1. If $f \neq 0$, then $|f|$ attains the maximum and the minimum in ∂D by the Maximum and the Minimum modulus principles. Hence if $|f(z)| = c$ for all $z \in \partial D$, then $|f|$ is a constant. Thus f is a constant by the Maximum modulus principle.

2. Let $A = \{z : |f(z)| < c\}$. By continuity of $|f|$, $A = \mathrm{Int}(A)$. Secondly $\{z : |f(z)| = c\} = \partial A$ by the Maximum modulus principle and continuity of $|f|$. Hence the assertion follows.

3. Let p be non-constant and $A = \{z : |p(z)| < c\}$. By the fact that $\lim_{|z|\to\infty} |p(z)| = \infty$ it is clear that A is bounded. A may be disconnected. In that case, $A = \cup\{A_j\}$, where the sets A_j are disjoint domains and $\partial A_j \subset \{z : |p(z)| = c\}$ for each j by the Exercise 2. (The sets A_j are called the *components* of A.) Let A_j be arbitrary. If A_j does not contain a zero of p, then p is a constant in A_j by the Exercise 1. Then, as a polynomial, p is a constant everywhere. This is a contradiction and the assertion follows.

4. Let $A = \{z : |p(z)| < c\}$. By the solution of the Exercises 2 and 3, it is clear that A is a union of disjoint bounded domains, the components of A, and $\partial A = \{z : |p(z)| = c\}$. Thus ∂A is a union of closed paths. By the Exercise 3 all components of A contain a zero of p. Polynomial p has finitely many zeros, thus the assertion follows. When c increases, the paths unite and finally we have only one large path, whose size increases all the time.

5. Function f/g is analytic in $\overline{D(0,r)}$. Hence by the Exercise 1 and the assumption $|f(z)/g(z)| = 1$ for all $z \in \partial D(0,r)$, $f/g \equiv \lambda$, where λ is a constant and $|\lambda| = 1$. Thus the assertion follows.

1. Simple calculations give the following inequalities:

$$
\begin{aligned}
M(r,f) &\leq \frac{2r}{r-R} \min_{z=R} \Re f(z) + \frac{R+r}{R-r}|f(0)|, \\
M(r,f) &\leq \frac{2r}{r-R} \min_{z=R} \Im f(z) + \frac{R+r}{R-r}|f(0)|, \\
M(r,f) &\leq \frac{2r}{R-r} \max_{z=R} \Im f(z) + \frac{R+r}{R-r}|f(0)|.
\end{aligned}
$$

2. By using

$$
g_2(z) = \frac{f^2(z)}{2A^2(R,f) - f^2(z)}
$$

instead of g in the proof of the BC-inequality, we obtain

$$
M(r,f) \leq \sqrt{\frac{2r}{R-r}}\Big(A(R,f) + |f(0)|\Big) + |f(0)|.
$$

3. Let $f(z) = -i\log(1-z)$ and $0 < r < R < 1$. Then $|f(0)| = 0$, $A(R,f) = \max_{|z|=R} \operatorname{Arg}(1-z) = C$, where $C \in (0, \frac{\pi}{2})$. Thus

$$M(r,f) \geq \log\frac{1}{1-r} = \frac{A(R,f)}{C}\log\frac{1}{1-r}.$$

Hence on the right hand side of a BC-type-inequality we have at least the factor $-\log(1-r)$. In the lectures and the solution of the Exercise 2 we had factors $1/(R-r)$ and $\sqrt{1/(R-r)}$ respectively. These factors increase faster than $-\log(1-r)$ as r grows.