On Becker's univalence criterion ☆

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A R T I C L E I N F O

A B S T R A C T

We study locally univalent functions f analytic in the unit disc $\mathbb{D}$ of the complex plane such that $|f''(z)/f'(z)|(1-|z|^2) \leq 1 + C(1-|z|)$ holds for all $z \in \mathbb{D}$, for some $C \in (0,\infty)$. If $C \leq 1$, then f is univalent by Becker's univalence criterion. We discover that for $C \in (1,\infty)$ the function f remains to be univalent in certain horodiscs. Sufficient conditions which imply that f is bounded, belongs to the Bloch space or belongs to the class of normal functions, are discussed. Moreover, we consider generalizations for locally univalent harmonic functions.

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1. Introduction

Let f be meromorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Then outside of the poles of f, we define its spherical derivative as

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

and its Schwarzian derivative as

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

If $z_0 \in \mathbb{D}$ is a pole of f, we define $f^{\#}(z_0) = \lim_{w \to z_0} f^{\#}(w)$ and $S(f)(z_0) = \lim_{w \to z_0} S(f)(w)$ along $w \in \mathbb{D}$ where $f(w) \neq \infty$. Both the Schwarzian derivative $S(f)$ and the pre-Schwarzian derivative $P(f) = f''/f'$ can be derived from the Jacobian $J_f = |f'|^2$ of f, namely

$$P(f) = \frac{\partial}{\partial z}(\log J_f), \quad S(f) = P(f)' - \frac{1}{2}P(f)^2. \tag{1}$$

A function f meromorphic in $\mathbb{D}$ is locally univalent, denoted by $f \in U^M_{\mathrm{loc}}$, if and only if its spherical derivative $f^\#$ is non-vanishing. Equivalently, the Schwarzian derivative $S(f)$ is an analytic function [23, p. 176]. According to the famous Nehari univalence criterion [21, Theorem 1], if $f \in U^M_{\mathrm{loc}}$ satisfies

$$|S(f)(z)|\,(1-|z|^2)^2 \le N, \quad z \in \mathbb{D}, \tag{2}$$

for $N = 2$, then f is univalent. The result is sharp by an example by Hille [15, Theorem 1].

Let $f \in U^M_{\mathrm{loc}}$ such that $f(a) = f(b)$ for some $a \ne b$. In this case, a paper by Binyamin Schwarz [24, proof of Theorem 1] shows that

$$\max_{\zeta \in \langle a,b \rangle} |S(f)(\zeta)|\,(1-|\zeta|^2)^2 > 2. \tag{3}$$

Here $\langle a, b \rangle = \{\varphi_a(\varphi_a(b)t) : 0 \le t \le 1\}$ is the hyperbolic segment between a and b and

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z} \tag{4}$$

is an automorphism of the unit disc. To see the veracity of (3), note first that the differential equation

$$y'' + py = 0, \tag{5}$$

where $p = \frac{1}{2}S(f)$ is analytic in $\mathbb{D}$, has two linearly independent solutions g, h such that $f = g/h$. Therefore, the assumption $f(a) = f(b) = \alpha$ for some $\alpha \in \mathbb{C}$ implies that $y(a) = y(b) = 0$ for the function $y = g - \alpha h$, which also is a solution of (5). Hence, if we assume

$$\max_{\zeta \in \langle a,b \rangle} |S(f)(\zeta)|\,(1-|\zeta|^2)^2 = 2 \max_{\zeta \in \langle a,b \rangle} |p(\zeta)|\,(1-|\zeta|^2)^2 \le 2,$$

then the reasoning in [24, pp. 161–162] implies a contradiction.

Condition (3) implies that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le N, \quad r_0 \le |z| < 1, \tag{6}$$

for $N = 2$ and some $r_0 \in (0,1)$, then f has finite valence [24, Corollary 1]. If (6) holds for $N < 2$, then f has a spherically continuous extension to $\overline{\mathbb{D}}$, see [9, Theorem 4].

Chuaqui and Stowe [6, p. 564] asked whether

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2 + C(1-|z|), \quad z \in \mathbb{D}, \tag{7}$$

where $C \in (0, \infty)$ is a constant, implies that f is of finite valence. The question remains open despite of some progress achieved in [12]. Steinmetz [25, p. 328] showed that if (7) holds, then f is normal, that is, the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$ is normal in the sense of Montel. Equivalently, $\|f\|_\mathcal{N} = \sup_{z \in \mathbb{D}} f^\#(z)(1-|z|^2) < \infty$ and we write $f \in \mathcal{N}$. A closely related concept is the Bloch space: a function f analytic in $\mathbb{D}$ belongs to the Bloch space $\mathcal{B}$ if $\|f\|_\mathcal{B} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|^2) < \infty$. For fundamental facts about Bloch and normal functions, see [1] and [23, Chapter 9].

A function f analytic in $\mathbb{D}$ is locally univalent, denoted by $f \in U^A_{\mathrm{loc}}$, if and only if $J_f = |f'|^2$ is non-vanishing. By the Cauchy integral formula, if g is analytic in $\mathbb{D}$, then

$$|g'(z)|(1-|z|^2)^2 \le 4 \max_{|\zeta| = \frac{1+|z|^2}{2}} |g(\zeta)|(1-|\zeta|^2), \quad z \in \mathbb{D}.$$

Consequently, the inequality

$$\|S(f)\|_{H_2^\infty} \leq 4\|P(f)\|_{H_1^\infty} + \frac{1}{2}\|P(f)\|_{H_1^\infty}^2$$

holds. Here, we denote $\|g\|_{H_p^\infty} = \sup_{z \in \mathbb{D}} |g(z)|(1-|z|^2)^p$ for $0 < p < \infty$. Thus, each one of the conditions (2), (6) and (7) holds if $|f''(z)/f'(z)|(1-|z|^2)$ is sufficiently small for $z \in \mathbb{D}$. Note also that conversely

$$\|P(f)\|_{H_1^\infty} \leq 2 + 2\sqrt{1 + \frac{1}{2}\|S(f)\|_{H_2^\infty}},$$

see [22, p. 133].

The famous Becker univalence criterion [3, Korollar 4.1] states that if $f \in U_{\mathrm{loc}}^A$ satisfies

$$|zP(f)(z)|\,(1-|z|^2) \leq \rho, \quad z \in \mathbb{D}, \tag{8}$$

for $\rho \leq 1$, then f is univalent in $\mathbb{D}$, and if $\rho < 1$, then f has a quasi-conformal extension to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For $\rho > 1$, condition (8) does not guarantee the univalence of f [4, Satz 6] which can in fact break brutally [10]. If (8) holds for $\rho \in (0, 2)$, then f is bounded, and in the case $\rho = 2$, f is a Bloch function.

Becker and Pommerenke proved recently that if

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < \rho, \quad r_0 \leq |z| < 1, \tag{9}$$

for $\rho < 1$ and some $r_0 \in (0, 1)$, then f has finite valence [5, Theorem 3.4]. However, the case of equality $\rho = 1$ in (9) is open and the sharp inequality corresponding to (3), in terms of the pre-Schwarzian, has not been found yet.

In this paper, we consider the growth condition

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \leq 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{10}$$

where $C \in (0, \infty)$ is an absolute constant, for $f \in U_{\mathrm{loc}}^A$. When (10) holds, we detect that f is univalent in horodiscs $D(ae^{i\theta}, 1-a)$, $e^{i\theta} \in \partial\mathbb{D}$, for some $a = a(C) \in [0, 1)$. Here $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ is the Euclidean disc with center $a \in \mathbb{C}$ and radius $r \in (0, \infty)$.

The remainder of this paper is organized as follows. In Section 2, we see that under condition (10) the function $f \in U_{\mathrm{loc}}^A$ is bounded. Weaker sufficient conditions which imply that the function f is either bounded, a Bloch function or a normal function are investigated. The main results concerning univalence are stated in Section 3 and proved in Section 4. Finally in Section 5 we state generalizations of our results to harmonic functions. Moreover, for sake of completeness, we discuss the harmonic counterparts of the results proven in [12].

2. Distortion theorems

Recall that each meromorphic and univalent function f in $\mathbb{D}$ satisfies (2) for $N = 6$. This is the converse of Nehari's theorem, discovered by Kraus [19]. In the same fashion, each analytic and univalent function f in $\mathbb{D}$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{11}$$

and hence (8) holds for $\rho = 6$, which is the converse of Becker's theorem [23, p. 21].

The class $\mathcal{S}$ consists of functions f univalent and analytic in $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Among all functions in $\mathcal{S}$, the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{(1-z)^2} - \frac{1}{1-z},$$

has the extremal growth. Namely, by inequality (11), each $f \in \mathcal{S}$ satisfies

$$|f^{(j)}(z)| \le k^{(j)}(|z|), \quad \left|\frac{f^{(j+1)}(z)}{f^{(j)}(z)}\right| \le \frac{k^{(j+1)}(|z|)}{k^{(j)}(|z|)}, \quad j = 0, 1, \tag{12}$$

for $z \in \mathbb{D} \setminus \{0\}$, see [23, p. 21]. Moreover, k satisfies condition (2), for $N = 6$, with equality for each $z \in \mathbb{D}$.

Bloch and normal functions emerge in a natural way as Lipschitz mappings. Denote the Euclidean metric by d_E, and define the hyperbolic metric in $\mathbb{D}$ as

$$d_H(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{D},$$

where $\varphi_z(w)$ is defined as in (4), and the chordal metric in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by setting

$$\chi(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \quad \chi(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}, \quad z \in \overline{\mathbb{C}}, w \in \mathbb{C}.$$

Then each $f \in \mathcal{B}$ is a Lipschitz function from $(\mathbb{D}, d_H)$ to $(\mathbb{C}, d_E)$ with a Lipschitz constant equal to $\|f\|_{\mathcal{B}}$, and each $f \in \mathcal{N}$ is a Lipschitz map from $(\mathbb{D}, d_H)$ to $(\overline{\mathbb{C}}, \chi)$ with constant $\|f\|_{\mathcal{N}}$, see [7, Theorems 5 and 10]. Here, we prove the first claim, the second claim can be verified in similar fashion. Let f be analytic in $\mathbb{D}$ such that

$$|f(z) - f(w)| \le M d_H(z, w), \quad z, w \in \mathbb{D}.$$

By letting $w \to z$, we obtain $|f'(z)|(1 - |z|^2) \le M$, for all $z \in \mathbb{D}$, and conclude that $\|f\|_{\mathcal{B}} \le M$. Conversely, if $f \in \mathcal{B}$, then

$$|f(z) - f(w)| \le \int_{\langle z, w \rangle} |f'(\zeta)||d\zeta| \le \sup_{\zeta \in \langle z, w \rangle} |f'(\zeta)|(1 - |\zeta|^2) d_H(z, w),$$

and we conclude that f is a Lipschitz map with a constant $M \le \|f\|_{\mathcal{B}}$.

In the same fashion as above, we deduce that

$$\left|\frac{f''(z)}{f'(z)}\right| \le \frac{B}{1 - |z|^2} + \frac{C(1 - |z|)}{1 - |z|^2}, \quad z \in \mathbb{D},$$

for some $B, C \in (0, \infty)$, is equivalent to

$$\left|\log \frac{f'(z)}{f'(w)}\right| \le B d_H(z, w) + C \left(1 - \frac{|z + w|}{2} + \frac{|z - w|}{2}\right) d_H(z, w), \quad z, w \in \mathbb{D}.$$

This follows from the fact that the hyperbolic segment $\langle z, w \rangle$ is contained in the disc $D((z + w)/2, |z - w|/2)$, which yields

$$1 - |\zeta| \le 1 - \frac{|z + w|}{2} + \frac{|z - w|}{2}, \quad \zeta \in \langle z, w \rangle.$$

For the convenience of the reader, we list some relationships between the classes $\mathcal{B}$ and $\mathcal{N}$ which can be found in [7]. By the Schwarz–Pick lemma, each bounded analytic function belongs to $\mathcal{B}$. If $f \in \mathcal{B}$, then

both $f \in \mathcal{N}$ and $e^f \in \mathcal{N}$. This is clear, since $\chi(z, w) \le d_E(z, w)$ for all $z, w \in \mathbb{C}$ and since the exponential function is Lipschitz from $(\mathbb{C}, d_E)$ to $(\overline{\mathbb{C}}, \chi)$. Moreover, since each rational function R is Lipschitz from $(\overline{\mathbb{C}}, \chi)$ to itself, $R \circ f \in \mathcal{N}$ whenever $f \in \mathcal{N}$. However, it is not clear under which assumptions $f^2 \in \mathcal{N}$ implies $f \in \mathcal{N}$.

If $f \in U_{\text{loc}}^M$ is univalent, then both $f, f' \in \mathcal{N}$ by the estimate

$$(f^{(j)})^{\#}(z) = \frac{|f^{(j+1)}(z)|}{1 + |f^{(j)}(z)|^2} \le \frac{1}{2} \left| \frac{f^{(j+1)}(z)}{f^{(j)}(z)} \right|$$

and (12). However, it is not clear if $f'' \in \mathcal{N}$. At least, each primitive g of a univalent function satisfies $g'' \in \mathcal{N}$. Recently, similar normality considerations which have connections to differential equations, were done in [11].

We denote by $\Delta_p(\alpha, \rho) = \{z \in \mathbb{D} : |\varphi_\alpha(z)| \le \rho\}$ a pseudo-hyperbolic disc with center $\alpha \in \mathbb{D}$ and radius $\rho \in (0, 1)$. This set is a Euclidean disc, that is,

$$\Delta_p(\alpha, \rho) = D(\xi, R) = \{z \in \mathbb{C} : |z - \xi| < R\} \tag{13}$$

where the center ξ and radius R are given by the well-known formulas

$$\xi = \frac{1 - \rho^2}{1 - \rho^2 |\alpha|^2} \alpha \quad \text{and} \quad R = \frac{1 - |\alpha|^2}{1 - \rho^2 |\alpha|^2} \rho,$$

respectively. Any number in the set $\{|\alpha|, \rho, |\xi|, R\}$ can be expressed in terms of two other numbers in the set, see [16, pp. 17–19], for example. In particular,

$$\rho = \rho(\alpha, \xi) = \rho(|\alpha|, |\xi|) = \sqrt{\frac{|\alpha| - |\xi|}{|\alpha|(1 - |\alpha||\xi|)}}, \tag{14}$$

whenever $\alpha \in \mathbb{D} \setminus \{0\}$, a fact needed in Section 4.

If $f \in U_{\text{loc}}^A$ and there exists $\delta \in (0, 1)$ such that f is univalent in each pseudo-hyperbolic disc $\Delta_p(a, \delta)$, for $a \in \mathbb{D}$, then f is called uniformly locally univalent. Binyamin Schwarz showed [24, Theorems 3–4] that this is the case if and only if $\sup_{z \in \mathbb{D}} |S(f)(z)|(1 - |z|^2)^2 < \infty$, or equivalently if $\log f' \in \mathcal{B}$. Consequently, the derivative of each uniformly locally univalent function is normal.

By using arguments similar to those in the proof of [5, Theorem 3.2] and in [18], we obtain the following result.

Theorem 1. *Let f be meromorphic in $\mathbb{D}$ such that*

$$\left| \frac{f''(z)}{f'(z)} \right| \le \varphi(|z|), \quad 0 \le R \le |z| < 1, \tag{15}$$

for some $\varphi : [R, 1) \to [0, \infty)$.

(i) *If*

$$\limsup_{r \to 1^-} (1 - r) \exp \left(\int_R^r \varphi(t) \, dt \right) < \infty, \tag{16}$$

then $\sup_{R < |z| < 1} |f'(z)|(1 - |z|^2) < \infty$.

(ii) *If*

$$\int\limits_{R}^{1} \exp\left(\int\limits_{R}^{s} \varphi(t)\, dt \right)\, ds < \infty, \tag{17}$$

then $\sup\limits_{R<|z|<1} |f(z)| < \infty.$

Proof. Let $\zeta \in \partial\mathbb{D}$. Let $R \le \rho < r < 1$ and note that f' is non-vanishing on the circle $|z| = \rho$. Then

$$\left| \log \frac{f'(r\zeta)}{f'(\rho\zeta)} \right| \le \int\limits_{\rho}^{r} \left| \frac{f''(t\zeta)}{f'(t\zeta)} \right|\, dt \le \int\limits_{\rho}^{r} \varphi(t)\, dt.$$

Therefore

$$|f'(r\zeta)| \le |f'(\rho\zeta)| \exp\left(\int\limits_{\rho}^{r} \varphi(t)\, dt \right),$$

which implies the first claim. By another integration,

$$|f(r\zeta) - f(\rho\zeta)| \le \int\limits_{\rho}^{r} |f'(s\zeta)|\, ds \le |f'(\rho\zeta)| \int\limits_{\rho}^{r} \exp\left(\int\limits_{\rho}^{s} \varphi(t)\, dt \right)\, ds.$$

Hence,

$$|f(z)| \le M(\rho, f) + M(\rho, f') \int\limits_{\rho}^{1} \exp\left(\int\limits_{\rho}^{s} \varphi(t)\, dt \right)\, ds < \infty$$

for $\rho < |z| < 1$. $\quad\square$

The assumptions in Theorem 4(i) and (ii) are satisfied, respectively, by the functions

$$\varphi(t) = \frac{2}{1 - t^2} = \left(\log \frac{1 + t}{1 - t} \right)'$$

and

$$\psi(t) = \frac{B}{1 - t^2} + \frac{C}{1 - t^2} \left(\log \frac{e}{1 - t} \right)^{-(1+\varepsilon)},$$

where $\varepsilon \in (0, \infty)$, $B \in (0, 2)$ and $C \in (0, \infty)$.

By Theorem 1, if f is meromorphic in $\mathbb{D}$ and satisfies (15) and (16) for some φ, then $f \in \mathcal{N}$. Moreover, if f is also analytic in $\mathbb{D}$, then $f \in \mathcal{B}$, and if (17) holds, then f is bounded.

3. Main results

Next we turn to present our main results. We consider Becker's condition in a neighborhood of a boundary point $\zeta \in \partial \mathbb{D}$ as well as univalence in certain horodiscs. Furthermore, we state some distortion type estimates similar to the converse of Becker's theorem. Some examples which concern the main results and the distribution of preimages of a locally univalent function are discussed.

Theorem 2. *Let $f \in U_{\mathrm{loc}}^A$ and $\zeta \in \partial \mathbb{D}$.*

If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that

$$\left| \frac{f''(w_n)}{f'(w_n)} \right| (1 - |w_n|^2) \to c \tag{18}$$

for some $c \in (6, \infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$.

Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (18) holds for some $c \in [1, \infty]$.

Example 3. It is clear that (18), $c \in (6, \infty)$, does not imply that f is of infinite valence. For example, the polynomial $f(z) = (1 - z)^{2n+1}$, $n \in \mathbb{N}$, satisfies the sharp inequality

$$\left| \frac{f''(z)}{f'(z)} \right| (1 - |z|^2) \le 4n, \quad z \in \mathbb{D},$$

although $f(z) = \varepsilon^{2n+1}$ has n solutions in $D(1, \delta) \cap \mathbb{D}$ for each $\varepsilon \in (0, \delta)$ when $\delta \in (0, 1)$ is small enough (depending on n).

The function $f(z) = (1 - z)^{-p}$, $p \in (0, \infty)$, satisfies the sharp inequality

$$\left| \frac{f''(z)}{f'(z)} \right| (1 - |z|^2) \le 2(p + 1), \quad z \in \mathbb{D},$$

and for each $p \in (2n, 2n + 2]$, $n \in \mathbb{N} \cup \{0\}$, the valence of f is $n + 1$ for suitably chosen points in the image set.

Under the condition (10), the function f is bounded, see Theorem 1 in Section 2. Condition (10) implies that f is univalent in horodiscs.

Theorem 4. *Let $f \in U_{\mathrm{loc}}^A$ and assume that (10) holds for some $C \in (0, \infty)$. If $C \in (0, 1]$, then f is univalent in $\mathbb{D}$. If $C \in (1, \infty)$, then there exists $a \in (0, 1)$, $a = a(C)$, such that f is univalent in all discs $D(ae^{i\theta}, 1 - a)$, $\theta \in [0, 2\pi)$. In particular, we can choose $a = 1 - (1 + C)^{-2}$.*

Let $f \in U_{\mathrm{loc}}^A$ be univalent in each horodisc $D(ae^{i\theta}, 1 - a)$, $\theta \in [0, 2\pi)$, for some $a \in (0, 1)$. By the proof of [12, Theorem 6], for each $w \in f(\mathbb{D})$, the sequence of pre-images $\{z_n\} \in f^{-1}(w)$ satisfies

$$\sum_{z_n \in Q} (1 - |z_n|)^{1/2} \le K\ell(Q)^{1/2} \tag{19}$$

for any Carleson square Q and some constant $K \in (0, \infty)$ depending on a. Here

$$Q = Q(I) = \left\{ re^{i\theta} \: : \: e^{i\theta} \in I, \; 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}$$

is called a Carleson square based on the arc $I \subset \partial\mathbb{D}$ and $|I| = \ell(Q)$ is the Euclidean arc length of I.

By choosing $Q = \mathbb{D}$ in (19), we obtain

$$n(f, r, w) = O\left(\frac{1}{\sqrt{1 - r}} \right), \quad r \to 1^-,$$

where $n(f, r, w)$ is the number of pre-images $\{z_n\} = f^{-1}(w)$ in the disc $\overline{D(0, r)}$. Namely, arrange $\{z_n\} = f^{-1}(w)$ by increasing modulus, and let $0 < |z_n| = r < |z_{n+1}|$ to deduce

$$(1 - r)^{1/2} n(f, r, w) \leq \sum_{k=0}^{n} (1 - |z_k|)^{1/2} \leq K\ell(\mathbb{D})^{1/2} < \infty$$

for some $K(a) \in (0, \infty)$.

Theorem 5. *Let* $f \in U_{\mathrm{loc}}^A$ *be univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C} e^{i\theta}, \frac{1}{1+C} \right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $C \in (0, \infty)$. *Then*

$$\left| \frac{f''(z)}{f'(z)} \right| (1 - |z|^2) \leq 2 + 4(1 + K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1 - |z|^2)$ *as* $|z| \to 1^-$.

In view of (11), Theorem 5 is sharp. Moreover, since (11) implies

$$\left| \frac{f''(z)}{f'(z)} \right| (1 - |z|) \leq \frac{4 + 2|z|}{1 + |z|} \leq 4$$

for univalent analytic functions f, the next theorem is sharp as well.

Theorem 6. *Let* $f \in U_{\mathrm{loc}}^A$ *be univalent in all Euclidean discs*

$$D(ae^{i\theta}, 1 - a) \subset \mathbb{D}, \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $a \in (0, 1)$. *Then*

$$\left| \frac{f''(z)}{f'(z)} \right| (1 - |z|) \leq 4, \quad a \leq |z| < 1. \tag{20}$$

Example 7. Let $f = f_{C,\zeta}$ be a locally univalent analytic function in $\mathbb{D}$ such that $f(-1) = 0$ and

$$f'(z) = -i \left(\frac{1 + z}{1 - z} \right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \partial\mathbb{D}, z \in \mathbb{D}.$$
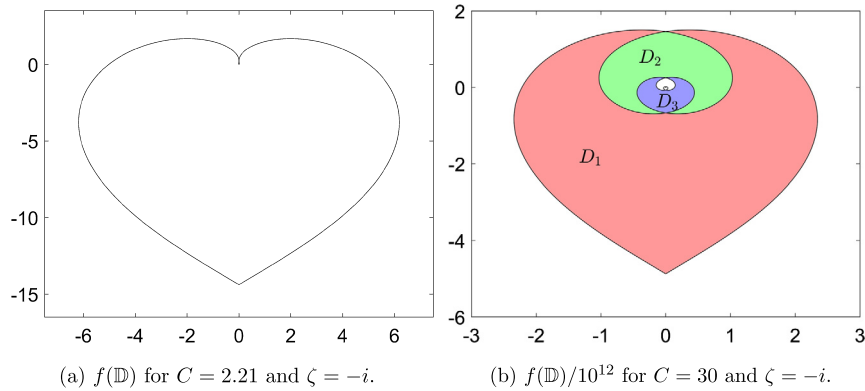
Then

Fig. 1. Shape of the image domain $f(\mathbb{D})$ for different values of C. In (a), $\partial f(\mathbb{D})$ is a simple closed curve. In (b), the valence of the simply connected domain D_j under f is j, for $j = 1, 2, 3$, respectively.

$$\frac{f''(z)}{f'(z)} = \frac{1}{1-z^2} + \frac{C\zeta}{2},$$

hence (10) holds and f is univalent in $\mathbb{D}$ if $C \leq 1$ by Becker's univalence criterion. If f is univalent, then $f - f(0) \in \mathcal{S}$ and we obtain for $\zeta = 1$,

$$1 \geq \frac{f'(x)}{k'(x)} = \frac{e^{\frac{Cx}{2}}(1-x)^{5/2}}{(1+x)^{1/2}} \sim \frac{1 + Cx/2}{1 + 3x}, \quad x \to 0^+.$$

Therefore, if $C > 6$, then f is not univalent.

The boundary curve $\partial f(\mathbb{D})$ has a cusp at $f(-1) = 0$. When $\zeta = -i$, the cusp has its worst behavior, and by numerical calculations the function f is not univalent if $C > 2.21$. Moreover, as C increases, the valence of f increases, see Fig. 1.

The curve $\{f(e^{it}) : t \in (0, \pi]\}$ is a spiral unwinding from $f(-1)$. We may calculate the valence of f by counting how many times $h(t) = \operatorname{Re}(f(e^{it}))$ changes its sign on $(0, \pi]$. Numerical calculations suggest that the valence of f is approximately equal to $\frac{100}{63}C$.

4. Proofs of main results

In this section, we prove the results stated in Section 3.

Proof of Theorem 2. To prove the first assertion, assume on the contrary that there exists $\delta > 0$ such that f is univalent in $D(\zeta, \delta) \cap \mathbb{D}$. Without loss of generality, we may assume that $\zeta = 1$. Let T be a conformal map of $\mathbb{D}$ onto a domain $\Omega \subset D(\zeta, \delta) \cap \mathbb{D}$ with the following properties:

(i) $T(\zeta) = \zeta$;
(ii) $\partial\Omega \supset \{e^{i\theta} : |\arg \zeta - \theta| < t\}$ for some $t > 0$;
(iii) $\left|\frac{T''(z)}{T'(z)}\right|(1 - |z|^2)^{\frac{1}{2}} \leq 1 - \rho$ for all $z \in \mathbb{D}$, where $\rho \in (0, 1)$ is any prescribed number.

The existence of such a map follows, for instance, by [8, Lemma 8]. Then

$$\left|\frac{f''(T(z))}{f'(T(z))}T'(z) + \frac{T''(z)}{T'(z)}\right|(1 - |z|^2) \leq 6, \quad z \in \mathbb{D},$$

by (11), since $f \circ T$ is univalent in $\mathbb{D}$. Moreover, $\frac{T''(z)}{T'(z)}(1 - |z|^2) \to 0$, as $|z| \to 1^-$, by (iii). Let $\{w_n\}$ be a sequence such that $w_n \to \zeta$, and define z_n by $T(z_n) = w_n$. Then $z_n \to \zeta$, and since T' belongs to the disc algebra by [8, Lemma 8], we have

$$1 < \frac{1 - |T(z_n)|^2}{|T'(z_n)|(1 - |z_n|^2)} \to 1^+, \quad n \to \infty.$$

For more details, see [12, p. 879]. It follows that

$$\limsup_{n\to\infty} \left|\frac{f''(w_n)}{f'(w_n)}\right| (1 - |w_n|^2)$$

$$= \limsup_{n\to\infty} \left|\frac{f''(T(z_n))}{f'(T(z_n))}\right| (1 - |T(z_n)|^2)$$

$$= \limsup_{n\to\infty} \left|\frac{f''(T(z_n))}{f'(T(z_n))}\right| |T'(z_n)|(1 - |z_n|^2)\frac{(1 - |T(z_n)|^2)}{|T'(z_n)|(1 - |z_n|^2)} \le 6,$$

which is the desired contradiction.

To prove the second assertion, assume on the contrary that (18) fails, so that there exist $\rho \in (0,1)$ and a constant $\delta \in (0,1)$ such that

$$\left|\frac{f''(z)}{f'(z)}\right| (1 - |z|^2) \le \rho, \quad z \in D(\zeta, \delta) \cap \mathbb{D}. \tag{21}$$

If $g = f \circ T$, then (21) and (i)–(iii) yield

$$\left|\frac{g''(z)}{g'(z)}\right| (1 - |z|^2) \le \left|\frac{f''(T(z))}{f'(T(z))}\right| |T'(z)|(1 - |z|^2) + \left|\frac{T''(z)}{T'(z)}\right| (1 - |z|^2)$$

$$\le \left|\frac{f''(T(z))}{f'(T(z))}\right| (1 - |T(z)|^2) + 1 - \rho \le 1$$

for all $z \in \mathbb{D}$. Hence g is univalent in $\mathbb{D}$ by Becker's univalence criterion, and so is f on Ω. This is clearly a contradiction. $\square$

Proof of Theorem 4. Assume that condition (10) holds for some $C \in (0,1]$. Now

$$\left|\frac{zf''(z)}{f'(z)}\right| (1 - |z|^2) \le |z|(1 + C(1 - |z|)) \le |z| + 1 - |z| = 1,$$

and hence f is univalent in $\mathbb{D}$ by Becker's univalence criterion.

Assume that (10) holds for some $C \in (1, \infty)$. It is enough to consider the case $\theta = 0$. Let $T(z) = a + (1-a)z$ for $z \in \mathbb{D}$, and $g = f \circ T$. Then

$$(1 - |z|^2)\left|\frac{g''(z)}{g'(z)}\right| = (1 - |z|^2)\left|\frac{f''(T(z))}{f'(T(z))}\right| |T'(z)|$$

$$= \left|\frac{f''(T(z))}{f'(T(z))}\right| (1 - |T(z)|^2)\frac{(1 - |z|^2)|T'(z)|}{1 - |T(z)|^2}$$

$$\le (1 + C(1 - |T(z)|))\frac{(1 - |z|^2)(1 - a)}{1 - |T(z)|^2}$$

$$\le (1 + C(1 - |a + (1-a)z|))\frac{(1 - |z|^2)(1 - a)}{1 - |a + (1-a)z|^2}.$$

By the next lemma, for $a = 1 - (1+C)^{-2}$, g is univalent in $\mathbb{D}$ and f is univalent in $D(a, 1-a)$. The assertion follows.   □

Lemma 8. *Let $C \in (1, \infty)$. Then, for $z \in \mathbb{D}$,*

$$\left(1 + C\left(1 - \left|\frac{C^2 + 2C}{C^2 + 2C + 1} + \frac{1}{(1+C)^2}z\right|\right)\right) \times \frac{(1 - |z|^2)\frac{1}{(1+C)^2}}{1 - \left|\frac{C^2+2C}{C^2+2C+1} + \frac{1}{(1+C)^2}z\right|^2} \leq 1.$$

Proof. Let $h : [0, 1) \to \mathbb{R}$, be defined by $h(t) = (1 + C(1 - t))/(1 - t^2)$. Then

$$h'(t) = \frac{-Ct^2 + 2(1+C)t - C}{(1 - t^2)^2} = 0$$

if and only if $t = t_C = \frac{1 + C - \sqrt{1 + 2C}}{C} \in (0, 1)$. Hence, h is strictly decreasing on $[0, t_C]$ and strictly increasing on $[t_C, 1]$. If

$$t = \left|\frac{C^2 + 2C}{C^2 + 2C + 1} + \frac{1}{(1+C)^2}z\right| \leq t_C,$$

then

$$h(t)(1 - |z|^2)\frac{1}{(1+C)^2} \leq h(0)(1 - |z|^2)\frac{1}{(1+C)^2} \leq \frac{1}{1+C} \leq 1.$$

On the other hand, if

$$t_C < t = \left|\frac{C^2 + 2C}{C^2 + 2C + 1} + \frac{re^{i\theta}}{(1+C)^2}\right| \leq \frac{C^2 + 2C + r}{C^2 + 2C + 1} = t',$$

then we obtain

$$h(t)\frac{(1 - |z|^2)}{(1+C)^2} \leq h(t')\frac{1 - r^2}{(1+C)^2} = \frac{(1+C)^2 + C(1-r)}{2(1+C)^2 - (1-r)}(1+r) \leq 1, \qquad (22)$$

provided that

$$k_C(r) = (1+r)\left[(1+C)^2 + C(1-r)\right] + 1 - r \leq 2(1+C)^2.$$

Since $k_C(1) \leq 2(1+C)^2$ and

$$k_C'(r) = (1+C)^2 + C(1-r) - C(1+r) - 1 > 0$$

for $r < 1 + C/2$, inequality (22) holds. This ends the proof of the lemma.   □

Proof of Theorem 5. Let $a \in \mathbb{D}$, $0 < C/(1+C) < |a| < 1$ and $g(z) = f(\varphi_a(r_a z))$, where $\varphi_a(z)$ is defined as in (4). By equations (13) and (14), we let

$$r_a = \rho\left(a, \frac{a}{|a|}\frac{C}{1+C}\right) = \sqrt{\frac{|a| - \frac{C}{1+C}}{|a|\left(1 - |a|\frac{C}{1+C}\right)}} \qquad (23)$$

and deduce for some $R \in (0, 1)$

$$\Delta_p(a, r_a) = D\left(\frac{a}{|a|} \frac{C}{1+C}, R\right) \subset D\left(\frac{a}{|a|} \frac{C}{1+C}, \frac{1}{1+C}\right).$$

Hence, g is univalent in $\mathbb{D}$. Now

$$\frac{g''(0)}{g'(0)} = \frac{f''(a)}{f'(a)}\varphi_a'(0)r_a + \frac{\varphi_a''(0)}{\varphi_a'(0)}r_a = -\frac{f''(a)}{f'(a)}(1 - |a|^2)r_a + 2\overline{a}r_a.$$

By (11), $|g''(0)/g'(0)| \le 4$ and therefore

$$\left|\frac{f''(a)}{f'(a)}(1 - |a|^2) - 2\overline{a}\right| \le \frac{4}{r_a},$$

which implies

$$\left|\frac{f''(a)}{f'(a)}\right|(1 - |a|^2) \le 2 + \frac{4}{r_a} = 2 + 4(1 + K(a)),$$

where

$$K(a) = \frac{1}{r_a} - 1 = \frac{1 - r_a^2}{r_a(1 + r_a)} \sim \frac{1}{2}(1 - r_a^2) = \frac{1}{2}\frac{\frac{C}{1+C}(1 - |a|^2)}{|a|\left(1 - |a|\frac{C}{1+C}\right)} \sim \frac{C}{2}(1 - |a|^2),$$

as $|a| \to 1^-$. □

Proof of Theorem 6. It suffices to prove (20) for $|z| = a$, since trivially f is univalent also in $D(be^{i\theta}, 1-b) \subset D(ae^{i\theta}, 1 - a)$ for $b \in (a, 1)$ and $e^{i\theta} \in \partial\mathbb{D}$. Moreover, by applying a rotation $z \mapsto \lambda z$, $\lambda \in \partial\mathbb{D}$, it is enough to prove (20) for $z = a$.

Let $T(z) = a + (1 - a)z$ for $z \in \mathbb{D}$. Now $g = f \circ T$ is univalent in $\mathbb{D}$ and by (11)

$$\left|\frac{f''(a)}{f'(a)}\right|(1 - a) = \left|\frac{f''(T(0))}{f'(T(0))}\right||T'(0)| = \left|\frac{g''(0)}{g'(0)}\right| \le 4.$$

The assertion follows. □

5. Generalizations for harmonic functions

Let f be a complex-valued and harmonic function in $\mathbb{D}$. Then f has the unique representation $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. In this case, $f = h + \overline{g}$ is orientation preserving and locally univalent, denoted by $f \in U_{\text{loc}}^H$, if and only if its Jacobian $J_f = |h'|^2 - |g'|^2 > 0$, by a result by Lewy [20]. In this case, $h \in U_{\text{loc}}^A$ and the dilatation $\omega_f = \omega = g'/h'$ is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into itself. Clearly $f = h + \overline{g}$ is analytic if and only if the function g is constant.

For $f = h + \overline{g} \in U_{\text{loc}}^H$, equation (1) yields the harmonic pre-Schwarzian and Schwarzian derivatives:

$$P(f) = P(h) - \frac{\overline{\omega}\,\omega'}{1 - |\omega|^2},$$

and

$$S(f) = S(h) + \frac{\overline{\omega}}{1 - |\omega|^2} \left(\frac{h''}{h'} \omega' - \omega'' \right) - \frac{3}{2} \left(\frac{\overline{\omega}\,\omega'}{1 - |\omega|^2} \right)^2.$$

This generalization of $P(f)$ and $S(f)$ to harmonic functions was introduced and motivated in [13].

There exists $\delta_0 \in (0, 2)$ such that if $f \in U_{\text{loc}}^H$ satisfies (2) for $N = \delta_0$, then f is univalent in $\mathbb{D}$, see [2] and [14]. The sharp value of δ_0 is not known. Moreover, if $f \in U_{\text{loc}}^H$ satisfies

$$|P(f)|\,(1 - |z|^2) + \frac{|\omega'(z)|(1 - |z|^2)}{1 - |\omega(z)|^2} \leq 1, \quad z \in \mathbb{D},$$

then f is univalent. The constant 1 is sharp, by the sharpness of Becker's univalence criterion. If one of these mentioned inequalities, with a slightly smaller right-hand-side constant, holds in an annulus $r_0 < |z| < 1$, then f is of finite valence [17].

Conversely to these univalence criteria, there exist absolute constants $C_1, C_2 \in (0, \infty)$ such that if $f \in U_{\text{loc}}^H$ is univalent, then (2) holds for $N = C_1$ and (8) holds for $\rho = C_2$, see [13]. The sharp values of C_1 and C_2 are not known.

By the above-mentioned analogues of Nehari's criterion, Becker's criterion and their converses, we obtain generalizations of the results in this paper for harmonic functions. Of course, the correct operators and constants have to be involved. Theorem 2 and its analogue [12, Theorem 1] for the Schwarzian derivative $S(f)$ are valid as well. Moreover, Theorems 4, 5, and 6 are valid. We leave the details for the interested reader.

We state the important generalization of [12, Theorem 3] for harmonic functions here. It gives a sufficient condition for the Schwarzian derivative of $f \in U_{\text{loc}}^H$ such that the preimages of each $w \in f(\mathbb{D})$ are separated in the hyperbolic metric. Here $\xi(z_1, z_2)$ is the hyperbolic midpoint of the hyperbolic segment $\langle z_1, z_2 \rangle$ for $z_1, z_2 \in \mathbb{D}$.

Theorem 9. *Let $f = h + \overline{g} \in U_{\text{loc}}^H$ such that*

$$|S_H(f)|(1 - |z|^2) \leq \delta_0(1 + C(1 - |z|)), \quad z \in \mathbb{D},$$

for some $C \in (0, \infty)$. Then each pair of points $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies

$$d_H(z_1, z_2) \geq \log \frac{2 - C^{1/2}(1 - |\xi(z_1, z_2)|)^{1/2}}{C^{1/2}(1 - |\xi(z_1, z_2)|)^{1/2}}. \tag{24}$$

Conversely, if there exists a constant $C \in (0, \infty)$ such that each pair of points $z_1, z_2 \in \mathbb{D}$ for which $f(z_1) = f(z_2)$ and $1 - |\xi(z_1, z_2)| < 1/C$ satisfies (24), then

$$|S_H(f)|(1 - |z|^2) \leq C_2(1 + \Psi_C(|z|)(1 - |z|)^{1/3}), \quad 1 - |z| < (8C)^{-1},$$

where Ψ_C is positive, and satisfies $\Psi_C(|z|) \to (2(8C)^{1/3})^+$ as $|z| \to 1^-$.

We have not found a natural criterion which would imply that $f = h + \overline{g} \in U_{\text{loc}}^H$ is bounded. However, the inequality $|g'(z)| < |h'(z)|$ can be utilized. A domain $D \subset \mathbb{C}$ is starlike if for some point $a \in D$ all linear segments $[a, z]$, $z \in D$, are contained in D. Let $h \in U_{\text{loc}}^A$ be univalent, let $h(\mathbb{D})$ be starlike with respect to $z_0 \in h(\mathbb{D})$ and $f = h + \overline{g} \in U_{\text{loc}}^H$. Then the function

$$z \mapsto \Omega(z) = \frac{g(z) - g(z_0)}{h(z) - h(z_0)}$$

maps $\mathbb{D}$ into $\mathbb{D}$. To see this, let $a \in \mathbb{D}$ and let $R = h^{-1}([h(z_0), h(a)])$ be the pre-image of the segment $[h(z_0), h(a)]$ under h. Then

$$|h(a) - h(z_0)| = \int_R |h'(\zeta)||d\zeta| \geq \left| \int_R g'(\zeta)\, d\zeta \right| = |g(a) - g(z_0)|.$$

Consequently, if $f = h + \overline{g} \in U_{\mathrm{loc}}^H$ is such that $h(\mathbb{D})$ is starlike and bounded, then $f(\mathbb{D})$ is bounded.

Acknowledgments

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