial integration and (7.1); see also [27, Proposition 6.1]. Now that $f\in\mathcal{B}\subset A^2_\omega$, we may repeat the proof from the beginning with $r=1$ to deduce the second part of the assertion. □

Remark 2. The proof of Theorem 10 shows that, in order to conclude $f\in\mathcal{B}$, it suffices to take the supremum in (7.6) over any annulus $R<|z|<1$ instead of $\mathbb{D}$.

We apply an operator theoretic argument to study the sharpness of Theorem 10. Let

$$I(A,\omega) = \limsup_{r\to 1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(r\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)$$

denote the left-hand side of (7.6), for short.

Theorem 11. *Let $\omega\in\mathcal{D}$ be normalized and $A\in\mathcal{H}(\mathbb{D})$. Then the following statements are equivalent:*

(i) $A\in\mathcal{L}^1$;
(ii) $I(A,\omega)<\infty$;
(iii) *the operator $S_A:\mathcal{B}\to\mathcal{B}$ is bounded.*

Proof. (i) $\Longrightarrow$ (ii): Observe that $\omega^\star(u)/(1-|u|^2)\asymp\widehat{\omega}(u)$ as $|u|\to 1^-$. By Fubini's theorem,

$$I(A,\omega)\lesssim \limsup_{r\to 1^-}\sup_{z\in\mathbb{D}}(1-|z|^2)\int_0^z |A(r\zeta)|\left(\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(u)\right|\widehat{\omega}(u)\,dm(u)\right)|d\zeta|,$$

where

$$\int_{\mathbb{D}}\left|(B_\zeta^\omega)'(u)\right|\widehat{\omega}(u)\,dm(u)\lesssim \int_0^{|\zeta|}\frac{\widehat{\widehat{\omega}}(t)\,dt}{\widehat{\omega}(t)(1-t)^2}\asymp\int_0^{|\zeta|}\frac{dt}{1-t^2}=\frac{1}{2}\log\frac{1+|\zeta|}{1-|\zeta|}, \quad \zeta\in\mathbb{D},$$

by [30, Theorem 1], Fubini's theorem and (7.1). It follows that $I(A,\omega)\lesssim\|A\|_{\mathcal{L}^1}<\infty$.

(ii) $\Longrightarrow$ (iii): This implication follows by an argument similar to the proof of Theorem 10. As in (7.7), we deduce

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B_\zeta^\omega)'(u)} A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\le I(A,\omega)<\infty,$$

and $A \in H_2^\infty$. Let $f \in \mathcal{B} \subset A^2_\omega$. The reproducing formula (7.3), Fubini's theorem and Lemma 9 imply

$$\begin{aligned}\left\|S_A(f)\right\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\int_0^z f(\zeta)A(\zeta)\,d\zeta\right| \lesssim \|f\|_{\mathcal{B}}\, I(A,\omega) + |f(0)|\cdot\|A\|_{H_2^\infty}\\ &\lesssim \big(\|f\|_{\mathcal{B}} + |f(0)|\big)\, I(A,\omega),\end{aligned}$$

and hence we deduce (iii).

(iii) $\Longrightarrow$ (i): By the assumption, there exists a constant $C>0$ such that

$$\sup_{z\in\mathbb{D}} |f(z)|\,|A(z)|(1-|z|^2)^2 = \left\|S_A(f)''\right\|_{H_2^\infty} \lesssim \left\|S_A(f)\right\|_{\mathcal{B}} \le C\big(\|f\|_{\mathcal{B}} + |f(0)|\big) \tag{7.9}$$

for any $f \in \mathcal{B}$. Consider the family of test functions

$$f_\zeta(z) = \log\frac{e}{1-\overline{\zeta}z}, \quad z,\zeta\in\mathbb{D},$$

for which $\sup_{\zeta\in\mathbb{D}}\|f_\zeta\|_{\mathcal{B}} \le 2$. By (7.9),

$$\left|\log\frac{e}{1-\overline{\zeta}z}\right|\,|A(z)|(1-|z|^2)^2 \le 3C, \quad z,\zeta\in\mathbb{D},$$

which gives (i) for $\zeta = z$. □

A close look at the proof of Theorem 11 implies

$$I(A,w) \asymp \sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u).$$

We obtain the following consequence of Theorem 10.

Corollary 12. *Let $\omega\in\mathcal{D}$ be normalized, and $A\in\mathcal{H}(\mathbb{D})$ such that*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u) \tag{7.10}$$

is sufficiently small. Then every solution of (1.2) *belongs to $\mathcal{B}$.*

Remark 3. In order to conclude that all solutions of (1.2) are in $\mathcal{B}$, it suffices to take the supremum in (7.10) over any annulus $R<|z|<1$ instead of $\mathbb{D}$.

The little Bloch space $\mathcal{B}_0$ consists of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\lim_{|z|\to1^-}|f'(z)|(1-|z|^2) = 0.$$

The following result is a counterpart of Theorem 10 concerning the little Bloch space.

Theorem 13. *Let $\omega\in\mathcal{D}$ be normalized, and $A\in\mathcal{H}(\mathbb{D})$ such that*

$$\lim_{|z|\to1^-}(1-|z|^2)\int_{\mathbb{D}}\left|\int_0^z \overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u) = 0.$$

Then every solution of (1.2) *belongs to $\mathcal{B}_0$.*

Proof. As in (7.7), we conclude

$$\lim_{|z|\to1^-}(1-|z|^2)\left|\int_0^z A(\zeta)\zeta\,d\zeta\right| = 0.$$

By the assumption and Remark 3, it follows that each solution f of (1.2) satisfies $f\in\mathcal{B}\subset A^2_\omega$. As in the proof of Theorem 10, we have

$$\begin{aligned}(1-|z|^2)|f'(z)| \le 4\,\|f\|_{\mathcal{B}}\,(1-|z|^2)&\int_{\mathbb{D}}\left|\int_0^z \overline{(B^\omega_\zeta)'(u)}A(\zeta)\,d\zeta\right|\frac{\omega^\star(u)}{1-|u|^2}\,dm(u)\\ &+ |f(0)|\,(1-|z|^2)\left|\int_0^z A(\zeta)\,d\zeta\right| + (1-|z|^2)|f'(0)|, \quad z\in\mathbb{D}.\end{aligned}$$

The assertion follows. □

If $A \in \mathcal{H}(\mathbb{D})$ and

$$\lim_{|z|\to 1^-} |A(z)|(1-|z|^2)^2 \log \frac{e}{1-|z|} = 0, \tag{7.11}$$

then every solution of (1.2) belongs to $\mathcal{B}_0$. Actually, $f \in \mathcal{B}$ by Remark 3. Therefore

$$f''(z) = -A(z) \int_{\mathbb{D}} \frac{f(u)}{(1-\overline{u}z)^2}\, dm(u), \quad z \in \mathbb{D}.$$

By applying Lemma 9 twice, we obtain

$$|f''(z)| \lesssim |A(z)| \left(|f(0)| + |f'(0)| + \|f''\|_{H_2^\infty} \int_{\mathbb{D}} \frac{(1-|u|^2)^2}{|1-\overline{u}z|^4}\, dm(u) \right), \quad z \in \mathbb{D}.$$

Since $f \in \mathcal{B}$, we deduce $f'' \in H_2^\infty$, and hence the argument above shows that $f \in \mathcal{B}_0$ by [41, Lemma 3.10 and Theorem 5.13].

The coefficient condition (7.11), which forces all solutions of (1.2) to be in $\mathcal{B}_0$, is sharp in the sense that it cannot be replaced by $A \in \mathcal{L}^1$. Indeed, the function $f(z) = \log(e/(1-z)) \in \mathcal{B} \setminus \mathcal{B}_0$ is a solution of (1.2) for

$$A(z) = \frac{-1}{(1-z)^2 \log(e/(1-z))}, \quad z \in \mathbb{D}.$$

8. Solutions of bounded and vanishing mean oscillation — parallel results

In this section, we consider two coefficient estimates, which are derived from the representation (5.1). These estimates give sufficient conditions for all solutions of (1.2) to be in BMOA or VMOA. Recall that, by (6.2) and (6.3), the measure $d\mu_f(z) = |f'(z)|^2(1-|z|^2)\, dm(z)$ satisfies

$$\|\mu_f\|_{\text{Carleson}} \lesssim \|f\|^2_{\text{BMOA}}. \tag{8.1}$$

Actually, $f \in \text{BMOA}$ if and only if μ_f is a Carleson measure [11, p. 231].

Theorem 14. *Let $A \in \mathcal{H}(\mathbb{D})$. If*

$$\limsup_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta} \right| dt \right)^2 (1-|\varphi_a(z)|^2)\, dm(z) \tag{8.2}$$

is sufficiently small, then all solutions of (1.2) *belong to* BMOA.

Proof. By applying (5.1) to $g \equiv 1$, we obtain

$$\left| \int_0^z A(r\zeta)\, d\zeta \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta}\, dt \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(r\zeta)\, d\zeta}{1-e^{-it}\zeta} \right| dt, \tag{8.3}$$

for $0 \le r \le 1$ and $z \in \mathbb{D}$. By (6.2) and (8.2), any second primitive of A belongs to BMOA.

Let f be a solution of (1.2). Then f_r is analytic in $\overline{\mathbb{D}}$ and satisfies $f_r''(\zeta) + r^2 A(r\zeta) f_r(\zeta) = 0$. We deduce (7.8). By (5.1) and Fubini's theorem,

$$\begin{aligned} f_r'(z) &= -\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \int_0^z \frac{r^2 A(r\zeta)}{1-e^{-it}\zeta}\, d\zeta\, dt + f_r'(0) \\ &= -\frac{r^2}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})}\, dt + f_r'(0), \quad z \in \mathbb{D}, \end{aligned}$$

where

$$g_{r,z}(w) = \overline{\int_0^z \frac{A(r\zeta)}{1-\overline{w}\zeta}\, d\zeta}, \quad w \in \mathbb{D}. \tag{8.4}$$

Since $f_r, g_{r,z} \in H^2$, Lemma 9 implies

$$\frac{1}{2\pi} \int_0^{2\pi} f_r(e^{it}) \overline{g_{r,z}(e^{it})}\, dt = 2 \int_{\mathbb{D}} f_r'(w) \overline{g_{r,z}'(w)} \log \frac{1}{|w|}\, dm(w) + f_r(0) \overline{g_{r,z}(0)}.$$

We deduce

$$|f_r'(z)|^2 \leq 8 \left| \int_{\mathbb{D}} f_r'(w)\overline{g_{r,z}'(w)} \log \frac{1}{|w|}\, dm(w) \right|^2 + 2 \left| f_r(0)\overline{g_{r,z}(0)} - f_r'(0) \right|^2, \quad z \in \mathbb{D}.$$

By the Hardy-Stein-Spencer formula

$$\int_{\mathbb{D}} \frac{|g_{r,z}'(w)|^2}{|g_{r,z}(w)|} \log \frac{1}{|w|}\, dm(w) \leq 2\, \|g_{r,z}\|_{H^1},$$

and hence by (8.1) and Carleson's theorem [8, Theorem 9.3], there exist absolute constants $0 < C < \infty$ and $0 < C' < \infty$ such that

$$\begin{aligned} \left| \int_{\mathbb{D}} f_r'(w)\overline{g_{r,z}'(w)} \log \frac{1}{|w|}\, dm(w) \right|^2 &\leq \int_{\mathbb{D}} \frac{|g_{r,z}'(w)|^2}{|g_{r,z}(w)|} \log \frac{1}{|w|}\, dm(w) \\ &\quad \cdot \int_{\mathbb{D}} |g_{r,z}(w)||f_r'(w)|^2 \log \frac{1}{|w|}\, dm(w) \\ &\leq 2\, \|g_{r,z}\|_{H^1} C'\, \|\mu_{f_r}\|_{\text{Carleson}}\, \|g_{r,z}\|_{H^1} \\ &\leq 2C\, \|g_{r,z}\|_{H^1}^2 \|f_r\|_{\text{BMOA}}^2. \end{aligned}$$

We have $|f_r'(z)|^2 \leq 16C\, \|g_{r,z}\|_{H^1}^2 \|f_r\|_{\text{BMOA}}^2 + 4\, |f_r(0)|^2 |g_{r,z}(0)|^2 + 4\, |f_r'(0)|^2$ for $z \in \mathbb{D}$, and by (6.2),

$$\begin{aligned} \|f_r\|_{\text{BMOA}}^2 \leq 64C\, \|f_r\|_{\text{BMOA}}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \|g_{r,z}\|_{H^1}^2 (1 - |\varphi_a(z)|^2)\, dm(z) \\ + 16\, |f_r(0)|^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g_{r,z}(0)|^2 (1 - |\varphi_a(z)|^2)\, dm(z) + 16\, |f_r'(0)|^2. \end{aligned}$$

By re-organizing terms and letting $r \to 1^-$, the assertion follows. □

Remark 4. The proof of Theorem 14 shows that, in order to conclude $f \in$ BMOA, it suffices to take the supremum in (8.2) over any annulus $R < |z| < 1$ instead of $\mathbb{D}$.

Theorem 15. *Let $A \in \mathcal{H}(\mathbb{D})$. If* (8.2) *is sufficiently small and*

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^z \frac{A(\zeta)d\zeta}{1 - e^{-it}\zeta} \right| dt \right)^2 (1 - |\varphi_a(z)|^2)\, dm(z) = 0,$$

then every solution of (1.2) *belongs to* VMOA.

Proof. First, by the assumption and (8.3), any second primitive of A belongs to VMOA. Let f be any solution of (1.2). By the assumption and Theorem 14, we have $f \in$ BMOA. As in the proof of Theorem 14, we obtain

$$|f'(z)|^2 \lesssim \|g_{1,z}\|_{H^1}^2 \|f\|_{\text{BMOA}}^2 + |g_{1,z}(0)|^2 |f(0)|^2 + |f'(0)|^2, \quad z \in \mathbb{D},$$

where $g_{1,z}$ is the function in (8.4). Hence, by (6.2),

$$\begin{aligned} \|f_a\|_{H^2}^2 \lesssim \|f\|_{\text{BMOA}}^2 \int_{\mathbb{D}} \|g_{1,z}\|_{H^1}^2 (1 - |\varphi_a(z)|^2)\, dm(z) \\ + |f(0)|^2 \int_{\mathbb{D}} |g_{1,z}(0)|^2\, (1 - |\varphi_a(z)|^2)\, dm(z) \\ + |f'(0)|^2 (1 - |a|^2) \int_{\mathbb{D}} \frac{1 - |z|^2}{|1 - \overline{a}z|^2}\, dm(z). \end{aligned}$$

The assertion follows by letting $|a| \to 1^-$. □

9. Hardy spaces

For $0 < p < \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

Proof of Theorem 4. The case $p = 2$ follows from the Littlewood-Paley identity by standard estimates, and if $k = 1$ then much more is true, see [26].

The following arguments rely on the representation (5.3) and on an application of the non-tangential maximal function (5.4). For $z \in \mathbb{D}$, let $I(z) = \{\zeta \in \mathbb{T} : z \in \Gamma(\zeta)\}$ and note that its Euclidean arc length satisfies $|I(z)| \asymp 1 - |z|^2$ for $z \in \mathbb{D}$.

(i) We proceed to prove the following preliminary estimate. If $0 < p < 2$, $k \in \mathbb{N}$ and $0 < r < 1$, then

$$\|f_r\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} \, dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p}}{\|f_r\|_{H^p}^{2-p}} \tag{9.1}$$

for all $f \in \mathcal{H}(\mathbb{D})$, $f \not\equiv 0$. Write $d\mu_r(z) = |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)} \, dm(z)$ for short. Fubini's theorem and Hölder's inequality (with indices $2/(2-p)$ and $2/p$) yield

$$\begin{aligned}
\|f_r\|_{H^p}^p &\asymp \int_{\mathbb{T}} \left(\int_{\Gamma(\zeta)} d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f_r^{(j)}(0)|^p \\
&\le \int_{\mathbb{T}} f_r^\star(\zeta)^{(2-p)\frac{p}{2}} \left(\int_{\Gamma(\zeta)} |f_r(z)|^{p-2} \, d\mu_r(z) \right)^{\frac{p}{2}} |d\zeta| + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
&\le \left(\int_{\mathbb{T}} f_r^\star(\zeta)^p \, |d\zeta| \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f_r(z)|^{p-2} \, d\mu_r(z) |d\zeta| \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p \\
&\lesssim \|f_r\|_{H^p}^{p(1-\frac{p}{2})} \left(\int_{\mathbb{D}} |f_r(z)|^{p-2} (1 - |z|^2) \, d\mu_r(z) \right)^{\frac{p}{2}} + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p,
\end{aligned}$$

where the last inequality follows from [11, pp. 55–56]. Estimate (9.1) follows by reorganizing the terms.

By a change of variable, we get

$$\begin{aligned}
&\int_{\mathbb{D}} |f_r(z)|^{p-2} |f_r^{(k)}(z)|^2 (1 - |z|^2)^{2(k-1)+1} \, dm(z) \\
&\qquad \le \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 \left(1 - |z|^2\right)^{2k-1} \, dm(z).
\end{aligned} \tag{9.2}$$

By means of (9.1) we conclude that, if (9.2) is finite then $f \in H^p$ and

$$\|f\|_{H^p}^p \lesssim \int_{\mathbb{D}} |f(z)|^{p-2} |f^{(k)}(z)|^2 \left(1 - |z|^2\right)^{2k-1} \, dm(z) + \frac{\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p}}{\|f\|_{H^p}^{2-p}}. \tag{9.3}$$

Cauchy's integral formula, and the estimate $|f(z)| \lesssim \|f\|_{H^p} (1 - |z|^2)^{-1/p}$ for $z \in \mathbb{D}$ [8, p. 36], give $|f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \cdot |f^{(j)}(0)|^p$ for $j = 0, 1, \ldots, k-1$, which implies

$$\left(\sum_{j=0}^{k-1} |f^{(j)}(0)|^p \right)^{2/p} \lesssim \sum_{j=0}^{k-1} |f^{(j)}(0)|^2 \lesssim \|f\|_{H^p}^{2-p} \sum_{j=0}^{k-1} |f^{(j)}(0)|^p. \tag{9.4}$$

Now (9.3) and (9.4) prove (2.9).

(ii) Let $2 < p < \infty$. We may assume that $f \in H^p$, for otherwise there is nothing to prove. Write $q = p-2$ and $d\mu(z) = |f^{(k)}(z)|^2(1-|z|^2)^{2(k-1)+1}\,dm(z)$, for short. Fubini's theorem, Hölder's inequality (with indices p/q and $p/(p-q)$) and [11, pp. 55–56] yield

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q\,d\mu(z) &\asymp \int_{\mathbb{D}}\left(\int_{I(z)}|d\zeta|\right)\frac{|f(z)|^q}{1-|z|^2}\,d\mu(z) = \int_{\mathbb{T}}\int_{\Gamma(\zeta)}\frac{|f(z)|^q}{1-|z|^2}\,d\mu(z)\,|d\zeta| \\ &\le \left(\int_{\mathbb{T}} f^\star(\zeta)^p\,|d\zeta|\right)^{\frac{q}{p}}\left(\int_{\mathbb{T}}\left(\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}\right)^{\frac{p}{p-q}}|d\zeta|\right)^{\frac{p-q}{p}} \\ &\lesssim \|f\|_{H^p}^{p-2}\left(\int_{\mathbb{T}}\left(\int_{\Gamma(\zeta)}|f^{(k)}(z)|^2(1-|z|^2)^{2(k-1)}\,dm(z)\right)^{\frac{p}{2}}|d\zeta|\right)^{\frac{2}{p}} \\ &\lesssim \|f\|_{H^p}^{p-2}\left(\|f\|_{H^p}^p - \sum_{j=0}^{k-1}|f^{(j)}(0)|^p\right)^{\frac{2}{p}} \lesssim \|f\|_{H^p}^p,\end{aligned}$$

and the assertion of (ii) follows.

(iii) If $f \in \mathcal{H}(\mathbb{D})$ is uniformly locally univalent, then $\sup_{z\in\mathbb{D}}|f''(z)/f'(z)|\,(1-|z|^2)$ is bounded by a constant depending on δ [39, Theorem 2]. Here $0 < \delta \le 1$ is a constant such that f is univalent in each pseudo-hyperbolic disc $\Delta(z,\delta)$ for $z \in \mathbb{D}$. Since

$$\left(\frac{f^{(k)}}{f'}\right)' = \frac{f^{(k+1)}}{f'} - \frac{f''}{f'}\cdot\frac{f^{(k)}}{f'}, \quad k \in \mathbb{N},$$

we conclude $\|f^{(k+1)}/f'\|_{H^\infty_k} < \infty$ for $k \in \mathbb{N}$ by induction. By means of the Hardy-Stein-Spencer formula, we deduce

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)|^{p-2}|f^{(k)}(z)|^2(1-|z|^2)^{2k-1}\,dm(z) \\ &\quad\lesssim \left\|\frac{f^{(k)}}{f'}\right\|_{H^\infty_{k-1}}^2\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\log\frac{1}{|z|}\,dm(z) \lesssim \|f\|_{H^p}^p,\end{aligned}$$

where the comparison constant depends on δ and p. This completes the proof of Theorem 4. □

9.1. **A class of functions for which Question 1 has an affirmative answer.** If $f \in \mathcal{H}(\mathbb{D})$ is non-vanishing, then $g = f^{(p-2)/2}f' \in \mathcal{H}(\mathbb{D})$ and $g' = \frac{p-2}{2}f^{\frac{p-4}{2}}(f')^2 + f^{\frac{p-2}{2}}f''$. The Hardy-Stein-Spencer formula (2.7) implies

$$\|f\|_{H^p}^p \le |f(0)|^p + C_1\,p^2\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)\,dm(z), \tag{9.5}$$

where $0 < C_1 < \infty$ is an absolute constant. By standard estimates, there exists another absolute constant $0 < C_2 < \infty$ such that

$$\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)\,dm(z) \le C_2\left(|g(0)|^2 + \int_{\mathbb{D}}|g'(z)|^2(1-|z|^2)^3\,dm(z)\right).$$

By (9.5), we deduce

$$\begin{aligned}\|f\|_{H^p}^p \le |f(0)|^p &+ C_1C_2\,p^2\left\|\frac{f'}{f}\right\|_{H^\infty_1}^{2-p}|f'(0)|^p + 2\,C_1C_2\,(p-2)^2\left\|\frac{f'}{f}\right\|_{H^\infty_1}^2\|f\|_{H^p}^p \\ &+ 2C_1C_2\,p^2\int_{\mathbb{D}}|f(z)|^{p-2}|f''(z)|^2(1-|z|^2)^3\,dm(z).\end{aligned}$$

In conclusion, if $f \in \mathcal{H}(\mathbb{D})$ is non-vanishing and $\|f'/f\|_{H^\infty_1} = \|\log f\|_{\mathcal{B}}$ is sufficiently small, then (2.8) holds with $C(p) \asymp p^2$ as $p \to 0^+$.

9.2. **Applications to differential equations.** Theorem 4 induces an alternative proof for a special case of [35, Theorem 1.7]).

Theorem A. *Let* $0 < p \leq 2$ *and* $A \in \mathcal{H}(\mathbb{D})$*. If* (4.1) *is sufficiently small (depending on p), then all solutions of* (1.2) *belong to* H^p*.*

Proof. Note that

$$\limsup_{r\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |A(rz)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)\, dm(z) \tag{9.6}$$

is at most a constant multiple of (4.1); compare to the proof of Theorem 3. Let f be a solution of (1.2). By Theorem 4(i), we deduce

$$\begin{aligned} \|f_r\|_{H^p}^p &\lesssim \int_{\mathbb{D}} |f_r(z)|^{p-2} r^2 |f''(rz)|^2(1-|z|^2)^3\, dm(z) + |f(0)|^p + |f'(0)|^p \\ &\lesssim \int_{\mathbb{D}} |f_r(z)|^p\, |A(rz)|^2(1-|z|^2)^3\, dm(z) + |f(0)|^p + |f'(0)|^p. \end{aligned}$$

If (9.6) is sufficiently small, then Carleson's theorem [8, Theorem 9.3] implies that $\|f_r\|_{H^p}$ is uniformly bounded for all sufficiently large $0 < r < 1$. By letting $r \to 1^-$, we obtain $f \in H^p$. □

An argument similar to the one above, taking advantage of Theorem 4(i), leads to a characterization of H^p solutions of (1.2): if $0 < p \leq 2$, f is a solution of (1.2) and $d\mu_A(z) = |A(z)|^2(1-|z|^2)^3\, dm(z)$ is a Carleson measure, then $f \in H^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p\, d\mu_A(z) < \infty. \tag{9.7}$$

For example, if f is a normal (in the sense of Lehto and Virtanen) solution of (1.2) and μ_A is a Carleson measure, then (9.7) holds for all sufficiently small $0 < p < \infty$ by [14, Corollary 9].

Remark 5. If Question 1 had an affirmative answer, then Theorem A would admit the following immediate improvement: if $A \in \mathcal{H}(\mathbb{D})$ such that (4.1) is finite, then all solutions of (1.2) belong to $\bigcup_{0<p<\infty} H^p$.

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