Fixed points and iteration

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1 Introduction

Iteration of rational and entire functions has been investigated atleast since the 1910's. The foundation was created by Fatou and Julia, see for example [8] and [11]. Some historical discussion is in [6].

Theory of fixed points is important for the theory of iteration. For example, it is vital for the paper [11] of Julia.

In Section 3 we recall the Fundamental theorem of algebra, Picard's theorem and some basics of Nevanlinna theory. In Section 4 we state some basic ideas about iteration and see that we should restrict ourselves to polynomials, rational functions or transcendental entire functions. In Section 5 we use the theory of Section 3 to prove some theorems about the existence of fixed points.

2 Notation

We use the following notation for functions $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$:

- $\mathcal{P}$ polynomials;
- $\mathcal{E}$ entire functions;
- $\mathcal{R}$ rational functions;
- $\mathcal{M}$ meromorphic functions;
- $\underline{\mathcal{E}} = \mathcal{E} \setminus \mathcal{P}$ transcendental entire functions;
- $\underline{\mathcal{M}} = \mathcal{M} \setminus \mathcal{R}$ transcendental meromorphic functions.

We denote by $\operatorname{Fix}(f, z_0)$ the exact order of the fixed point z_0 of function f. See Section 4.

3 Existence of zeros

In this section, we recall the Fundamental theorem of algebra, Picard's theorem and some basics of Nevanlinna theory.

3.1 Fundamental theorem of algebra

Theorem 3.1 is essential for the study of polynomials and rational functions. It carries over to rational functions, as Corollary 3.2 shows. Corollary 3.2 and some ideas about rational functions are needed in the proof of Theorem 5.3. We settle for reviewing these ideas in Example 3.3.

Theorem 3.1 (Fundamental theorem of algebra) *Let $f \in \mathcal{P}$ and $d = \deg(f) \geq 1$. Let $a \in \mathbb{C}$. Now the number of a-points of f is d CM.*

Corollary 3.2 *Let $f \in \mathcal{R}$ and $d = \deg(f) \geq 1$. Let $a \in \widehat{\mathbb{C}}$. Now the number of a-points of f is d CM.*

Example 3.3 *Let*

$$f(z) = \frac{z(z-1)^2}{(z-2)(z-3)^3}.$$

Now the degree of the numerator is 3 and the degree of the denumerator is 4. Hence $\deg(f) = 4$.

Function f attains ∞ once at $z = 2$ and three times at $z = 3$. Hence f attains ∞ exactly four times.

Function f attains 0 once at $z = 0$, twice at $z = 1$ and once at $z = \infty$. Hence f attains 0 exactly four times.

Function f attains 3 once in each of the points $z \approx 1.8, 5.1, 2.2 - i, 2.2 + i$. Hence f attains 3 exactly four times.

The multiplicity of an a-point b of f can be detected by inspecting the derivatives of f at b. That is, if

$$f(b) - a = f'(b) = \ldots = f^{(m-1)}(b) = 0 \neq f^{(m)}(b),$$

then b is an a-point of multiplicity m for f. Now, for target ∞, we see that

$$\left(\frac{1}{f}\right)(2) = 0 \neq \left(\frac{1}{f}\right)'(2)$$

and

$$\left(\frac{1}{f}\right)(3) = \left(\frac{1}{f}\right)'(3) = \left(\frac{1}{f}\right)''(3) = 0 \neq \left(\frac{1}{f}\right)'''(3).$$

For target 0, we see that

$$f(0) = 0 \neq f'(0)$$

and

$$f(1) = f'(1) = 0 \neq f''(1)$$

and

$$f(\infty) = 0 \neq f'(\infty).$$

For target 3

$$f(z) - 3 = 0 \neq f'(z)$$

for each of the points $z \approx 1.8, 5.1, 2.2 - i, 2.2 + i$.

3.2 Picard's theorems

Theorem 3.4 is needed for the proof of Theorems 5.7 and 5.9. Theorem 3.4 has tree simple corollaries, which we state here. These results are a special case of the results of Section 3.3.

Theorem 3.4 (Picard) *Let f be analytic in $D(z_0, r) \setminus \{z_0\}$ with essential singularity at z_0. Let $a, b \in \widehat{\mathbb{C}}$ be distinct. Now*

$$\#\{z \in D(z_0, s) \setminus \{z_0\} : f(z) \in \{a, b\}\} = \infty, \quad 0 < s < r.$$

Corollary 3.5 *Let $f \in \underline{\mathcal{E}}$ and let $a, b \in \mathbb{C}$ be distinct. Now*

$$\#\{z \in \mathbb{C} : f(z) \in \{a, b\}\} = \infty.$$

Proof. Let $g(z) = f(1/z)$. Now g is analytic in $D(0, 1) \setminus \{0\}$ with essential singularity at $z = 0$. Apply Theorem 3.4 to g. □

Corollary 3.6 *Let $f \in \underline{\mathcal{M}}$ and let $a, b, c \in \widehat{\mathbb{C}}$ be distinct. Now*

$$\#\{z \in \mathbb{C} : f(z) \in \{a, b, c\}\} = \infty.$$

Proof. We can assume that $a, b, c \in \mathbb{C}$, by applying a Möbius transformation, if necessary. Assume that the equations $f(z) = a$, $f(z) = b$ have only $n, m \in \mathbb{N}_0$ solutions, respectively. Now, consider the transcendental meromorphic function

$$g = \frac{(f - a)(c - b)}{(f - b)(c - a)}.$$

Function g has only n zeros that occur, when $f(z) = a$. Similarly g has only m poles. Hence $g(1/z)$ is analytic in $D(0, r) \setminus \{0\}$ for some $r > 0$ and has atmost n zeros in $D(0, r) \setminus \{0\}$. By Theorem 3.4, g attains 1 infinitely often in $D(0, r) \setminus \{0\}$ and hence f attains c infinitely often. □

By combining Corollary 3.2 and Corollary 3.7, we get Corollary 3.7.

Corollary 3.7 *Let $f \in \mathcal{M}$ and let $a, b, c \in \widehat{\mathbb{C}}$ be distinct. If*

$$\#\{z \in \mathbb{C} : f(z) \in \{a, b, c\}\} = 0,$$

then f is a constant.

3.3 Nevanlinna theory

We recall the basic theorems of Nevanlinna theory and state them in the most suitable way for Section 5.

In Nevanlinna theory, we measure the growth of a meromorphic function with a characteristic function T. The growth of a function inside a disc $D(0, r)$ is related to the number of zeros the function has inside the disc. Hence, by measuring a function with T, we get quantitative versions of the results of Sections 3.1 and 3.2.

Theorem 3.8 is an analogue of Corollary 3.2. It states that if for a meromorphic function f equality $f(z) = a$ happens rarely, then $|f - a|$ is small on average.

Theorems 3.9 and 3.10 say that in some sence there are only a few values that a meromorphic function does not attain. Theorem 3.11 is a generalization of Theorem 3.9 from constant targets to slowly moving targets.

The functions $n, N, \overline{n}, \overline{N}, m, T, S, \delta, \theta$ and Θ are defined as in [12]. This section follows [12].

Theorem 3.8 (First Main Theorem) *Let $f \in \mathcal{M}$ be non-constant and $a \in \mathbb{C}$. Now*

$$T(r, f) = T\left(r, \frac{1}{f-a}\right) + O(1)$$

for all $r \geq 0$. [12, p. 9]

Theorem 3.9 (Second Main Theorem) *Let $f \in \mathcal{M}$ be non-constant and $a_1, \ldots, a_q \in \mathbb{C}$ be $q \geq 2$ distinct points. Then*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r, f).$$

[12, p. 18, Corollary 6.5.]

Since $\overline{N} \leq N \leq T$, we have

$$(q-1)T(r, f) \leq T(r, f) + \sum_{i=1}^{q} N\left(r, \frac{1}{f-a_i}\right) + S(r, f),$$

which gives for $f = h$

$$\sum_{j=1}^{q} N(h, a_j, r) = \sum_{j=1}^{q} N\left(r, \frac{1}{h-a_j}\right) \geq (q-2)T(h, r) - o(T((h, r)), \tag{3.1}$$

for all $r \geq r_0$ outside a set E of finite Lebesgue measure. This is needed in the proof of Lemma 5.13.

Theorem 3.9 is needed to prove Theorem 3.10. Theorem 3.10 is an improvement of Corollary 3.6.

Theorem 3.10 (Nevanlinna's theorem on deficient values) *Let $f \in \underline{\mathcal{M}}$. Now the set of values $a \in \mathbb{C}$ for which*

$$\Theta(a, f) > 0$$

is countable and

$$\sum_{a \in \widehat{\mathbb{C}}} (\delta(a, f) + \theta(a, f)) \leq \sum_{a \in \widehat{\mathbb{C}}} \Theta(a, f) \leq 2.$$

[12, p. 10, Theorem 6.11]

Theorem 3.11 *[12, Theorem 6.14., p. 21] Let $f \in \mathcal{M}$ be non-constant and let a_1, a_2, a_3 be distinct small functions with respect to f. Then*

$$T(r, f) \leq \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

4 Basics of iteration

Let X be a set and $f : X \to X$ a function. We set $f_1 = f$ and set inductively

$$f_{n+1} = f_n \circ f$$

for $n \in \mathbb{N}$. A point $z_0 \in X$ is a *fixed point of order* n for f if $f_n(z_0) = z_0$. (If $n = 1$, we simply say, that z_0 is a fixed point for f.) In this case

$$\text{Fix}\,(f, z_0) = \min\{n \in \mathbb{N} : \quad f_n(z_0) = z_0\}$$

is the *exact order* of the fixed point z_0 of f.

Now z_0 is a fixed point of order m for f if and only if $\text{Fix}\,(f, z_0) \mid m$. For this, let $f_m(z_0) = z_0$ and let $\text{Fix}\,(f, z_0) = n$. Now $n \leq m$ and hence $m = qn + r$ for some $q \in \mathbb{N}, r \in \mathbb{N}_0, 0 \leq r < n$. Now $r = 0$. Otherwise we would have

$$z_0 = f_m(z_0) = f_r(f_{qn}(z_0)) = f_r(f_n(f_n(\cdots(f_n(z_0)))) = f_r(z_0),$$

a contradiction.

For a fixed $\mathcal{S} \in \{\mathcal{P}, \mathcal{E}, \underline{\mathcal{E}}, \mathcal{R}\}$ and $f, g \in \mathcal{S}$ we have $f \circ g \in \mathcal{S}$. Hence, in this case $\mathcal{S}$ is closed with respect to composition of functions. However, in general it is not possible to iterate in $\underline{\mathcal{M}}$ as Example 4.1 shows. The author has seen Example 4.2 on many occasions and it gave him the idea about Example 4.1.

Example 4.1 *Let $f : \widehat{\mathbb{C}} \to \widehat{C}$,*

$$f(z) = \frac{\sin z}{z^2}.$$

Then f is meromorphic, but f_2 is not.

Proof. Clearly f has a pole of order 1 at the origin and ∞ is an essential singularity for the sine function. Hence the origin is an essential singularity of

$$\sin \frac{\sin z}{z^2}.$$

Now

$$f(f(z)) = \frac{\sin \frac{\sin z}{z^2}}{\left(\frac{\sin z}{z^2}\right)^2}$$

is not meromorphic, since if it was,

$$\sin \frac{\sin z}{z^2} = f(f(z)) \cdot \left(\frac{\sin z}{z^2}\right)^2$$

would also be. □

Example 4.2 *Let* $f \in \mathcal{R}, g \in \underline{\mathcal{M}}$. *Now* $f \circ g \in \underline{\mathcal{M}}$, *but* $g \circ f \notin \mathcal{M}$ *is possible. To see the first, note that* g *is a quotient of two entire functions and hence so is* f. *For the latter, take* $g(z) = \sin z$ *and* $f(z) = \frac{1}{z}$, *which gives* $g(f(z)) = \sin \frac{1}{z}$.

5 Existence of fixed points

5.1 Polynomials and rational functions

For this section, let $f \in \mathcal{R}$, $d = \deg(f)$ and $n \in \mathbb{N}$. When does f have fixed points of exact order n?

First we discuss some basics in Example 5.1 and see an exceptional polynomial in Example 5.2. Then we prove Theorem 5.3 which shows that a polynomial has fixed points of all exact orders with possibly one exception. Theorem 5.4 and Example 5.5 give a complete solution of the question for rational functions, but would require extra work, see [2].

Example 5.1 *Every polynomial has a fixed point in* $\mathbb{C}$. *If the polynomial is a constant, the constant is a fixed point. Let* p *be a polynomial with degree* $d = \deg(p) = 1$. *If* $p(z) - z \equiv 0$, *then every point is a fixed point. Otherwise* $p(z) - z$ *has exactly one solution. If* $d = \deg(p) \geq 2$, *then* p *has exactly* d *fixed points CM.*

Since the composition of two polynomials is a polynomial, every polynomial has a fixed point of order n *for every* $n \in \mathbb{N}$. *If* p *is a polynomial with* $d = \deg(p) \geq 2$, *then* $\deg(p_n) = d^n$. *We see that* p_n *has* d^n *fixed points CM, that is,* p *has* d^n *fixed points of order* n *CM.*

The number of fixed points of exact order n is always atmost d^n. There migth be none, as Example 5.2 shows.

Example 5.2 *[1, p. 284] Consider* $f(z) = z^2 - z$. *Now* $f(z) = z$ *implies*

$$f(z) - z = z^2 - 2z = z(z-2) = 0,$$

giving $\mathit{Fix}\,(f, 0) = \mathit{Fix}\,(f, 2) = 1$. *Moreover*

$$f(f(z)) - z = (z^2 - z)^2 - (z^2 - z) - z = z^3(z-2)$$

shows that $\mathit{Fix}\,(f, z) \mid 2$ *implies* $z \in \{0, 2\}$ *and hence* $\mathit{Fix}\,(f, z) = 1 < 2$.

We prove Theorem 5.3, which originates in [1, p. 280]. Our proof is from [10, p. 52]. We state Theorem 5.4 and Example 5.5, which give in some sence a complete solution.

Theorem 5.3 (Baker 1960) *Let* $f \in \mathcal{P}$, $\deg(f) \geq 2$ *and* $n, k \in \mathbb{N}$. *Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) \in \{n, k\}\} \geq 1.$$

Proof. Let $d = \deg(f) \geq 2$. Now $f_n(z)$ has degree d^n and so $f(z)$ has fix-points of every order n. In particular $f(z)$ has a fix-point of exact order 1. Suppose that $f(z)$ has no fix-points of exact orders n, k, where $n > k \geq 2$. We consider

$$\varphi(z) = \frac{f_n(z) - z}{f_{n-k}(z) - z}.$$

Now $\varphi(z) = 0$ has roots only where $f_n(z) = z$, and these roots occur only when $f_j(z) = z$ for some $j < n$. Also, if j is the exact order of z, then the numbers $z_t = f_t(z)$ form a cycle of j points and so if $z_n = z_0 = z$, $j \mid n$. If $n = 3$, we must have $j = 1$, so that there are at most d distinct zeros of $\varphi(z)$. If $n = 4$, any fix-point of exact order 1 is a fix-point of order 2, so that since $j = 1$ or 2, $\varphi(z)$ has at most d^2 distinct zeros. If $n > 4$, we must have $j \leq n - 3$ (assuming $n = jk$, $k \geq 2$, $j \in \{n-1, n-2\}$, we get $n > n$), and so $\varphi(z)$ has at most

$$\sum_{j=1}^{n-3} d^j = \frac{d^{n-2} - 1}{d - 1} \leq d^{n-2}$$

distinct zeros. Thus in all cases $\varphi(z)$ has at most d^{n-2} distinct zeros.

Again $\varphi(z) = 1$ implies $f_{n-k}(z) = f_n(z) = f_k(f_{n-k}(z))$, so that $\zeta = f_{n-k}$ is a fix point of f_k and so a fix-point of f_j, for some divisor j of k with $1 \leq j < k$. Thus

$$f_j(f_{n-k}(z)) = f_{n-k+j}(z) = f_{n-k}(z).$$

The polynomial $f_{n-k+j}(z) - f_{n-k}(z)$ has degree d^{n-k+j} so that the number of different 1-points of $\varphi(z)$ is at most

$$\sum_j d^{n-k+j} \leq \sum_{j=1}^{k-2} d^{n-k+j} \leq d^{n-1}, \quad k \geq 3,$$

and

$$\sum_j d^{n-k+j} = d^{n-1}, \quad k = 2.$$

Let N be the number of zeros of $\varphi'(z)$ CM. Then we deduce that the total number of distinct solutions of the equation $\varphi(z) = 0, 1$ is at most $d^{n-1} + d^{n-2}$ and so the total number of solutions CM is at most

$$N + d^{n-1} + d^{n-2}.$$

Suppose, that $\varphi(z)$ has q finite poles CM. Then $\varphi(z)$ has a pole of order $d^n - d^{n-k}$ at ∞, and $\varphi(z)$ has $d^n - d^{n-k} + q$ poles and so $2(d^n - d^{n-k} + q)$ zeros and ones altogether in the closed plane CM. Also $\varphi'(z)$ has a pole of order $d^n - d^{n-k} - 1$ at ∞ and at most $2q$ finite poles. Thus

$$N \leq d^n - d^{n-k} + 2q - 1.$$

Hence

$$2(d^n - d^{n-k} + q) \leq N + d^{n-1} + d^{n-2} \leq d^n - d^{n-k} + d^{n-1} + d^{n-2} + 2q - 1.$$

Thus

$$d^n \leq d^{n-1} + d^{n-2} + d^{n-k} - 1 \leq 2d^{n-1} - 1 \leq d^n - 1.$$

This is a contradiction. □

Theorem 5.3 can be modified for rational functions. We obtain Theorem 5.4. The exceptions mentioned in Theorem 5.4 can occur, as Example 5.5 shows.

Theorem 5.4 (Baker 1964) *[2, p. 620] Except in the cases*

$$\begin{aligned} n &= 2, \quad d \in \{2,3,4\} \\ n &= 3, \quad d = 2 \end{aligned}$$

a rational function f of order $d > 1$ has fixed points of exact order n for all $n > 1$.

Example 5.5 (Baker 1965) *[2, p. 621] The exceptions mentioned in Theorem 5.4 can occur.*
$n = 2, d = 2$. Let $f(z) = z^2 - z$ as in Example 5.2.
$n = 2, d = 3$. Let

$$f(z) = \frac{2}{3z^2} + \frac{z}{3} = z - \frac{2(z^3 - 1)}{3z^2}.$$

$n = 2, d = 4$. Let

$$f(z) = -z\frac{1 + 2z^3}{1 - 3z^3}$$

$n = 3, d = 2$. Let

$$f(z) = z + (\eta - 1)\frac{z^2 - 1}{2z}, \qquad \textit{where} \qquad \eta = \exp\frac{2\pi i}{3}.$$

5.2 Transcendental entire functions

For this section, let $f \in \underline{\mathcal{E}}$ and $n \in \mathbb{N}$. We study, when f has fixed points of order n.

First we consider an exceptional function in Example 5.6. Then we follow Fatou [8] and prove Theorems 5.7 and 5.9 by using Picard's theorem. With Nevanlinna theory, we can prove Theorem 5.11. In the end, we state Theorems 5.14 and 5.15, which are out of our scope.

Example 5.6 *Let $f(z) = e^z + z$. Now f has no fixed points.*

Theorem 5.7 (Fatou 1926) *Let $f \in \underline{\mathcal{E}}$ with*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) = 1\} = 0.$$

Now

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f_2, z) = 1\} = \infty.$$

Proof. Consider

$$\varphi(z) = \frac{f_2(z) - z}{f(z) - z},$$

which is entire. Now φ omits 1, since $\varphi(z) = 1$ implies $f(f(z)) = f(z)$, which is $f(w) = w$ for $w = f(z)$. Hence, by Picard's theorem, $\varphi(z) = 0$ has infinitely many solutions. □

Corollary 5.8 *Let $f \in \underline{\mathcal{E}}$. Now f_2 has a fixed point.*

Theorem 5.9 (Fatou 1926) *[8, p. 345] Let $f \in \underline{\mathcal{E}}$, $p \in \mathbb{N}$ and suppose*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f_p, z) = 1\} = 0.$$

Now

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f_{p+1}, z) = 1\} = \infty.$$

Proof. Consider

$$\varphi(z) = \frac{f_{p+1}(z) - z}{f(z) - z},$$

which is entire. As in the proof of Theorem 5.7, φ omits 1 and Picard's theorem yields the claim. □

Corollary 5.10 *Let $f \in \underline{\mathcal{E}}$ and $p \in \mathbb{N}$. Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) \mid p \text{ or } \mathit{Fix}\,(f, z) \mid p + 1\} \geq 1.$$

Theorem 5.11 (Baker 1960) *[1] Let $f \in \underline{\mathcal{E}}$ and $n, k \in \mathbb{N}$. Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) \in \{n, k\}\} = \infty.$$

Theorem 5.11 was preceded by Theorem 5.12, which we state here to obtain historical perspective. We prove Theorem 5.11 by following [7, p. 146]. First we need Lemma 5.13.

Theorem 5.12 (Rosenbloom 1948) *[13] Let $f \in \underline{\mathcal{E}}$ and $n \geq 2$. Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) \mid n\} = \infty.$$

Lemma 5.13 *[7, p. 150, Lemma 5.10.4.] Let $g, h \in \underline{\mathcal{E}}$ and $f = g \circ h$. Then, there exists an exceptional set E of finite Lebesgue measure such that*

$$\lim_{\substack{r \to \infty \\ r \notin E}} \frac{T(h, r)}{T(f, r)} = 0.$$

The claim of Lemma 5.13 holds even without the exceptional set E, but in this case the proof is harder, see [7, p. 147, Lemma 5.10.3.].

Proof. Because $g \in \underline{\mathcal{E}}$, there exists a point $b \in \mathbb{C}$ such that $g(z) = b$ has infinitely many solutions, $a_1, a_2, \ldots$. If no such would b exist, every value $b \in \mathbb{C}$ were deficient for g, a contradiction with Theorem 3.10. Now, let $q \in \mathbb{N}, q \geq 3$. From the First Main Theorem we get

$$T(r, f) + O(1) \geq N(r, f, b) \geq \sum_{j=1}^{q} N(h, a_j, r).$$

For each q we can use the Second Main Theorem to find a radius $r_q > 0$ and an exceptional set E_q with Lebesgue measure $\leq 2^{-q}$ such that

$$\sum_{j=1}^{q} N(h, a_j, r) \geq (q-2)T(h, r) - o(T(h, r))$$

for all $r \geq r_q$ outside E_q. Thus

$$\lim_{\substack{r \to \infty \\ r \notin E_q}} \frac{T(h, r)}{T(f, r)} \leq \frac{1}{q-2}.$$

Let

$$E = \bigcup_{q=3}^{\infty} E_q.$$

Let $\varepsilon > 0$. Now there exists $q \in \mathbb{N}$ such that

$$\frac{1}{q-2} < \varepsilon.$$

Now, for all $r \geq r_q, r \notin E$, we have

$$\frac{T(h, r)}{T(f, r)} = \frac{1}{q-2} < \varepsilon.$$

The assertion follows. □

Proof of Theorem 5.11.[7, p. 149] Suppose that f has only a finite number of fixed points of exact order n, and call them $\zeta_1, \ldots, \zeta_q$. It suffices to prove that f has infinitely many fixed points of exact order k for all $k > n$. So, let $k > n$, and let z_0 be a solution of the equation $f_k(z) = f_{k-n}(z)$. Then

$$f_n(f_{k-n}(z_0)) = f_{k-n}(z_0)$$

and so $\zeta = f_{k-n}(z_0)$ is a fixed point of order n. Thus, either ζ is one of the ζ_j, or else ζ is a fixed point of exact order strictly less than n.

If ζ is not among the ζ_j, then note that

$$f_{k-n+m}(z_0) = f_{k-n}(z_0),$$

where $m \leq n-1$ is the exact order of the fixed point ζ. Thus, any solution to $f_k(z) = f_{k-n}(z)$ is either a solution to $f_{k-n}(z) = \zeta_j$ for some j, or a solution to $f_{k-n+m}(z) = f_{k-n}(z)$ for some $1 \leq m \leq n-1$. Thus,

$$\overline{N}(f_k - f_{k-n}, 0, r) \leq \sum_{m=1}^{n-1} \overline{N}(f_{k-n+m} - f_{k-n}, 0, r) + \sum_{j=1}^{q} \overline{N}(f_{k-n}, \zeta_j, r)$$

$$\leq O\left(\sum_{l=1}^{k-1} T(f_l, r)\right),$$

where the last inequality follows from the First Main Theorem. Clearly

$$f_k = f_l \circ f_{k-l}, \quad 1 \leq l \leq k-1,$$

and f_l and f_{k-l} are transcendental. Hence, we know that

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{T(f_l, r)}{T(f_k, r)} = 0$$

for an exceptional set E of finite Lebesgue measure. The set E can be taken as common for all values $1 \leq l \leq k-1$, by taking an union. Thus the functions

$$a_1(z) = z, \quad a_2(z) = f_{k-n}(z), \quad \text{and} \quad a_3(z) = \infty$$

are all slowly moving with respect to f_k. By Theorem 3.11 we have a sequence of radii $r_i \to \infty$ such that

$$T(f_k, r_i) \leq \sum_{j=1}^{3} \overline{N}(f_k, a_j, r_i) + o(T(f_k, r_i)).$$

However, f_k has no poles since it is entire, and we just saw that

$$\overline{N}(f_k, f_{k-n}, r) = o(T(f_k, r)).$$

Thus, $T(f_k, r_i) \leq \overline{N}(f_k, a_1, r_i) + o(T(f_k, r_i))$. Now, again by the First Main Theorem and Lemma 5.13, the number of fixed points of order strictly less than k is $o(T(f_k, r))$, and so f has infinitely many fixed points of exact order k for all $k > n$. □

Theorem 5.14 improves Theorem 5.11, but is out of our scope. Theorem 5.15 is an example about the case, when two transcendental entire functions are composed. Theorem 5.15 is discussed for example in [3, 4, 9].

Theorem 5.14 (Bergweiler 1991/Baker 1967) *[5, p. 3, Theorem 1] Let $f \in \underline{\mathcal{E}}$ and $n \geq 2$. Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(f, z) = n\} = \infty.$$

Theorem 5.15 (Bergweiler 1990/Gross 1966) *[3, 4] Let $h, g \in \underline{\mathcal{E}}$. Now*

$$\#\{z \in \mathbb{C} : \mathit{Fix}\,(h \circ g, z) = 1\} = \infty.$$

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