Chapter 13

1. Since $f'(0) = 1$ and $|f'(z)| \le M \in (0, \infty)$ $\forall z \in \mathbb{D}$, then $M \ge 1$.

2. Let f be analytic in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = a \in \mathbb{C} \setminus \{0\}$ and $|f'(z)| \le M \in (0, \infty)$ $\forall z \in \mathbb{D}$. If $g(z) := \frac{f(z)}{a}$, then g is analytic in $\mathbb{D}$, $g(0) = 0$, $g'(0) = 1$ and $|g'(z)| \le \frac{M}{|a|} \in (0, \infty)$ $\forall z \in \mathbb{D}$. Now Lemma 13.3 implies that $D\left(0, \frac{1}{2(\frac{M}{|a|}+1)}\right) \subset g(\mathbb{D}) = \frac{f(\mathbb{D})}{a}$, and thus

$$D\left(0, \frac{|a|^2}{2(M + |a|)}\right) \subset f(\mathbb{D}).$$

3. Let $f : D(a, r) \to \mathbb{C}$ be analytic such that $|f'(z) - f'(a)| < |f'(a)|$ for all $z \in D(a, r) \setminus \{a\}$. Then it is clear that $f'(z) \ne 0$ for all $z \in D(a, r)$, and so f is univalent in $D(a, r)$.

Chapter 14

1. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ be arbitrary, and denote

$$a^+_{n,m} = \log(\sqrt{n} + \sqrt{n-1}) + 2\pi i m$$

and

$$a^-_{n,m} = \log(\sqrt{n} - \sqrt{n-1}) + 2\pi i m.$$

It suffices to show that $|a^+_{n,m} - a^+_{n+1,m+1}| < 10$, $|a^-_{n,m} - a^-_{n+1,m+1}| < 10$ and $|a^+_{1,m} - a^-_{1,m+1}| < 10$. The last one is trivial, since

$$|a^+_{1,m} - a^-_{1,m+1}| = |i2\pi(m - (m+1))| = 2\pi < 10.$$

To prove the first one, write

$$\begin{aligned} |a^+_{n,m} - a^+_{n+1,m+1}| &= \left|\log \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} - i2\pi\right| \\ &\le \log \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n} + \sqrt{n-1}} + 2\pi. \end{aligned}$$

Now

$$1 < \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} \leq \sqrt{2}+1 < e^{10-2\pi}$$

(if $g(x) = \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x}+\sqrt{x-1}}$, then

$$g'(x) = \frac{\left(\frac{1}{2\sqrt{x+1}}+\frac{1}{2\sqrt{x}}\right)(\sqrt{x}+\sqrt{x-1}) - (\sqrt{x+1}+\sqrt{x})\left(\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x}+\sqrt{x-1})^2} < 0,$$

so $g(x) \leq g(1)$ for all $x \geq 1$). Thus $|a^+_{n,m} - a^+_{n+1,m+1}| < \log e^{10-2\pi} + 2\pi = 10$. Similarly

$$\begin{aligned} |a^-_{n,m} - a^-_{n+1,m+1}| &= \left|\log \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} - i2\pi\right| \\ &\leq \log \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} + 2\pi. \end{aligned}$$

Since

$$\begin{aligned} \frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} &= \left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n+1}-\sqrt{n}} \frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n}}\right) \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}} \\ &= \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}, \end{aligned}$$

the calculation done above shows that $|a^-_{n,m} - a^-_{n+1,m+1}| < \log e^{10-2\pi} + 2\pi = 10$, and we are done.

2. The exercise was to prove the following lemma needed in the proof of Schottky's theorem.

Lemma 1. *Let $z \in \mathbb{D}$ and let H be an analytic function in $\mathbb{D}$ such that $H'(z) \neq 0$. Then the range of the function*

$$h(\xi) = \frac{H(\xi) - H(z)}{H'(z)}, \qquad \xi \in D(z, 1-|z|),$$

covers a disc of radius $\frac{1-|z|}{13}$.

Proof. Consider the function

$$f(\xi) = \frac{h\left(\xi(1-|z|)+z\right)}{1-|z|} = \frac{H\left(\xi(1-|z|)+z\right) - H(z)}{(1-|z|)H'(z)}, \qquad \xi \in \mathbb{D}.$$

Then f is analytic in $\mathbb{D}$ and

$$f'(\xi) = \frac{H'\left(\xi(1-|z|)+z\right)(1-|z|)}{(1-|z|)H'(z)} = \frac{H'\left(\xi(1-|z|)+z\right)}{H'(z)},$$

and so $f'(0) = 1$. Thus, by the proof of Bloch-Landau's theorem, the range of f contains a disc of radius $\frac{1}{13}$. Hence, the range of h contains a disc of radius $\frac{1-|z|}{13}$. □

Chapter 15

1. Let D be a simply connected domain and suppose that f is an analytic function on D which does not attain 0 or 1. Then, by Picard's little theorem, $D \neq \mathbb{C}$ if f is non-constant. Hence we can without loss of generality suppose that $D = \mathbb{D}$, and so the assumptions of Schottky's theorem hold. On the other hand, by the proof of the Schottky's theorem, there exists $g \in H(\mathbb{D})$ such that

$$\log f = \frac{\pi i}{2}(e^{2g} + 2 + e^{-2g}),$$

and so

$$\begin{aligned} f &= \exp\left(\frac{\pi i}{2}(e^{2g} + 2 + e^{-2g})\right) \\ &= -\exp\left(i\pi\frac{e^{2g} + e^{-2g}}{2}\right) \\ &= -\exp(i\pi\cosh(2g)). \end{aligned}$$