On Becker's univalence criterion

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Abstract

We study locally univalent functions f analytic in the unit disc $\mathbb{D}$ of the complex plane such that

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D},$$

holds for all $z \in \mathbb{D}$, for some $0 < C < \infty$. If $C \le 1$, then f is univalent by Becker's univalence criterion. We discover that for $1 < C < \infty$ the function f remains to be univalent in certain horodiscs. Sufficient conditions which imply that f is bounded, belongs to the Bloch space or belongs to the class of normal functions, are discussed. Moreover, we consider generalizations for locally univalent harmonic functions.

Introduction

Let us recall some classical univalence criteria. From now on, for simplicity, let f be analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Moreover, assume that f is locally univalent, that is, $f'(z) \neq 0$ for $z \in \mathbb{D}$.

The Schwarzian derivative of f is defined by setting

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Since f' is nonvanishing, $S(f)$ is an analytic function.

According to the famous Nehari univalence criterion [10, Theorem 1], if

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2, \quad z \in \mathbb{D}, \tag{1}$$

then f is univalent. The result is sharp by an example by Hille [7, Theorem 1].

Binyamin Schwarz [12] showed that if $f(a) = f(b)$ for some $a \neq b$, then

$$\max_{\zeta \in \langle a,b\rangle} |S(f)(\zeta)|\,(1-|\zeta|^2)^2 > 2. \tag{2}$$

Here $\langle a, b\rangle = \{\varphi_a(\varphi_a(b)t) : 0 \le t \le 1\}$ is the hyperbolic segment between a and b and

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$$

is an automorphism of the unit disc. Condition (2) implies that if

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2, \quad r_0 \le |z| < 1, \tag{3}$$

for some $0 < r_0 < 1$, then f has finite valence [12, Corollary 1].

Chuaqui and Stowe [4, p. 564] asked whether

$$|S(f)(z)|\,(1-|z|^2)^2 \le 2 + C(1-|z|), \quad z \in \mathbb{D}, \tag{4}$$

where $0 < C < \infty$ is a constant, implies that f is of finite valence. The question remains open despite of some progress achieved by Gröhn and Rättyä in [6]. Steinmetz [13, p. 328] showed that if (4) holds, then f is normal, that is, the family $\{f \circ \varphi_a : a \in \mathbb{D}\}$ is normal in the sense of Montel. Equivalently, $\sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2) < \infty$.

The pre-Schwarzian derivative of f is defined as $P(f) = f''/f'$. Conditions (1)-(4) have analogues stated in terms of the pre-Schwarzian derivative.

The famous Becker univalence criterion [1, Korollar 4.1], states that if

$$|zP(f)|\,(1-|z|^2) \le \rho, \quad z \in \mathbb{D}, \tag{5}$$

for $\rho \le 1$, then f is univalent in $\mathbb{D}$. The right-hand-side constant 1 is sharp, see [2, Satz 6] and [5].

Becker and Pommerenke proved recently that if

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) < \rho, \quad r_0 \le |z| < 1, \tag{6}$$

for $\rho < 1$ and some $r_0 \in (0,1)$, then f has finite valence [3, Theorem 3.4].

It is an open problem, what happens in the case of equality $\rho = 1$ in (6). Moreover, the sharp inequality corresponding to (2), in terms of the pre-Schwarzian, has not been found yet.

In this paper, we consider the growth condition

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{7}$$

where $0 < C < \infty$ is an absolute constant. Analogously to the Chuaqui-Stowe question, the most interesting question is whether (7) implies that f is of finite valence. We have obtained some partial results.

Conversely to Becker's criterion, each analytic and univalent function f in $\mathbb{D}$ satisfies (5) for $\rho = 6$. This follows from the sharp inequality

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{4|z|}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{8}$$

see [11, p. 21]. Also condition (7) implies growth estimates for f. These estimates may be calculated analogously to [3] and [9]. In particular, condition (7) implies that f is bounded. Slightly relaxed versions of inequality (7) imply that

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty \quad \text{or} \quad \sup_{z\in\mathbb{D}} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2) < \infty.$$

See [8] for details.

Results

First, we state a local version of Becker's univalence criterion. By Becker's criterion and its converse, the following result is sharp.

Theorem 1 *Let f be analytic and locally univalent in $\mathbb{D}$ and let $\zeta \in \partial\mathbb{D}$. If there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that*

$$\left|\frac{f''(w_n)}{f'(w_n)}\right|(1-|w_n|^2) \to c \tag{9}$$

for some $c \in (6, \infty]$, then for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$.
Conversely, if for each $\delta > 0$ there exists a point $w \in f(\mathbb{D})$ such that at least two of its distinct preimages belong to $D(\zeta, \delta) \cap \mathbb{D}$, then there exists a sequence $\{w_n\}$ of points in $\mathbb{D}$ tending to ζ such that (9) holds for some $c \in [1, \infty]$.

Theorem 2 *Let f be analytic and locally univalent in $\mathbb{D}$. Assume that*

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 1 + C(1-|z|), \quad z \in \mathbb{D}, \tag{10}$$

for some $0 < C < \infty$. If $0 < C \le 1$, then f is univalent in $\mathbb{D}$. If $1 < C < \infty$, then f is univalent in all discs

$$D(ae^{i\theta}, 1-a), \quad 0 \le \theta < 2\pi,$$

where $a = 1 - (1+C)^{-2} \in (0,1)$.

Theorem 3 *Let f be analytic in $\mathbb{D}$ and univalent in all Euclidean discs*

$$D\left(\frac{C}{1+C}e^{i\theta}, \frac{1}{1+C}\right), \quad e^{i\theta} \in \partial\mathbb{D},$$

for some $0 < C < \infty$. Then

$$\left|\frac{f''(z)}{f'(z)}\right|(1-|z|^2) \le 2 + 4(1+K(z)), \quad z \in \mathbb{D},$$

where $K(z) \asymp (1-|z|^2)$ as $|z| \to 1^-$.

Example 4 Let $f = f_{C,\zeta}$ be a locally univalent analytic function in $\mathbb{D}$ such that $f(-1) = 0$ and

$$f'(z) = -i\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} e^{\frac{C\zeta z}{2}}, \quad \zeta \in \partial\mathbb{D}, z \in \mathbb{D}.$$

Then

$$\frac{f''(z)}{f'(z)} = \frac{1}{1-z^2} + \frac{C\zeta}{2},$$

hence (7) holds and f is univalent in $\mathbb{D}$ if $C \le 1$ by Becker's univalence criterion. If f is univalent, then we obtain for $\zeta = 1$,

$$1 \ge \frac{|f'(x)|}{|k'(x)|} = \frac{e^{\frac{Cx}{2}}(1-x)^{5/2}}{(1+x)^{1/2}} \sim \frac{1+Cx/2}{1+3x}, \quad x \to 0^+,$$

where $k(z) = z/(1-z)^2$ is the Koebe function, see [11, p. 21].
Therefore, if $C > 6$, then f is not univalent.
The boundary curve $\partial f(\mathbb{D})$ has a cusp at $f(-1) = 0$. When $\zeta = -i$, the cusp has its worst behavior, and by numerical calculations the function f is not univalent if $C > 2.21$.
Moreover, as C increases, the valence of f increases. see Figures 1 and 2.
The curve $\{f(e^{it}) : t \in (0, \pi]\}$ is a spiral unwinding from $f(-1)$. We may calculate the valence of f by counting how many times $h(t) = \mathrm{Re}(f(e^{it}))$ changes its sign on $(0, \pi]$. Numerical calculations suggest that the valence of f is approximately equal to $\frac{100}{63}C$.

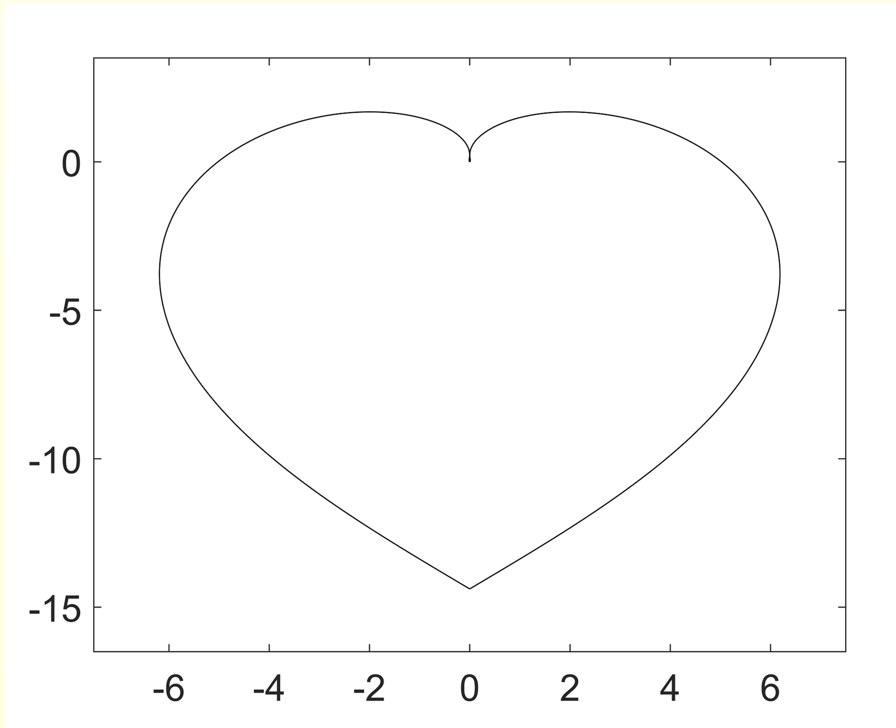

Figure 1. Image domain $f(\mathbb{D})$ for $C = 2.21$ and $\zeta = -i$. Here $\partial f(\mathbb{D})$ is a simple closed curve and f is univalent.

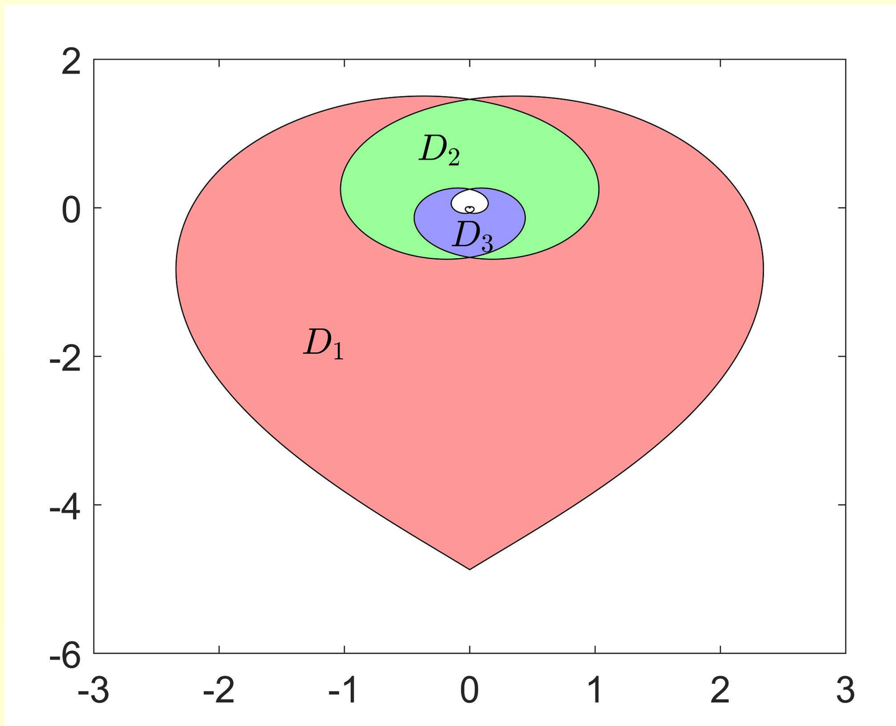

Figure 2. Image domain $f(\mathbb{D})/10^{12}$ for $C = 30$ and $\zeta = -i$. Here $\partial f(\mathbb{D})$ intersects itself multiple times. The valence of the simply connected domain D_j under f is j, for $j = 1, 2, 3$, respectively.

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