

Linear differential equations with solutions in the growth space H^∞_ω

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Abstract

We give sufficient conditions for analytic coefficients ${\cal A}_k$ of

 $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \dots + A_1(z)f' + A_0(z)f = A_n(z)$

such that all solutions or their derivatives belong to H^{∞}_{ω} . Here H^{∞}_{ω} consists of those analytic functions f in the unit disc \mathbb{D} for which $|f(z)|\omega(z)$ is uniformly bounded, and $\omega : \mathbb{D} \to (0,\infty)$ is radial and measurable and satisfies certain regularity conditions.

The following theorem is a simplified version of our main result.

Theorem 1. Let f be a solution of (1) where $A_n \equiv 0$, and suppose that ω is a weight satisfying (3). Denote $\omega_0 = \omega$ and $\omega_k(z) = \omega(z)(1 - |z|)^k$ for $k \in \mathbb{N}$. Then the following assertions hold:

(a) If ω_k satisfies (4) for all $k = 1, 2, \ldots, n$, and

$$E := P_n \left(\|A_0\|_{H_n^{\infty}} + m \sum_{k=1}^{n-1} k! (1+\varepsilon)^k \|A_k\|_{H_{n-k}^{\infty}} \right) < 1,$$

By Theorem 2 we easily obtain the following result regarding the important special case (2) of (1), where the coefficient A is given by a power series.

Corollary 3. Let f be a solution of the differential equation (2), where $A(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$. Then the following assertions hold: (a) If $\alpha \in (0,1)$ and $|a_k| < \alpha(1-\alpha) \frac{\Gamma(k+\alpha+1)}{k! \Gamma(\alpha+1)}$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^{\alpha}$. (b) If $|a_k| < \frac{1}{k!} \int_1^2 \frac{\Gamma(k+x)}{\Gamma(x)} dx$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}$.

INTRODUCTION We study the growth of solutions of the differential equation $f^{(n)} + A_{n-1}(z)f^{(n-1)} + \dots + A_1(z)f' + A_0(z)f = A_n(z), \quad n \ge 2, (1)$

where $A_0(z), A_1(z), \ldots, A_n(z)$ are analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane \mathbb{C} , denoted by $A_0, A_1, \ldots, A_n \in \mathcal{H}(\mathbb{D})$ for short. In particular, we are interested in the case where the solutions or their derivatives belong to

$$H^{\infty}_{\omega} = \left\{ g \in \mathcal{H}(\mathbb{D}) : \|g\|_{H^{\infty}_{\omega}} := \sup_{z \in \mathbb{D}} |g(z)|\omega(z) < \infty \right\}.$$

Here ω is a (radial) weight, which means that $\omega : \mathbb{D} \to (0, \infty)$ is bounded, measurable and satisfies $\omega(z) = \omega(|z|)$ for all z. The case $\omega(z) = (1 - |z|)^p$, $p \in (0, \infty)$ is denoted simply by H_p^{∞} . We also consider the derivatives of the solutions and denote $\mathcal{B}^{\alpha} = \{f : f' \in H_{\alpha}^{\infty}\}$ with $\mathcal{B} = \mathcal{B}^1$ being the classical Bloch space.

The growth of solutions of (1) depends almost entirely on the growth of the coefficients A_k . Consider, for example, the differential equation

f'' + A(z)f = 0.

Then all solutions are bounded when $A \in H_{2-\varepsilon}^{\infty}$ for any $\varepsilon \in (0,2)$ [6, Corollary 3.16]. On the other hand, if $A \in H_{2+\varepsilon}^{\infty} \setminus \bigcup_{p<2+\varepsilon} H_p^{\infty}$ for $\varepsilon \in (0,\infty)$, then the order of growth of any nontrivial solution is $\varepsilon/2$ [2, Theorem 1.4(c)]. To ensure that the growth of the solutions is somewhere between these two extremal cases, the growth condition for the coefficient A(z) needs to be more delicate. For example, if $||A||_{H_2^{\infty}}$ or $\sup_{z\in\mathbb{D}}[|A(z)|(1-|z|)^2|\log(1-|z|)|]$ is small enough, then all solutions belong to H_p^{∞} or \mathcal{B} (respectively), see Example 5. where m is as in (3), then $f \in H^{\infty}_{\omega}$. (b) If ω_k satisfies (4) for all $k = 1, 2, \dots, n-1$, and $F := P_{n-1} \left(\|A_0\|_{H^{\infty}_{\omega_{n-1},\omega}} + \|A_1\|_{H^{\infty}_{n-1}} \right)$

 $F := P_{n-1} \left(\|A_0\|_{H^{\infty}_{\omega_{n-1},\omega}} + \|A_1\|_{H^{\infty}_{n-1}} + m \sum_{k=1}^{n-2} k! (1+\varepsilon)^k \|A_{k+1}\|_{H^{\infty}_{n-k-1}} \right) < 1,$

where m is as in (3), then $f' \in H^{\infty}_{\omega}$.

We make the following remarks about Theorem 1.

- (i) Theorem 1 generalizes to the non-homogenous equation (1):
 - (1) In Theorem 1(a) the condition $A_n \equiv 0$ can be replaced by the condition $A_n \in H^{\infty}_{\omega_n}$.
 - (2) In Theorem 1(b) the condition $A_n \equiv 0$ can be replaced by the condition $A_n \in H^{\infty}_{\omega_{n-1}}$.
- (ii) If one of the following conditions holds, then the assumption that ω satisfies (3) is not necessary.

(1) In Theorem 1(a) $A_{n-1} \equiv A_{n-2} \equiv \ldots \equiv A_1 \equiv 0$. (2) In Theorem 1(b) $A_{n-1} \equiv A_{n-2} \equiv \ldots \equiv A_2 \equiv 0$.

Special cases and examples

(c) If $\alpha \in (1, \infty)$ and $|a_k| < \alpha(\alpha - 1)(1 + k)$ for $k \in \mathbb{N} \cup \{0\}$, then $f \in \mathcal{B}^{\alpha}$.

Using [7, Theorem 14] we can also state the following consequence of Theorem 2 which concerns the case where A(z) is a gap series.

Corollary 4. Let f be a solution of the differential equation (2), where

$$A(z) = C \sum_{k=0}^{\infty} a_k z^{n_k}, \quad 1 < q \leq \frac{n_{k+1}}{n_k}, \quad k, n_k \in \mathbb{N},$$

and C > 0 is a constant independent of z. Then the following assertions hold:

- (a) If C is small enough and $\limsup_{k\to\infty} |a_k| n_k^{-1-\alpha} < \infty$ for $\alpha \in (0,1)$, then $f \in \mathcal{B}^{\alpha}$.
- (b) If C is small enough and $\limsup_{k\to\infty} |a_k| n_k^{-2} \log n_k < \infty$, then $f \in \mathcal{B}$.
- (c) If C is small enough and $\limsup_{k\to\infty} |a_k| n_k^{-2} < \infty$, then $f \in \mathcal{B}^{\alpha}$ for $\alpha \in (1, \infty)$.

We conclude with an example showing that Theorems 1 and 2 are sharp in the sense that the assumptions E < 1 and F < 1 cannot be relaxed to $E < 1 + \varepsilon$ and $F < 1 + \varepsilon$, respectively, for any $\varepsilon \in (0, \infty)$.

Conditions on weights

We consider weights ω and ω_k that satisfy the conditions

 $\limsup_{r \to 1^{-}} \frac{\omega(r)}{\omega\left(\frac{1+\varepsilon r}{1+\varepsilon}\right)} < m$

for some constants $\varepsilon \in (0,\infty)$ and $m=m(\omega,\varepsilon)\in (0,\infty)$, and

$$\limsup_{r \to 1^{-}} \int_0^r \frac{ds}{\omega_k(s)} \,\omega_{k-1}(r) < M_k < \infty, \quad k = 1, 2, \dots, n, \quad (4)$$

where $M_k = M_k(\omega_k, \omega_{k-1}) > 0$. Regarding the constants M_k we also write $P_n := \prod_{k=1}^n M_k$. It should be noted that conditions (3) and (4) are independent and have the following properties.

(i) It is possible that (3) holds for some ε but not for all. However, if (3) holds for some ε , then it holds for some arbitrarily small ε .

(ii) If ω is nonincreasing and (3) holds for some ε , then it holds for all

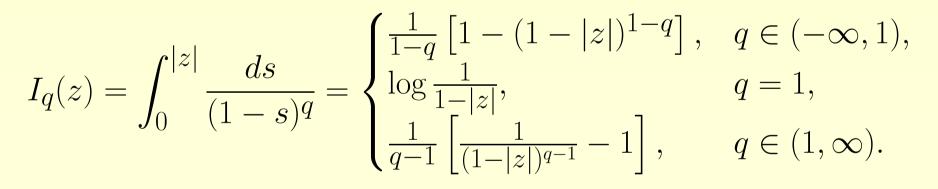
We denote by
$$H_{p,q}^{\infty}$$
 the space of functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|_{H^{\infty}_{p,q}} = \sup_{z \in \mathbb{D}} \left[|g(z)|(1-|z|)^p I_q(z) \right] < \infty,$$

where $p,q\in(0,\infty)$ and

(2)

(3)



Note that $H_{p,q}^{\infty} = H_{p+1-q}^{\infty}$ when 1 < q < p+1, and $H_{p,q}^{\infty} = H_p^{\infty}$ when $q \in (0,1)$ and $p \in (0,\infty)$.

The following result is an important special case of Theorem 1.

Theorem 2. Let f be a solution of the differential equation (1) where $A_n \equiv 0$. Then the following assertions hold: (a) If, for $p \in (0, \infty)$,

$$\begin{split} E &:= \prod_{j=1}^{n} \frac{1}{p+j-1} \left(\|A_0\|_{H_n^{\infty}} + \sum_{k=1}^{n-1} k! \frac{(k+p)^{k+p}}{k^k p^p} \|A_k\|_{H_{n-k}^{\infty}} \right) \\ &< 1, \end{split}$$
then
$$\|f\|_{H_p^{\infty}} \leq \frac{|f(0)| + \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{1}{p+j-1} |f^{(k)}(0)|}{1-E}. \end{split}$$

Example 5. Let us consider the differential equation (2). (a) If $A(z) = -(p + \alpha)(p + \alpha + 1)(1 - z)^{-2}$ for $p \in (0, \infty)$ and $\alpha \in [0, \infty)$, then (2) has a solution base $\{f_1, f_2\}$, where

 $f_1(z) = (1-z)^{-p-\alpha}$ and $f_2(z) = (1-z)^{p+\alpha+1}$.

Hence, if $\alpha = 0$, then $||A||_{H_2^{\infty}}/p(p+1) = 1$ and all solutions belong to H_p^{∞} space. On the other hand, for any $\varepsilon \in (0, \infty)$, we find $\alpha = \alpha(\varepsilon) \in (0, \infty)$ such that $||A||_{H_2^{\infty}}/p(p+1) \in (1, 1+\varepsilon)$ and $f_1 \notin H_p^{\infty}$.

(b) If $A(z) = -\alpha(1-z)^{-2} \left((\alpha - 1) \left(\log \frac{e}{1-z} \right)^{-2} + \left(\log \frac{e}{1-z} \right)^{-1} \right)$ for $\alpha \in [1, \infty)$, then (2) has a solution base $\{f_1, f_2\}$, where

 $f_1(z) = \left(\log \frac{e}{1-z}\right)^{\alpha}$ and $f_2(z) = f_1(z) \int_0^z \frac{d\zeta}{f_1(\zeta)^2}.$

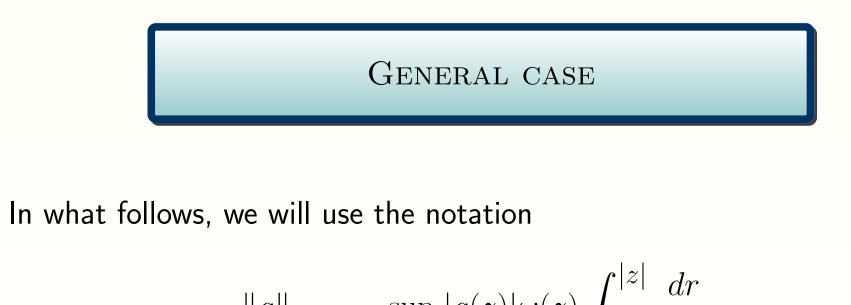
Here $|f'_2(z)| \leq (\log \frac{e}{2})^{-\alpha} |f'_1(z)| + (\log \frac{e}{2})^{-2\alpha}$ for $z \in \mathbb{D}$. Hence, for $\alpha = 1$, we have $||A||_{H^{\infty}_{2,1}} = 1$ and that all solutions belong to \mathcal{B} . However, for any $\varepsilon \in (0,\infty)$, we find $\alpha = \alpha(\varepsilon) \in (1,\infty)$ such that $||A||_{H^{\infty}_{2,1}} \in (1,1+\varepsilon)$ and $f_1 \notin \mathcal{B}$.

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 ε . Hence, in this case (3) is equivalent to the doubling condition $\omega(r) \leq m\omega\left(\frac{1+r}{2}\right)$ when $r \in [0,1)$ is large enough.

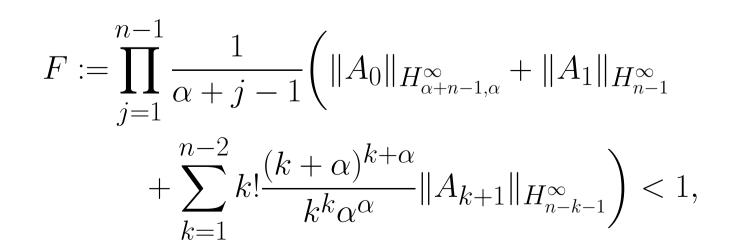
(iii) If $\mu : [0,1) \to (0,\infty)$ is nonincreasing and ω satisfies (4), then $\omega_k \mu$ satisfies (4) with $M' \leq M_k$. Moreover, if $\omega \mu^k$ satisfies (4) for k = 1, then it satisfies the condition for all $k \in \mathbb{N}$ with a nonincreasing sequence of constants $(M_k)_{k=1}^{\infty}$. The typical example $\omega_k(r) = \omega(r)(1-r)^k$ is obtained by the choice $\mu(r) = 1-r$.



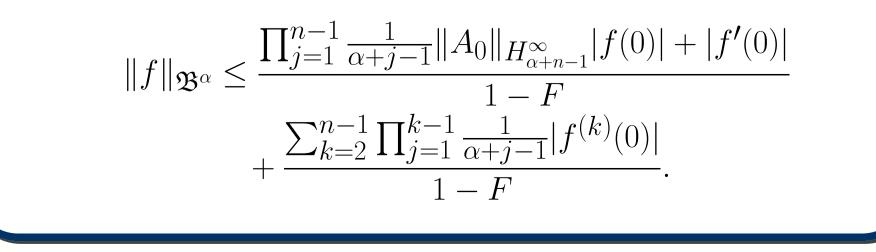
$$\|g\|_{H^{\infty}_{\omega,\mu}} = \sup_{z \in \mathbb{D}} |g(z)|\omega(z) \int_{0}^{r} \frac{ar}{\mu(r)}$$

where $g \in \mathcal{H}(\mathbb{D})$ and ω and μ are weights.

(b) If, for $\alpha \in (0,\infty)$,



then



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