A generalization of Livingston's coefficient inequalities for functions with positive real part

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Abstract

For functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ holomorphic in the unit disk, satisfying $\mathrm{Re}\, p(z) > 0$, we generalize two inequalities proved by Livingston [2, 3] and simplify their proofs. One of our results is $|p_n - w p_k p_{n-k}| \le 2\max\{1, |1-2w|\}$, $w \in \mathbb{C}$, while the other involves certain determinants with entries the coefficients p_n.
Next, we consider the functionals $\Phi_{\alpha,\beta}(p) = p_3 - \alpha\, p_1 p_2 - \beta\, p_1^3$ and study the problem of determining the set $\{(\alpha,\beta) \in \mathbb{R}^2 : |\Phi_{\alpha,\beta}(p)| \le 2, \text{ for all } p \in \mathcal{P}\}$.

Let $\mathcal{P}$ denote the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ which are holomorphic in the unit disk $\mathbb{D}$ and satisfy $\mathrm{Re}\, p(z) > 0$. For the Taylor coefficients of functions in $\mathcal{P}$ we have

Carathéodory (1911): $|p_n| \le 2, \quad n \ge 1,$

Livingston (1969): $|p_n - p_k p_{n-k}| \le 2, \quad 0 \le k \le n,$

(1985): $|A_{k,n}(1)| \le 2, \quad k \ge 0,\ n \ge 1,$

where $A_{k,n}(w)$, $w \in \mathbb{C}$ is defined as the $(k+1)\times(k+1)$ determinant:

$$A_{k,n}(w) = \begin{vmatrix} p_{n+k} & p_{n+k-1} & p_{n+k-2} & \cdots & p_{n+1} & p_n \\ wp_1 & 1 & 0 & \ldots & 0 & 0 \\ wp_2 & wp_1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ wp_{k-1} & wp_{k-2} & wp_{k-3} & \ldots & 1 & 0 \\ wp_k & wp_{k-1} & wp_{k-2} & \ldots & wp_1 & 1 \end{vmatrix}.$$

Some examples of initial $A_{k,n}$'s are:

$$\begin{aligned} &A_{0,n} = p_n, \qquad A_{1,n} = p_{n+1} - wp_1p_n, \qquad A_{2,1} = p_3 - 2wp_1p_2 - w^2p_1^3, \\ &A_{2,2} = p_4 - wp_1p_3 - wp_2^2 + w^2p_1^2p_2, \qquad A_{3,1} = p_4 - 2wp_1p_3 - wp_2^2 + 3w^2p_1^2p_2 - w^3p_1^4. \end{aligned} \tag{1}$$

Our main theorems are

Theorem 1. *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then* $|p_n - wp_kp_{n-k}| \le 2\max\{1, |1-2w|\}$.

Theorem 2. *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then* $|A_{k,n}(w)| \le 2\max\{1, |1-2w|^k\}$.

Both theorems are accompanied by discussions of the equality case: for Theorem 1 we provide a complete characterization while for Theorem 2 we provide a characterization in case $|1-2w| < 1$ and a sufficient condition when $|1-2w| \ge 1$. Thinking that this would be too technical for a poster session, we avoid it, and instead we simply consider the case $w = 1$ in Theorem 1. About this case, which is Livingston's original functional, it was mentioned in [2] that equality holds for rotations of the half plane function $\frac{1+z}{1-z}$. We will see that this condition is far from being necessary. We first need some notation.

Denote by $U_n = \{e^{2k\pi i/n} : k = 1, 2, \ldots, n\}$ the set of n-th roots of unity. For $n = 0$ we understand U_0 as $\mathbb{T} = \partial\mathbb{D}$. Also, for a set $E \subset \mathbb{C}$ and a number $a \in \mathbb{C}$ we write $aE = \{az : z \in E\}$.

Herglotz representation: For every $p \in \mathcal{P}$ there is a unique probability measure μ (the *Herglotz measure*) supported on $\mathbb{T}$, such that

$$p(z) = \int_{\mathbb{T}} \frac{1+\lambda z}{1-\lambda z} d\mu(\lambda), \quad z \in \mathbb{D}.$$

Theorem 1 (continuation). Let $p \in \mathcal{P}$, μ be its Herglotz measure and $1 \le k \le n-1$. Then $|p_n - p_kp_{n-k}| = 2$ if and only if one of the following holds: either

(i) $p_k = 0$ and $supp(\mu) \subseteq e^{i\varphi}U_n$ for some $\varphi \in [0, 2\pi)$; or

(ii) $p_k \ne 0$ and

$$supp(\mu) \subseteq (e^{i\varphi}U_{n-2k} \cap e^{i\vartheta_1}U_k) \cup (e^{i\varphi}U_{n-2k} \cap e^{i\vartheta_2}U_k)$$

for some φ, ϑ_1 and ϑ_2 in $[0, 2\pi)$. Except for the degenerate case where the support of μ consists of only one point, the total mass of the measure on each of the two sets of the union is equal to $1/2$.

Proof of the inequality in Theorem 1. First we note that $|1-2w| \le 1$ if and only if $|w|^2 \le \mathrm{Re}\, w$. From the Herglotz representation we easily deduce that $p_n = 2\int_{\mathbb{T}} \lambda^n d\mu(\lambda)$. We compute

$$\begin{aligned} |p_n - wp_kp_{n-k}| &= \left| 2\int_{\mathbb{T}} \lambda^n d\mu(\lambda) - 2wp_k \int_{\mathbb{T}} \lambda^{n-k} d\mu(\lambda) \right| \\ &\le 2\int_{\mathbb{T}} |\lambda^n - wp_k\lambda^{n-k}| d\mu(\lambda) \\ &\le 2\left(\int_{\mathbb{T}} |\lambda^k - wp_k|^2 d\mu(\lambda) \right)^{1/2} \\ &= 2\left(\int_{\mathbb{T}} 1 - 2\mathrm{Re}\,(wp_k\lambda^{-k}) + |wp_k|^2 d\mu(\lambda) \right)^{1/2} \\ &= 2\Big(1 - 2\mathrm{Re}\,(wp_k\overline{p_k}/2) + |wp_k|^2\Big)^{1/2} \\ &= 2\Big(1 + (|w|^2 - \mathrm{Re}\, w)|p_k|^2\Big)^{1/2} \\ &\le 2\max\{1, |1-2w|\}. \end{aligned}$$

Here we used the triangle and Cauchy-Schwarz inequalities. At the last step, in case $|1-2w| > 1$, we made use of Carathéodory's inequality $|p_n| \le 2$. □

A related problem. By the term *rotation* of a function p we mean the function $p_\lambda(z) = p(\lambda z)$, $\lambda \in \mathbb{T}$. We say that a functional $\Phi : \mathcal{P} \to \mathbb{C}$ is *homogeneous* if $\Phi(p_\lambda) = \lambda^n \Phi(p)$ for some $n \ge 1$. We call the number n the *degree of homogeneity* of Φ. We consider homogeneous coefficient functionals of degree n in which p_n appears multiplied by 1. All examples in (1) are of this kind. In general they involve all combinations whose indices sum up to n:

$$\Phi(p) = p_n - w_1\, p_1\, p_{n-1} - w_2\, p_1^2\, p_{n-2} - w_3\, p_2\, p_{n-2} - \ldots - w_m\, p_1^n.$$

We call *minimal* a functional Φ of this kind that satisfies $|\Phi(p)| \le 2$ for all p in $\mathcal{P}$. (No smaller bound is possible since $\frac{1+z^n}{1-z^n} = 1 + 2z^n + \ldots$ belongs to $\mathcal{P}$.)

Looking at Theorems 1 and 2 we are naturally led to the following

Problem. *Characterize all minimal homogeneous coefficient functionals in $\mathcal{P}$.*

Suppose Φ is a homogeneous functional of degree n. If $n = 1$ then $\Phi(p) = p_1$ and it is minimal by Carathéodory's inequality. If $n = 2$ then $\Phi(p) = p_2 - wp_1^2$ and Theorem 1 provides the answer to our problem:

$$\{w \in \mathbb{C} : |p_2 - wp_1^2| \le 2, \text{ for all } p \in \mathcal{P}\} = \{w \in \mathbb{C} : |1-2w| \le 1\}.$$

For $n = 3$ we write $\Phi_{\alpha,\beta}(p) = p_3 - \alpha\, p_1p_2 - \beta\, p_1^3$. As a first step we consider α, β in $\mathbb{R}$. Let

$$E = \{(\alpha,\beta) \in \mathbb{R}^2 : |\Phi_{\alpha,\beta}(p)| \le 2, \text{ for all } p \in \mathcal{P}\}.$$

Proposition. *The set E is convex and invariant under the affine map $T(\alpha,\beta) = (2-\alpha,\ \alpha+\beta-1)$. It satisfies the inclusions $E_1 \subseteq E \subseteq E_2$, where*

$$\begin{aligned} E_1 &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \in [0,2],\ -\frac{\alpha}{2} \le \beta \le \frac{\alpha - \alpha^2}{2}\right\}, \\ E_2 &= \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \in [0,2],\ -\frac{\alpha}{2} \le \beta \le b(\alpha)\right\} \end{aligned}$$

and

$$b(\alpha) = \begin{cases} 1 - \alpha - (3-2\alpha)^3/27, & \text{for } \alpha \in [0, 3/4], \\ (1-\alpha)/2, & \text{for } \alpha \in [3/4, 5/4], \\ (1-2\alpha)^3/27, & \text{for } \alpha \in [5/4, 2]. \end{cases}$$

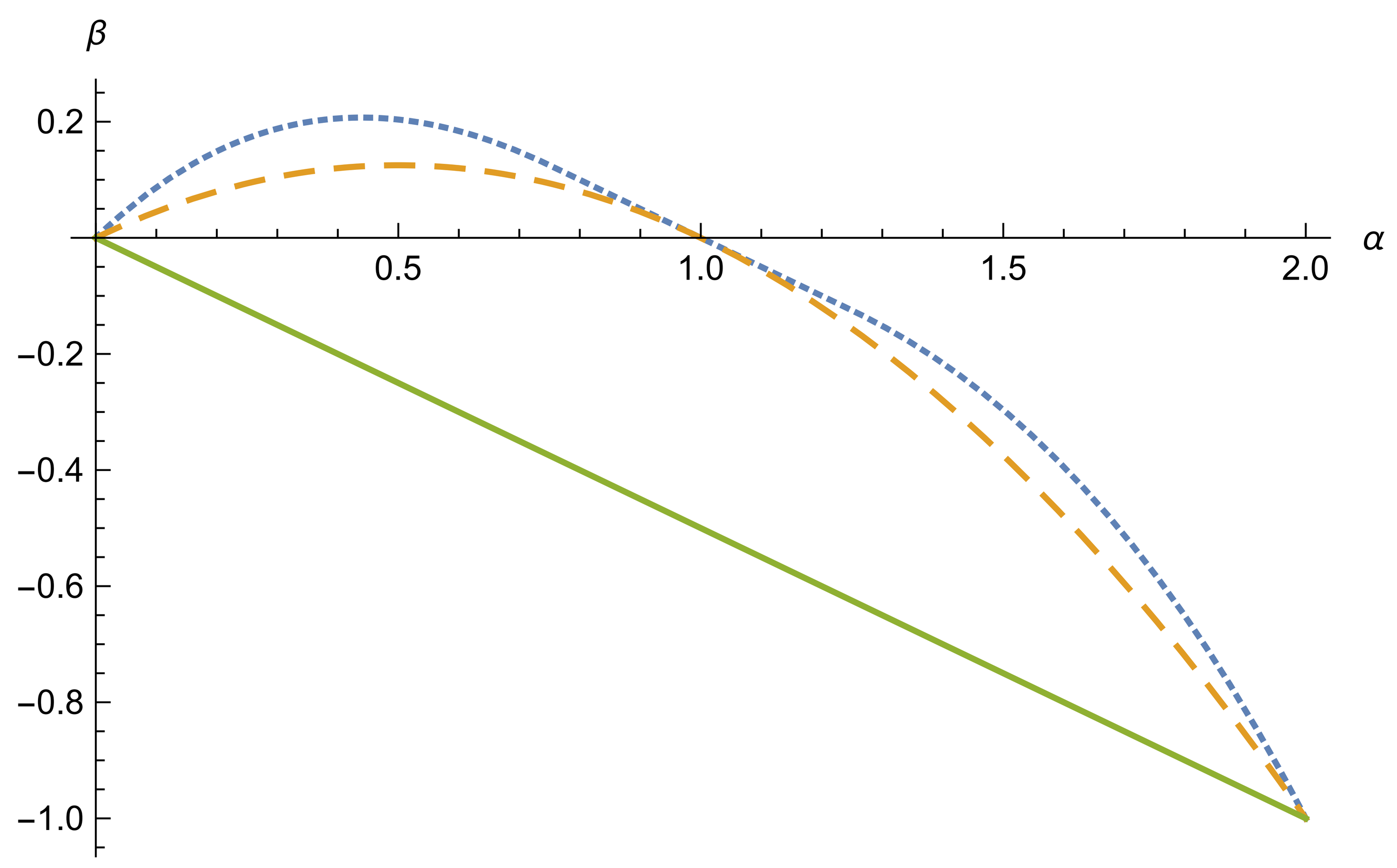

Figure 1: solid (green): lower bound of E, thick dashing (orange): upper bound of E_1, thin dashing (blue): the curve $b(\alpha)$.

Sketch of the Proof. A typical rotation argument allows as to consider $\mathrm{Re}\,\Phi_{\alpha,\beta}$, which is linear with respect to α and β and thus E is convex.

That the set E is invariant under the transformation T comes from the fact that $\mathcal{P}$ is invariant under inversion: $p \in \mathcal{P}$ if and only if $1/p \in \mathcal{P}$.

By Theorem 1 we have that $[0,1] \times \{0\} \subset E$. Also, $T(0,0) = (2,-1)$ must be in E. Hence, by convexity, the triangle with vertices $(0,0)$, $(1,0)$, $(2,-1)$ is contained in E. The fact that $E_1 \subseteq E$ follows from a rather technical lemma.

To determine the set E_2 we first see that E is contained in the strip $\left\{(\alpha,\beta) \in \mathbb{R}^2 : -\frac{\alpha}{2} \le \beta \le \frac{1-\alpha}{2}\right\}$, simply by choosing the half-plane function $\frac{1+z}{1-z}$. Next we prove that E is contained in the vertical strip $[0,2] \times \mathbb{R}$ by making use of $p_\zeta(z) = \frac{1+z\varphi_\zeta(z)}{1-z\varphi_\zeta(z)}$, $\zeta \in \mathbb{D}$, where $\varphi_\zeta(z) = \frac{\zeta - z}{1-\bar{\zeta}z}$ is a disk automorphism. Finally, we let $t \in [0, 1/2)$ and consider the singular measure on $\mathbb{T}$ with point masses on 1 and -1, having weights $1-t$ and t respectively. The function having this as its Herglotz measure is

$$p_t(z) = (1-t)\,\frac{1+z}{1-z} + t\,\frac{1-z}{1+z}.$$

Now, for every t, the inequality $|\Phi_{\alpha,\beta}(p_t)| \le 2$ shows that points (α,β) in E must lie bellow a certain straight line. The envelope of this family of lines yields the remaining of the boundary of E_2, that is, the cubic polynomials in $b(\alpha)$. □

Conjecture. $E = E_2$

References

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