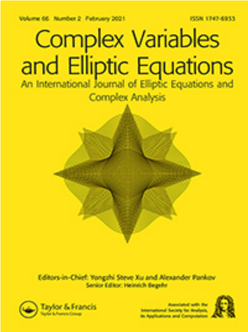

Taylor & Francis
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Complex Variables and Elliptic Equations
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

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COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
2021, VOL. 66, NO. 2, 194–208
https://doi.org/10.1080/17476933.2020.1711744

Linear differential equations with solutions lying in weighted Fock spaces

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ABSTRACT

Sufficient conditions for coefficients of the non-homogeneous linear complex differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = A_k(z)$$

are found such that all solutions belong to some weighted Fock spaces, where $A_j(z)$ are entire functions, $j = 0, 1, \ldots, k$. Furthermore, sufficient conditions for the coefficient $A(z)$ such that all solutions of the special second order equation

$$f'' + A(z)f = 0$$

belong to some weighted Fock spaces are given by the Bergman reproducing kernel, where $A(z)$ is an entire function.

ARTICLE HISTORY
Received 27 September 2018
Accepted 28 December 2019

COMMUNICATED BY
H. Boas

KEYWORDS
Linear complex differential equation; entire function; weighted Fock space

AMS SUBJECT CLASSIFICATIONS
34M10; 30D35

1. Introduction and main results

One of main objectives in the research of the non-homogeneous linear complex differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = A_k(z) \tag{1}$$

with analytic coefficients in the complex domain is to consider the relations between the growth of coefficients and the growth of solutions. Many results on the case of fast growing solutions have been obtained by Nevanlinna theory and Wiman-Valiron theory. On the other hand, some other methods are needed in dealing with slowly growing solutions. There are some useful and powerful techniques, for instance, Herold's comparison theorem [10,11], Gronwall's lemma [14], Picard's successive approximations [4,6] and some methods based on Carleson measures [13,21,22].

In recent years, the research of Equation (1) in function spaces has been widely concerned, where $A_j(z)$ are analytic in the unit disc, $j = 0, 1, \ldots, k$ and see [7,12,16,18]. For the

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case of the complex plane $\mathbb{C}$, in [24], the homogeneous linear complex differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

is considered, where $A_j(z)$ are entire functions, $j = 0, 1, \ldots, k-1$. The relations between coefficients and solutions in Fock type spaces are obtained and for more details see [24, Theorems 2.1 and 3.1].

It is well known that differential equations play an important role in the spectrum analysis of differential operators (see [1,2,9]). Recently, differential operators on weighted Fock spaces have been studied and refer to [19,20].

Motivated by the study of weighted Fock spaces and complex differential equations, sufficient conditions for coefficients of Equation (1) are found such that all solutions belong to some weighted Fock spaces, where $A_j(z)$ are entire functions, $j = 0, 1, \ldots, k$. Furthermore, similar to Section 7 in [7], sufficient conditions for the coefficient $A(z)$ such that all solutions of the second order linear differential equation

$$f'' + A(z)f = 0 \tag{2}$$

belong to some weighted Fock spaces are shown, where $A(z)$ is an entire function. In this paper, we use some methods and ideas from [5,7,11,15,16,23,24] to deal with them.

The classical Fock space is defined as follows. Let $g(r) = e^{-(1/2)r^2}$ for $r \in [0, \infty)$. For $p \in [1, \infty]$, the space L_g^p consists of those functions f, Lebesgue measurable in $\mathbb{C}$, for which

$$\|f\|_p := \left(\int_{\mathbb{C}} \left|f(z)e^{-(1/2)|z|^2}\right|^p \,\mathrm{d}m(z)\right)^{1/p} < \infty, \quad p \in [1, \infty),$$

and

$$\|f\|_p := \sup_{z\in\mathbb{C}}\{|f(z)|e^{-(1/2)|z|^2}\} < \infty, \quad p = \infty.$$

Here $\mathrm{d}m(z)$ denotes the classical Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$ in $\mathbb{C}$. The Fock space F^p consists of all entire functions in L_g^p and refer to [25]. In particular, the space F^2 is a closed subspace of the Hilbert space L_g^2 with the inner product:

$$\langle f, h\rangle := \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{h(z)}e^{-|z|^2}\,\mathrm{d}m(z), \quad f, h \in L_g^2.$$

The Fock-Sobolev space is widely studied and was first introduced in [3]. For $p \in [1, \infty]$ and $m \in \mathbb{N}$, the Fock-Sobolev space $F^{p,m}$ is a subspace of F^p consisting of all entire functions f such that

$$\|f\|_{F^{p,m}} := \sum_{\alpha\le m} \|f^{(\alpha)}\|_p < \infty,$$

where $\|\cdot\|_p$ is the norm in F^p. It follows from [3, Theorem A] that $f \in F^{p,m}$ if and only if $z^m f(z) \in F^p$. Namely, there is a positive constant C such that

$$C^{-1}\|z^m f\|_p \le \sum_{\alpha\le m} \|f^{(\alpha)}\|_p \le C\|z^m f\|_p.$$

In [5], the weighted Fock space is studied intensively.

Let $\phi : [0, \infty) \to (0, \infty)$ be a twice continuously differentiable function with its Laplacian satisfying $\Delta\phi(|z|) > 0$ and we can extend ϕ to $\mathbb{C}$ by $\phi(z) = \phi(|z|)$. And there exist a positive differentiable decreasing function $\tau : \mathbb{C} \to \mathbb{R}^+$ with $\tau(r) = \tau(|z|) = \tau(z)$ and a constant $C \in (0, \infty)$ such that $\tau(z) = C$ for $0 \le |z| < 1$, and for $|z| \ge 1$,

$$C^{-1}(\Delta\phi(|z|))^{-1/2} \le \tau(z) \le C(\Delta\phi(|z|))^{-1/2},$$
$$\lim_{r\to\infty} \tau(r) = 0, \quad \lim_{r\to\infty} \tau'(r) = 0.$$

Moreover, suppose that either there exists a constant $\eta > 0$ such that $\tau(r)r^{\eta}$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The set $\mathcal{I}$, consisting of all ϕ satisfying the above-mentioned conditions, is called the class of rapidly increasing functions. It is obvious that $\phi(r) = r^{\alpha} \in \mathcal{I}$ for $\alpha > 2$, and $\phi(r) = e^{\beta r} \in \mathcal{I}$ for $\beta > 0$.

For $p \in [1, \infty]$ and $\phi \in \mathcal{I}$, the weighted Fock space F^p_ϕ consists of all entire functions f with

$$\|f\|_{F^p_\phi} := \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)}\, \mathrm{d}m(z)\right)^{1/p} < \infty, \quad p \in [1, \infty),$$

and

$$\|f\|_{F^p_\phi} := \sup_{z\in\mathbb{C}} \left\{|f(z)|e^{-\phi(z)}\right\} < \infty, \quad p = \infty.$$

Clearly, it is the classical Fock space when $\phi(z) = |z|^2/2$ and it is the Fock-Sobolev space when $\phi(z) = |z|^2/2 - m \log|z|$.

Let the point evaluation $L_\zeta(f) = f(\zeta)$ for $f \in F^2_\phi$ and $\zeta \in \mathbb{C}$. It follows from [5] that the point evaluation L_ζ is a bounded linear functional in F^2_ϕ. And there exists a reproducing kernel $K_\zeta \in F^2_\phi$ with $\|K_\zeta\|_{F^2_\phi} = \|L_\zeta\|$ such that

$$f(\zeta) = L_\zeta(f(z)) = \int_{\mathbb{C}} f(z)\overline{K_\zeta(z)} e^{-2\phi(z)}\, \mathrm{d}m(z).$$

Moreover, if

$$\Gamma = \left\{e_n(z) = z^n \delta_n^{-1} : \quad n \in \mathbb{N}\right\}$$

is an orthonormal basis of F^2_ϕ, where $\delta_n^2 = 2\pi \int_0^\infty r^{2n+1} e^{-2\phi(r)}\, \mathrm{d}r$, then

$$K_\zeta(z) = \sum_{n=0}^{\infty} \langle K_\zeta, e_n\rangle e_n(\zeta) = \sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(z)}.$$

Similar to the weighted Fock space, we can define the weighted Fock-Sobolev space as follows.

For $p \in [1,\infty)$, $q \in \mathbb{R}$ and $\phi \in \mathcal{I}$, the weighted Fock-Sobolev space $F_{\phi}^{p,q}$ consists of those entire functions f, for which

$$\|f\|_{F_{\phi}^{p,q}} := \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} \phi^q(z)\, \mathrm{d}m(z) \right)^{1/p} < \infty.$$

The weighted Fock-Sobolev space is the weighted Fock space when $q = 0$.

In this paper, the above-mentioned spaces are uniformly called weighted Fock spaces for convenience.

In [16,23], they studied the sufficient conditions for solutions of the linear complex differential equation in Hardy type space and the $F(p,q,s)$ space by the characterization of higher derivatives respectively. In [15], a characterization of the entire function $f \in F^p$ is obtained by the higher derivative $f^{(m)}$, where $p \in [1,\infty)$ and $m \in \mathbb{N}$. Similarly, we give the following Theorems 1.1 and 1.2.

Theorem 1.1: *Let $A_j(z)$ be entire functions, $j = 0, 1, \ldots, k$. Suppose that for every $p \geq 1$ there exist positive constants C and D_l, depending on p and k, such that*

$$C \sum_{l=0}^{k-1} D_l \sup_{z \in \mathbb{C}} \left\{ \frac{|A_l(z)|}{(1+|z|)^{k-l}} \right\} < 1$$

and the kth order primitive function $\varphi_k(z)$ of $A_k(z)$ belongs to F^p. Then all solutions of Equation (1) belong to F^p.

Theorem 1.2: *Let $A_j(z)$ be entire functions, $j = 0, 1, \ldots, k$. Suppose that for every $p \geq 1$ there exist positive constants E_j and C_l, depending on p and k, such that*

$$\sum_{l=0}^{k} C_l \left(\sum_{j=0}^{k-1} \sup_{z \in \mathbb{C}} \left\{ \frac{|A_j|}{(1+|z|)^{k-l-j}} \right\} E_j \right) < 1$$

and the kth order primitive function $\varphi_k(z)$ of $A_k(z)$ belongs to $F^{p,k}$. Then all solutions of Equation (1) belong to $F^{p,k}$.

In [24], one sufficient condition such that all coefficients of the homogeneous linear complex differential equation belong to Fock-type spaces is obtained. Thus, we try to find sufficient conditions such that all coefficients of Equation (1) belong to some weighted Fock spaces. However, it is difficult to deal with this problem. Here we only obtain the following Remark 1.1 and its proof is omitted.

Remark 1.1: Suppose that $A_j(z)$ are constant functions, $j = 0, 1, \ldots, k-1$ and $A_k(z)$ is an entire function. If Equation (1) has a solution $f \in F^{p,k}$, then $A_k(z)$ is in F^p.

Next, the ideas of Theorems 1.3 and 1.4 come from [7, Section 7], and sufficient conditions for the coefficient $A(z)$ such that all solutions of Equation (2) belong to some weighted Fock spaces are shown by the Bergman reproducing kernel.

Theorem 1.3: *Let ϕ, φ be in the class $\mathcal{I}$ and A be an entire function. Suppose that $|\int_0^z A(\zeta)\,\mathrm{d}\zeta|e^{-\varphi(z)}$ is bounded in $z \in \mathbb{C}$ and*

$$T_K(A) = \sup_{z\in\mathbb{C}} \left\{ \left| \int_{\mathbb{C}} \left(\int_0^z \overline{K'_\zeta(\eta)} A(\zeta)\,\mathrm{d}\zeta \right) \frac{e^{-2\phi(\eta)+\varphi(\eta)}}{(1+\phi'(\eta))^2} \mathrm{d}m(\eta) \right| e^{-\varphi(z)} \right\} < 1.$$

Then the derivative f' of each solution f of Equation (2) *belongs to F^∞_φ.*

Theorem 1.4: *Let ϕ be in the class $\mathcal{I}$ and there exists $r_0 > 0$ such that $\phi'(r) \neq 0$ for $r > r_0$. Moreover, assume that ϕ satisfies*

$$\lim_{r\to\infty} \frac{re^{-p\phi(r)}}{\phi'(r)} = 0,$$

and

$$-\infty < \liminf_{r\to\infty} \frac{1}{r} \left(\frac{r}{\phi'(r)} \right)' \leq \limsup_{r\to\infty} \frac{1}{r} \left(\frac{r}{\phi'(r)} \right)' < p,$$

where $p \geq 1$. If A is an entire function and

$$Z_K(A) = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} e^{-2\phi(\eta)} \left| \int_0^z \overline{K_\zeta(\eta)} A(\zeta)\,\mathrm{d}\zeta \right|^2 \mathrm{d}m(\eta) \right) \frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2} \mathrm{d}m(z)$$

is sufficiently small, then all solutions f of Equation (2) *belong to F^2_ϕ.*

The structure of this paper is arranged as follows. Sufficient conditions for coefficients such that all solutions of Equation (1) belong to some weighted Fock spaces are obtained in Section 2. Furthermore, sufficient conditions for the coefficient $A(z)$ such that all solutions of Equation (2) belong to some weighted Fock spaces are shown in Section 3.

2. Sufficient conditions for solutions of Equation (1) to be in some weighted Fock spaces

In this section, we consider sufficient conditions for coefficients of Equation (1) such that all solutions belong to some weighted Fock spaces.

The following lemma is the characterization of the Fock norm $\|f\|_p$ needed in the proofs of Theorems 1.1 and 1.2.

Lemma 2.1 ([15]): *Suppose that $p \in [1, \infty)$ and $m \in \mathbb{N}$. Then, for any entire function $f(z)$, there exists a positive constant C, depending on p and m, such that*

$$C^{-1}\|f\|_p \leq \sum_{\alpha \leq m-1} |f^{(\alpha)}(0)| + \left(\int_{\mathbb{C}} \left| f^{(m)}(z) \frac{e^{-(1/2)|z|^2}}{(1+|z|)^m} \right|^p \mathrm{d}m(z) \right)^{1/p} \leq C\|f\|_p.$$

We now start to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1**:** It is known that if $A_j(z)$ are entire functions, $j = 0, 1, \ldots, k$, then all solutions f of Equation (1) are entire functions [17]. By Lemma 2.1,

$$
\begin{aligned}
\|f\|_p &\le C\left(\sum_{\alpha\le k-1} |f^{(\alpha)}(0)| + \left(\int_{\mathbb{C}} \left| f^{(k)}(z)\frac{1}{(1+|z|)^k} e^{-(1/2)|z|^2}\right|^p \, \mathrm{d}m(z)\right)^{1/p}\right) \\
&= C_1 + C\left(\int_{\mathbb{C}} \left| f^{(k)}(z)\frac{1}{(1+|z|)^k} e^{-(1/2)|z|^2}\right|^p \, \mathrm{d}m(z)\right)^{1/p},
\end{aligned}
$$

where $C_1 = C(\sum_{\alpha\le k-1} |f^{(\alpha)}(0)|)$. By Equation (1) and the Minkowski inequality,

$$
\begin{aligned}
\|f\|_p &\le C_1 + C\left(\int_{\mathbb{C}} \left| \left(\sum_{l=0}^{k-1} A_l(z) f^{(l)}(z) - A_k(z)\right) \frac{e^{-(1/2)|z|^2}}{(1+|z|)^k}\right|^p \, \mathrm{d}m(z)\right)^{1/p} \\
&\le C_1 + C\sum_{l=0}^{k-1}\left(\int_{\mathbb{C}} \left| A_l(z) f^{(l)}(z) \frac{e^{-(1/2)|z|^2}}{(1+|z|)^k}\right|^p \, \mathrm{d}m(z)\right)^{1/p} \\
&\quad + C\left(\int_{\mathbb{C}} \left| \frac{A_k(z) e^{-(1/2)|z|^2}}{(1+|z|)^k}\right|^p \, \mathrm{d}m(z)\right)^{1/p} \\
&\le C_1 + C\sum_{l=0}^{k-1} \sup_{z\in\mathbb{C}}\left\{\frac{|A_l(z)|}{(1+|z|)^{k-l}}\right\}\left(\int_{\mathbb{C}} \left| f^{(l)}(z) \frac{e^{-(1/2)|z|^2}}{(1+|z|)^l}\right|^p \, \mathrm{d}m(z)\right)^{1/p} \\
&\quad + C\left(\int_{\mathbb{C}} \left| \frac{A_k(z) e^{-(1/2)|z|^2}}{(1+|z|)^k}\right|^p \, \mathrm{d}m(z)\right)^{1/p}.
\end{aligned}
$$

Using Lemma 2.1 again,

$$
\|f\|_p \le C_1 + C\left(\sum_{l=0}^{k-1} \sup_{z\in\mathbb{C}}\left\{\frac{|A_l(z)|}{(1+|z|)^{k-l}}\right\} D_l \|f\|_p + D_k \|\varphi_k\|_p\right),
$$

where $\varphi_k(z)$ is the kth primitive function of $A_k(z)$. Then,

$$
\|f\|_p \left(1 - C\left(\sum_{l=0}^{k-1} \sup_{z\in\mathbb{C}}\left\{\frac{|A_l(z)|}{(1+|z|)^{k-l}}\right\} D_l\right)\right) \le C_1 + C D_k \|\varphi_k\|_p.
$$

If $\|f\|_p = \infty$, it is in contradiction to the condition of Theorem 1.1. Therefore,

$$
\|f\|_p \le \frac{C_1 + C D_k \|\varphi_k\|_p}{1 - C\sum_{l=0}^{k-1} D_l \sup_{z\in\mathbb{C}}\left\{\frac{|A_l(z)|}{(1+|z|)^{k-l}}\right\}} < +\infty,
$$

and then $f \in F^p$. ■

***Proof of Theorem* 1.2:** If f is a solution of Equation (1), then f is an entire function. By Lemma 2.1, Equation (1) and the Minkowski inequality,

$$\begin{aligned}
\|f(z)\|_{F^{p,k}} &\leq \sum_{l=0}^{k} C_l \left(\sum_{\alpha=l}^{k-1} |f^{(\alpha)}(0)| + \left(\int_{\mathbb{C}} \left| f^{(k)}(z) \frac{e^{-(1/2)|z|^2}}{(1+|z|)^{k-l}} \right|^p \mathrm{d}m(z) \right)^{1/p} \right) \\
&\leq D_1 + \sum_{l=0}^{k} C_l \sum_{j=0}^{k-1} \left(\int_{\mathbb{C}} \left| \frac{A_j(z) f^{(j)}(z) e^{-(1/2)|z|^2}}{(1+|z|)^{k-l}} \right|^p \mathrm{d}m(z) \right)^{1/p} \\
&\quad + \sum_{l=0}^{k} C_l \left(\int_{\mathbb{C}} \left| \frac{A_k(z) e^{-(1/2)|z|^2}}{(1+|z|)^{k-l}} \right|^p \mathrm{d}m(z) \right)^{1/p} \\
&\leq D_1 + \sum_{l=0}^{k} C_l \sum_{j=0}^{k-1} \sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} \left(\int_{\mathbb{C}} \left| \frac{f^{(j)}(z) e^{-(1/2)|z|^2}}{(1+|z|)^{j}} \right|^p \mathrm{d}m(z) \right)^{1/p} \\
&\quad + \sum_{l=0}^{k} C_l \left(\int_{\mathbb{C}} \left| \frac{A_k(z) e^{-(1/2)|z|^2}}{(1+|z|)^{k-l}} \right|^p \mathrm{d}m(z) \right)^{1/p},
\end{aligned}$$

where $D_1 = \sum_{l=0}^{k} C_l (\sum_{\alpha=l}^{k-1} |f^{(\alpha)}(0)|)$. Using Lemma 2.1 again,

$$\begin{aligned}
\|f\|_{F^{p,k}} &\leq D_1 + \sum_{l=0}^{k} C_l \sum_{j=0}^{k-1} \left(\sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} E_j \|f\|_p \right) + \sum_{l=0}^{k} C_l F_l \|\varphi_k^{(l)}\|_p \\
&\leq D_1 + \sum_{l=0}^{k} C_l \left(\sum_{j=0}^{k-1} \sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} E_j \right) \|f\|_{F^{p,k}} + \sum_{l=0}^{k} C_l F_l \|\varphi_k^{(l)}\|_p \\
&\leq D_1 + \sum_{l=0}^{k} C_l \left(\sum_{j=0}^{k-1} \sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} E_j \right) \|f\|_{F^{p,k}} + G\|\varphi_k\|_{F^{p,k}},
\end{aligned}$$

where $G = \max_{0\leq l\leq k}\{C_l F_l\}$. Then,

$$\|f\|_{F^{p,k}} \left(1 - \sum_{l=0}^{k} C_l \left(\sum_{j=0}^{k-1} \sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} E_j \right) \right) \leq D_1 + G\|\varphi_k\|_{F^{p,k}}.$$

If $\|f\|_{F^{p,k}} = \infty$, it is in contradiction to the condition of Theorem 1.2. Therefore,

$$\|f\|_{F^{p,k}} \leq \frac{D_1 + G\|\varphi_k\|_{F^{p,k}}}{1 - \sum_{l=0}^{k} C_l \left(\sum_{j=0}^{k-1} E_j \sup_{z\in\mathbb{C}} \left\{ \frac{|A_j(z)|}{(1+|z|)^{k-l-j}} \right\} \right)} < +\infty,$$

and then $f \in F^{p,k}$. ■

In [11], the growth of solutions of Equation (1) is obtained as follows. In order to state the following Theorem A, the notion is needed

$$\delta = \begin{cases} 0, & \text{if } A_k \equiv 0, \\ 1, & \text{otherwise.} \end{cases}$$

Theorem A ([11]): *Let $f(z)$ be a solution of Equation (1) in the disk $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, where $0 < R \le \infty$, and $0 \le k_c \le k$ be the number of nonzero coefficients $A_j(z)$, $j = 0, 1, \dots, k-1$. If R_0 is a positive real number such that there exists some $A_j(R_0 e^{i\theta}) \neq 0$, then, for all $R_0 < r < R$,*

$$|f(re^{i\theta})| \le C\left(\max_{0\le x\le r} |A_k(xe^{i\theta})| + 1\right) \exp\left(\int_0^r \left(\delta + k_c \max_{0\le j\le k-1} |A_j(se^{i\theta})|^{1/(k-j)}\right) \mathrm{d}s\right),$$

where C is some positive constant depending on the values of the derivatives of $f(z)$ and the values of $A_j(z)$ at $R_0 e^{i\theta}$, $j = 0, 1, \dots, k$.

Here we give an application of Theorem A in weighted Fock spaces.

Example 2.1: Suppose that $A_k(z)$ is a nonconstant entire function and there exists a non-constant function among entire functions $A_j(z)$, $j = 0, 1, \dots, k-1$. If ϕ is in the class $\mathcal{I}$ and there exists a sufficiently large r_0 such that for $r > r_0$,

$$|A_j(re^{i\theta})| \le \frac{\phi^{1/2}(r)}{r}, \quad j = 0, 1, \dots, k,$$

then all solutions of Equation (1) belong to $F^{p,q}_\phi$.

The following lemmas are important in the proof of Example 2.1.

Lemma 2.2: *Suppose that ϕ is in the class $\mathcal{I}$. Then*

$$\lim_{r\to\infty} \frac{\phi(r)}{r^2} = \infty.$$

Proof: By L'Hospital's rule,

$$\begin{aligned} \lim_{r\to\infty} \frac{\phi(r)}{r^2} &= \lim_{r\to\infty} \frac{\phi'(r)}{2r} = \lim_{r\to\infty} \frac{r\phi'(r)}{2r^2} = \lim_{r\to\infty} \frac{(r\phi'(r))'}{4r} \\ &= \lim_{r\to\infty} \frac{1}{4}\left(\phi''(r) + \frac{\phi'(r)}{r}\right) = \lim_{r\to\infty} \frac{1}{4}\Delta\phi(r). \end{aligned}$$

Since $C^{-1}(\Delta\phi(|z|))^{-1/2} \le \tau(z)$ and $\tau(z)$ decreases to 0 as $|z| \to \infty$,

$$\lim_{r\to\infty} \frac{\phi(r)}{r^2} = \infty. \qquad \blacksquare$$

Lemma 2.3 ([8]): *Suppose that ϕ is in the class $\mathcal{I}$ and $f(z)$ is an entire function. Then, for any $R>0$, there eixits a constant $C>0$ such that*

$$\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)}\phi^q(z)\,\mathrm{d}m(z) \le C\int_{|z|\ge R} |f(z)|^p e^{-p\phi(z)}\phi^q(z)\,\mathrm{d}m(z).$$

***Proof of Example* 2.1:** If f is a solution of Equation (1), then f is an entire function. By Theorem A,

$$|f(re^{i\theta})| \le C\left(\max_{0\le x\le r}|A_k(xe^{i\theta})|+1\right)\exp\left(\int_0^r\left(\delta+k_c\max_{0\le j\le k-1}|A_j(se^{i\theta})|^{1/(k-j)}\right)\mathrm{d}s\right).$$

Since $A_k(z)$ is not a constant function and there exists a nonconstant function among $A_j(z)$, $j=0,\dots,k-1$, for sufficiently large $r>r_1$,

$$|f(re^{i\theta})| \le C\left(2\max_{0\le x\le r}|A_k(xe^{i\theta})|\right)\exp\left(\int_0^r\left(2k_c\max_{0\le j\le k-1}|A_j(se^{i\theta})|^{1/(k-j)}\right)\mathrm{d}s\right).$$

Since $|A_j(re^{i\theta})|\le \phi^{1/2}(r)/r$ for $r>r_0$, for $r>R=\max\{r_0,r_1,1\}$,

$$|f(re^{i\theta})| \le 2CD\frac{\phi^{1/2}(r)}{r}\exp\left(2k_c\max_{0\le j\le k-1}\int_R^r\left(\frac{\phi^{1/2}(s)}{s}\right)^{1/(k-j)}\mathrm{d}s\right),$$

where

$$D=\exp\left(\int_0^R\left(2k_c\max_{0\le j\le k-1}|A_j(se^{i\theta})|^{1/(k-j)}\right)\mathrm{d}s\right).$$

Since

$$\max_{0\le j\le k-1}\int_R^r\left(\frac{\phi^{1/2}(s)}{s}\right)^{1/(k-j)}\mathrm{d}s \le \phi^{1/2}(r)\int_R^r s^{-1/k}\,\mathrm{d}s \le \phi^{1/2}(r)\frac{k}{k-1}r^{1-1/k},$$

we have

$$|f(re^{i\theta})| \le D_1\frac{\phi^{1/2}(r)}{r}\exp\left(D_2\phi^{1/2}(r)r^{1-1/k}\right),\quad R<r<\infty,$$

where $D_1=2CD$ and $D_2=2k_c(k/(k-1))$. By Lemma 2.2 and $\phi\in\mathcal{I}$, there exists $R'>0$ such that $\phi(r)>r^2$ for $r>R'$. By Lemma 2.3,

$$\begin{aligned}
\|f\|^p_{F^{p,q}_\phi} &= \int_{\mathbb{C}}|f(z)|^p e^{-p\phi(z)}\phi^q(z)\,\mathrm{d}m(z)\\
&\le M\int_{|z|\ge R_1}|f(z)|^p e^{-p\phi(z)}\phi^q(z)\,\mathrm{d}m(z)\\
&\le M\int_0^{2\pi}\int_{R_1}^{\infty}\left(D_1\frac{\phi^{1/2}(r)}{r}\exp\left(D_2\phi^{1/2}(r)r^{1-1/k}\right)\right)^p e^{-p\phi(r)}\phi^q(r)r\,\mathrm{d}r\,\mathrm{d}\theta\\
&\le MD_1^p\int_0^{2\pi}\int_{R_1}^{\infty}\left(\frac{\phi^{p/2+q}(r)}{r^{p-1}}\exp\left(p\left(D_2\phi^{1/2}(r)r^{1-1/k}-\phi(r)\right)\right)\right)\mathrm{d}r\,\mathrm{d}\theta
\end{aligned}$$

$$\leq 2\pi M D_1^p \int_{R_1}^{\infty} \left(\frac{\phi^{p/2+q}(r)}{r^{p-1}} \exp\left(p\left(D_2\phi^{1/2}(r)r^{1-1/k} - \phi(r)\right)\right)\right) \mathrm{d}r,$$

where $R_1 = \max\{R, R'\}$. Since $D_2 r^{1-1/k} - \phi^{1/2}(r) < -1$ for $r > R_1$,

$$\|f\|_{F_\phi^{p,q}}^p \leq 2\pi M D_1^p \int_{R_1}^{\infty} \left(\frac{\phi^{p/2+q}(r)}{r^{p-1}} e^{-p\phi^{1/2}(r)} \right) \mathrm{d}r < \infty.$$

Therefore, $f \in F_\phi^{p,q}$. ■

3. Sufficient conditions for solutions of Equation (2) to be in some weighted Fock spaces

The research of Equation (2) in function spaces has been widely concerned. Sufficient conditions for the coefficient function $A(z)$ such that all solutions of Equation (2) belong to Hardy spaces are first found by Pommerenke [22]. Later, many results of sufficient conditions on all solutions belonging to some other function spaces are obtained and see [12–14,16] for details.

In [7], sufficient conditions for the coefficient $A(z)$ such that all solutions of Equation (2) belong to Bloch spaces are shown by the reproducing formula of weighted Bergman spaces. Thus, we try to generalize the method of [7] to weighted Fock spaces. Luckily, the following Littlewood-Paley type formula of some weighted Fock spaces is obtained in [5].

Littlewood-Paley Type Formula Suppose that ϕ is in the class $\mathcal{I}$ and there exists $r_0 > 0$ such that $\phi'(r) \neq 0$ for $r > r_0$. Moreover, assume that ϕ satisfies

$$\lim_{r\to\infty} \frac{re^{-p\phi(r)}}{\phi'(r)} = 0,$$

and

$$-\infty < \liminf_{r\to\infty} \frac{1}{r}\left(\frac{r}{\phi'(r)}\right)' \leq \limsup_{r\to\infty} \frac{1}{r}\left(\frac{r}{\phi'(r)}\right)' < p,$$

where $p \geq 1$. Then for any entire function $f(z)$,

$$C^{-1}\|f\|_{F_\phi^p}^p \leq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p \frac{e^{-p\phi(|z|)}}{(1+\phi'(|z|))^p} \mathrm{d}m(z) \leq C\|f\|_{F_\phi^p}^p,$$

where C is positive constant only depending on p.

The following lemma is essential in the proofs of Theorems 1.3 and 1.4.

Lemma 3.1 ([5]): *Suppose that ϕ is in the class $\mathcal{I}$ and $f, h \in F_\phi^2$. Then*

$$\langle f, h\rangle = f(0)\overline{h(0)} + \int_{\mathbb{C}} f'(z)\overline{h'(z)}(1+\phi'(z))^{-2}e^{-2\phi(z)}\,\mathrm{d}m(z).$$

Now we start to give the proofs of Theorems 1.3 and 1.4.

***Proof of Theorem* 1.3:** By all solutions f of Equation (2),

$$f'(z) = -\int_0^z f(\zeta)A(\zeta)\,\mathrm{d}\zeta + f'(0), \quad z \in \mathbb{C}.$$

If g satisfies the reproducing formula, $g \in F^2_\phi$. Since f is an entire function, there exists a finite Taylor expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$ such that $f_n = \sum_{j=0}^{n} a_j z^j \in F^2_\phi$. Thus,

$$f'(z) = -\int_0^z \lim_{n\to\infty} f_n(\zeta)A(\zeta)\,\mathrm{d}\zeta + f'(0), \quad z \in \mathbb{C}.$$

By the reproducing formula, Fubini's theorem and $\phi \in \mathcal{I}$,

$$\begin{aligned} f'(z) &= -\int_0^z \left(\lim_{n\to\infty}\int_{\mathbb{C}} f_n(\eta)\overline{K_\zeta(\eta)}e^{-2\phi(\eta)}\,\mathrm{d}m(\eta)\right)A(\zeta)\,\mathrm{d}\zeta + f'(0) \\ &= -\int_{\mathbb{C}} \lim_{n\to\infty} f_n(\eta)e^{-2\phi(\eta)}\left(\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta) + f'(0). \end{aligned}$$

Using $K_\zeta(0) = \sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(0)} = \delta_0^{-2}$ and Lemma 3.1,

$$\begin{aligned} f'(z) &= -\int_{\mathbb{C}} \lim_{n\to\infty} f_n'(\eta)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\left(1+\phi'(\eta)\right)^{-2}e^{-2\phi(\eta)}\,\mathrm{d}m(\eta) \\ &\quad - \lim_{n\to\infty} f_n(0)\left(\int_0^z \overline{K_\zeta(0)}A(\zeta)\,\mathrm{d}\zeta\right) + f'(0) \\ &= -\int_{\mathbb{C}} f'(\eta)\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)(1+\phi'(\eta))^{-2}e^{-2\phi(\eta)}\,\mathrm{d}m(\eta) \\ &\quad - f(0)\left(\int_0^z \delta_0^{-2}A(\zeta)\,\mathrm{d}\zeta\right) + f'(0). \end{aligned}$$

It follows that

$$\begin{aligned} |f'(z)|e^{-\varphi(z)} &\le e^{-\varphi(z)}\left|\int_{\mathbb{C}}\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\frac{e^{-2\phi(\eta)+\varphi(\eta)}}{(1+\phi'(\eta))^2}\mathrm{d}m(\eta)\right| \\ &\quad \cdot \sup_{\eta\in\mathbb{C}}\left\{|f'(\eta)|e^{-\varphi(\eta)}\right\} + \left|f(0)\int_0^z \delta_0^{-2}A(\zeta)\,\mathrm{d}\zeta\right|e^{-\varphi(z)} + |f'(0)|. \end{aligned}$$

Then,

$$\begin{aligned} \|f'\|_{F^\infty_\varphi} &\le \sup_{z\in\mathbb{C}}\left\{\left|\int_{\mathbb{C}}\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\frac{e^{-2\phi(\eta)+\varphi(z)}}{(1+\phi'(\eta))^2}\mathrm{d}m(\eta)\right|e^{-\varphi(z)}\right\} \\ &\quad \cdot \|f'\|_{F^\infty_\varphi} + \sup_{z\in\mathbb{C}}\left\{\left|f(0)\left(\int_0^z \delta_0^{-2}A(\zeta)\,\mathrm{d}\zeta\right)\right|e^{-\varphi(z)}\right\} + |f'(0)|. \end{aligned}$$

Thus,

$$\|f'\|_{F^\infty_\varphi}\left(1-\sup_{z\in\mathbb{C}}\left\{\left|\int_{\mathbb{C}}\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\frac{e^{-2\phi(\eta)+\varphi(\eta)}}{(1+\phi'(\eta))^2}\mathrm{d}m(\eta)\right|e^{-\varphi(z)}\right\}\right)$$
$$\le \sup_{z\in\mathbb{C}}\left\{\left|f(0)\left(\int_0^z \delta_0^{-2}A(\zeta)\,\mathrm{d}\zeta\right)\right|e^{-\varphi(z)}\right\}+|f'(0)|.$$

If $\|f'\|_{F^\infty_\varphi}=\infty$, it is in contradiction to the condition of Theorem 1.3. Therefore,

$$\|f'\|_{F^\infty_\varphi}\le\frac{1}{1-T_K(A)}\left(\sup_{z\in\mathbb{C}}\left\{\left|f(0)\left(\int_0^z \delta_0^{-2}A(\zeta)\,\mathrm{d}\zeta\right)\right|e^{-\varphi(z)}\right\}+|f'(0)|\right)<\infty,$$

and $f'\in F^\infty_\varphi$. ∎

We have two natural corollaries by Theorem 1.3 and their proofs are omitted.

Corollary 3.2: *Let ϕ be in the class $\mathcal{I}$ and A be an entire function. Suppose that $|\int_0^z A(\zeta)\,\mathrm{d}\zeta|e^{-\phi(z)}$ is bounded in $z\in\mathbb{C}$ and*

$$X_K(A)=\sup_{z\in\mathbb{C}}\left\{\left|\int_{\mathbb{C}}\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\left(1+\phi'(\eta)\right)^{-2}e^{-\phi(\eta)}\,\mathrm{d}m(\eta)\right|e^{-\phi(z)}\right\}<1.$$

Then the derivative f' of each solution f of Equation (2) *belongs to F^∞_ϕ.*

Corollary 3.3: *Let ϕ be in the class $\mathcal{I}$ and A be an entire function. Suppose that $|\int_0^z A(\zeta)\,\mathrm{d}\zeta|e^{-(1/2)|z|^2}$ is bounded in $z\in\mathbb{C}$ and*

$$Y_K(A)=\sup_{z\in\mathbb{C}}\left\{\left|\int_{\mathbb{C}}\left(\int_0^z \overline{K'_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\frac{e^{-2\phi(\eta)+(1/2)|\eta|^2}}{(1+\phi'(\eta))^2}\mathrm{d}m(\eta)\right|e^{-(1/2)|z|^2}\right\}<1.$$

Then the derivative f' of each solution f of Equation (2) *belongs to F^∞.*

Proof of Theorem* 1.4*: By the proof of Theorem 1.3,

$$f'(z)=-\int_{\mathbb{C}}\lim_{n\to\infty}f_n(\eta)e^{-2\phi(\eta)}\left(\int_0^z\overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta)+f'(0)$$
$$=-\int_{\mathbb{C}}f(\eta)e^{-2\phi(\eta)}\left(\int_0^z\overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta)+f'(0).$$

Then,

$$|f'(z)|^2\le 2\left(\left|\int_{\mathbb{C}}f(\eta)e^{-2\phi(\eta)}\left(\int_0^z\overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta)\right|^2+|f'(0)|^2\right).$$

Since the condition of Theorem 1.4 satisfies the Littlewood-Paley type formula,

$$\begin{aligned}\|f\|_{F_\phi^2}^2 &\le C\left(|f(0)|^2 + \int_{\mathbb{C}} |f'(z)|^2 \frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2} \mathrm{d}m(z)\right)\\ &\le 2C\left(|f(0)|^2 + \int_{\mathbb{C}} \left|\int_{\mathbb{C}} f(\eta)e^{-2\phi(\eta)}\left(\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta)\right|^2\right.\\ &\quad \left.\cdot\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z) + |f'(0)|^2\int_{\mathbb{C}}\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z)\right).\end{aligned}$$

By the Littlewood-Paley type formula,

$$\int_{\mathbb{C}}\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z) \le C\|z\|_{F_\phi^2}^2.$$

By Lemma 2.2, there exists a positive number M such that $\|z\|_{F_\phi^2}^2 < M$. Thus,

$$\begin{aligned}\|f\|_{F_\phi^2}^2 &\le P + 2C\int_{\mathbb{C}}\left|\int_{\mathbb{C}} f(\eta)e^{-2\phi(\eta)}\left(\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}m(\eta)\right|^2 \frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z)\\ &\le P + 2C\int_{\mathbb{C}}\left(\int_{\mathbb{C}}\left|f(\eta)e^{-2\phi(\eta)}\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right|\mathrm{d}m(\eta)\right)^2 \frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z),\end{aligned}$$

where $P = 2C(|f(0)|^2 + |f'(0)|CM)$. By the Cauchy-Schwarzian inequality,

$$\begin{aligned}\|f\|_{F_\phi^2}^2 &\le P + 2C\int_{\mathbb{C}}\left(\int_{\mathbb{C}}\left|f(\eta)\right|^2 e^{-2\phi(\eta)}\,\mathrm{d}m(\eta)\right)\\ &\quad\cdot\left(\int_{\mathbb{C}} e^{-2\phi(\eta)}\left|\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right|^2 \mathrm{d}m(\eta)\right)\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z)\\ &\le P + 2C\|f\|_{F_\phi^2}^2\int_{\mathbb{C}}\left(\int_{\mathbb{C}} e^{-2\phi(\eta)}\left|\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right|^2 \mathrm{d}m(\eta)\right)\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z).\end{aligned}$$

Since

$$Z_K(A) = \int_{\mathbb{C}}\left(\int_{\mathbb{C}} e^{-2\phi(\eta)}\left|\int_0^z \overline{K_\zeta(\eta)}A(\zeta)\,\mathrm{d}\zeta\right|^2 \mathrm{d}m(\eta)\right)\frac{e^{-2\phi(|z|)}}{(1+\phi'(|z|))^2}\mathrm{d}m(z),$$

then

$$\|f\|_{F_\phi^2}^2\,(1 - 2CZ_K(A)) \le P.$$

If $\|f\|_{F_\phi^2}^2 = \infty$, it is in contradiction to the condition of Theorem 1.4. Therefore,

$$\|f\|_{F_\phi^2}^2 < \frac{P}{1-2CZ_K(A)} < \infty,$$

and $f \in F_\phi^2$. ■

Acknowledgments

We thank the referees for many important and useful comments. The first author would like to thank Department of Physics and Mathematics, University of Eastern Finland, for providing a good environment during the preparation of this work.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The third author is supported by the National Natural Science Foundation of China (Grant No. 11861023), and the Foundation of Science and Technology project of Guizhou Province of China (Grant No. [2018]5769-05).

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