1. Let $g(x) = \log f(x)$. Becauce $g''(x) = \frac{f''(x)f(x)-(f'(x))^2}{f(x)^2}$, g' is non-decreasing if and only if $f''(x)f(x) - (f'(x))^2 \geq 0$. Thus f is logarithmically convex if and only if $f''(x)f(x) - (f'(x))^2 \geq 0$ by the Proposition 3.3.

2. Suppose that $f : (a, b) \to \mathbb{R}$ is convex function, that is,

$$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$$

for all $x_1, x_2 \in (a, b)$, $x_1 < x_2$, and $0 < t < 1$. Let $a < x_1 < x_2 < x_3 < b$, and choose $t = \frac{x_2-x_1}{x_3-x_1}$. Then $1 - t = \frac{x_3-x_2}{x_3-x_1}$ and $x_2 = tx_3 + (1-t)x_1$, and thus

$$(x_3 - x_1)f(x_2) \leq (x_2 - x_1)f(x_3) + (x_3 - x_2)f(x_1) + x_2f(x_2) - x_2f(x_2),$$

from which we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

On the other hand,

$$f(x_2) \leq \frac{x_2 - x_1}{x_3 - x_1}f(x_3) + \frac{(x_3 - x_1) - (x_2 - x_1)}{x_3 - x_1}f(x_1),$$

and thus

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

By applying these inequalities to points $a < x_1 < x_2 < x < x+h_1 < x+h_2 < b$ we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x + h_1) - f(x)}{h_1} \leq \frac{f(x + h_2) - f(x)}{h_2}.$$

Hence the function $F_1(h) = \frac{f(x+h)-f(x)}{h}$ is bounded bellow and increasing in some interval $(0, \delta)$ and thus the limit $\lim_{h\to 0^+} F_1(h) = f'_+(x)$ exists. Similarly, by writing the convextity condition as

$$f(x_2) \leq \frac{(x_3 - x_1) - (x_3 - x_2)}{x_3 - x_1}f(x_3) + \frac{x_3 - x_2}{x_3 - x_1}f(x_1),$$

we obtain

$$\frac{f(x_3) - f(x_2)}{x_3 - x_2} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

Hence, if $a < x - h_2 < x - h_1 < x < x_1 < x_2 < b$, we have

$$\frac{f(x) - f(x - h_2)}{h_2} \leq \frac{f(x) - f(x - h_1)}{h_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Thus the function $F_2(h) = \frac{f(x)-f(x-h)}{h}$ is bounded above and decreasing in some interval $(0, \delta)$ and hence the limit $\lim_{h\to 0^+} F_2(h) = f'_-(x)$ exists.

Now let $x \in (a, b)$. Since we know that $f'_+(x)$ and $f'_-(x)$ exist, we may write

$$\lim_{h\to 0^+} f(x + h) - f(x) = \left(\lim_{h\to 0^+} \frac{f(x + h) - f(x)}{h}\right)\left(\lim_{h\to 0^+} h\right) = 0$$

and

$$\lim_{h\to 0^-} f(x+h)-f(x) = \left(\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0^-} h\right) = 0.$$

Hence f is continuous at x. If f is convex in a closed interval $[a,b]$, it is not nessessarily continuous at the endpoints a and b. An easy counterexample is the function $f:[0,1]\to\mathbb{R}$ such that $f(0)=f(1)=1$ and $f(x)=0$ for all $0<x<1$.

3. a) Let $f:[a,b]\to\mathbb{R}$ be convex, $x_1,...,x_n\in[a,b]$ and $t_1,...,t_n\geq 0$ such that $\sum_{i=1}^n t_i = 1$. If $n=1$, the assertion is trivially true and if $n=2$ the assertion is true by the definition of convex functions. Suppose $f(\sum_{i=1}^n t_i x_i)\leq\sum_{i=1}^n t_i f(x_i)$ for some $n\in\mathbb{N}$. Now

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} t_i x_i\right) &= f\left(t_{n+1}x_{n+1} + (1-t_{n+1})\sum_{i=1}^{n}\frac{t_i x_i}{1-t_{n+1}}\right) \\ &\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1})f\left(\sum_{i=1}^{n}\frac{t_i x_i}{1-t_{n+1}}\right) \\ &\leq t_{n+1}f(x_{n+1}) + (1-t_{n+1})\sum_{i=1}^{n}\frac{t_i}{1-t_{n+1}}f(x_i) \\ &= \sum_{i=1}^{n+1} t_i f(x_i), \end{aligned}$$

since $\frac{t_1+...+t_n}{1-t_{n+1}} = 1$.

Conversely suppose $f(\sum_{i=1}^n t_i x_i)\leq\sum_{i=1}^n t_i f(x_i)$ for any points $x_1,...,x_n\in[a,b]$ and the real numbers $t_1,...,t_n$ with $\sum_{i=1}^n t_1 = 1$. Then $f(tx_2+(1-t)x_1)\leq tf(x_2)+(1-t)f(x_1)$ for all $x_1,x_2\in[a,b]$ and $0\leq t\leq 1$ since $t+(1-t)=1$. So f is convex.

b) Suppose that $A\subset\mathbb{C}$ is convex. Again, the assertion is true for $n=1$ trivially and for $n=2$ by the definition of convexity, so suppose that, for some $n\in\mathbb{N}$, $\sum_{i=1}^n t_i z_i\in A$ holds for all $z_1,\ldots,z_n\in A$ and $t_z,\ldots,t_n\geq 0$ such that $\sum_{i=1}^n t_i = 1$. Then, if $z_1,\ldots,z_{n+1}\in A$ and $t_z,\ldots,t_{n+1}\geq 0$ such that $\sum_{i=1}^{n+1} t_i = 1$, we have

$$\sum_{i=1}^{n+1} t_i z_i = t_{n+1}z_{n+1} + (1-t_{n+1})\sum_{i=1}^{n}\frac{t_i}{1-t_{n+1}}z_i\in A,$$

because $\sum_{i=1}^n \frac{t_i}{1-t_{n+1}} = 1$ and thus $\sum_{i=1}^n \frac{t_i}{1-t_{n+1}}z_i\in A$.

Conversely, suppose that $\sum_{i=1}^n t_i z_i\in A$ for all $z_1,\ldots,z_n\in A$ and $t_z,\ldots,t_n\geq 0$ such that $\sum_{i=1}^n t_i = 1$. Then, by choosing $n=2$ and $t_2=t$ we have $tz_2+(1-t)z_1\in A$, and thus A is convex.

4. Let $f:[a,b]\to\mathbb{R}$ be differentiable convex function and $a\leq x_1<x_2\leq b$. Let

$t_1, t_2 \in (0,1)$, and define $h_1 = t_1(x_2 - x_1)$ and $h_2 = t_2(x_2 - x_1)$. Then

$$\begin{aligned}
\frac{f(x_1+h_1)-f(x_1)}{h_1} &= \frac{f(t_1x_2+(1-t_1)x_1)-f(x_1)}{h_1} \\
&\le \frac{t_1f(x_2)+(1-t_1)f(x_1)-f(x_1)}{h_1} \\
&= \frac{f(x_2)-f(x_1)}{h_1/t_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{h_2/t_2} \\
&= \frac{f(x_2)-(t_2f(x_1)+(1-t_2)f(x_2)}{h_2} \\
&\le \frac{f(x_2)-f(t_2x_1+(1-t_2)x_2)}{h_2} \\
&= \frac{f(x_2)-f(x_2-h_2)}{h_2}.
\end{aligned}$$

By letting $h_1 \to 0$ we have

$$f'(x_1) \le \frac{f(x_2)-f(x_2-h_2)}{h_2},$$

and by then letting $h_2 \to 0$, we obtain $f'(x_1) \le f'(x_2)$.

Suppose then that $f : [a,b] \to \mathbb{R}$ is differentiable such that f' is increasing, and let $a \le x_1 < x_2 < x_3 \le b$. By the mean value theorem, there exist $y_1 \in (x_1, x_2)$ and $y_2 \in (x_2, x_3)$ such that

$$f'(y_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \text{ and } f'(y_2) = \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

Hence

$$\begin{aligned}
\frac{f(x_2)-f(x_1)}{x_2-x_1} &\le \frac{f(x_2)-f(x_1)}{x_2-x_1} + \left(\frac{f(x_3)-f(x_2)}{x_3-x_2} - \frac{f(x_2)-f(x_1)}{x_2-x_1}\right)\frac{x_3-x_2}{x_3-x_1} \\
&= \frac{f(x_2)-f(x_1)}{x_2-x_1} + \frac{(x_2-x_1)f(x_3)-(x_3-x_1)f(x_2)+(x_3-x_2)f(x_1)}{(x_2-x_1)(x_3-x_1)} \\
&= \frac{f(x_3)-f(x_1)}{x_3-x_1}.
\end{aligned}$$

By defining $t = \frac{x_2-x_1}{x_3-x_1}$, we obtain $x_2 = tx_3 + (1-t)x_1$ and

$$f(tx_3+(1-t)x_1) \le tf(x_3)+(1-t)f(x_1),$$

and hence f is convex.

5. Let $f : [a,b] \to \mathbb{R}_+$ be logarithmically convex. Becauce $g(x) = e^x$ is increasing and convex (g' is increasing), we have

$$f(tx_2+(1-t)x_1) = e^{\log f(tx_2+(1-t)x_1)} \le e^{t\log f(x_2)+(1-t)\log f(x_1)} = tf(x_2)+(1-t)f(x_1)$$

$\forall x_1, x_2 \in [a,b]$, and $t \in [0,1]$. So f is convex. On the other hand, $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^2$ is convex (f' is increasing), but $\log x^2$ is not ($f''(x)f(x) - (f'(x))^2 = 4x - 4x^2 < 0$ when $x \notin (0,1]$).

6. Let $0 < R_1 < R_2 < \infty$ and suppose that f is analytic in $A(0; R_1, R_2)$. Let $G = \{x + iy : \log R_1 < x < \log R_2\}$ and $R_1 < r_1 \le r \le r_2 < R_2$. Now the function e^z maps G onto $A(0; R_1, R_2)$ and ∂G onto $\partial A(0; R_1, R_2)$, and f is continuous in $\overline{A(0; r_1, r_2)}$. Consider the function $g(z) = f(e^z)$, which is now analytic in G, continuous in $\overline{G_{r_1,r_2}} \subset G$, where $G_{r_1,r_2} = \{x + iy : \log r_1 < x < \log r_2\}$, and thus also bounded in G_{r_1,r_2}.

Define the function $M : [\log r_1, \log r_2] \to \mathbb{R}$ by

$$M(x) = \sup_{-\infty < y < \infty} |g(x + iy)|.$$

By Theorem 3.4 we know that $\log M$ is a convex function, and hence

$$\log M(\log r) \le t \log M(r_1) + (1-t) \log M(\log r_2),$$

where $t = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}$ and therefore $1 - t = \frac{\log r - \log r_1}{\log r_2 - \log r_1}$. Now, by the definitions of M and g, we have

$$M(\log r) = \sup_{-\infty<y<\infty} |f(e^{\log r + iy})| = \sup_{-\infty<y<\infty} |f(re^{iy})| = \max_{z \in \partial D(0,r)} |f(z)| = M(r, f),$$

and the assertion follows.

1. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and $0 < r < 1$. Let $R_1 \in (0, r)$, $R_2 \in (r, 1)$ and $r_1, r_2 \in (R_1, R_2)$ such that $r_1 < r < r_2$. Then

$$\log M(r, f) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1, f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2, f)$$

by Hadamard's three circles theorem. From this we have

$$\begin{aligned} \log M(r, f) &\le \frac{(\log r_2 - \log r_1) - (\log r - \log r_1)}{\log r_2 - \log r_1} \log M(r_1, f) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2, f) \\ &= (\log r - \log r_1) \frac{\log M(r_2, f) - \log M(r_1, f)}{\log r_2 - \log r_1} + \log M(r_1, f), \end{aligned}$$

and thus

$$\frac{\log M(r, f) - \log M(r_1, f)}{\log r - \log r_1} \le \frac{\log M(r_2, f) - \log M(r_1, f)}{\log r_2 - \log r_1}.$$

Similar calculation (write the coefficient of $\log M(r_2, f)$ as $\frac{(\log r_2 - \log r_1) - (\log r_2 - \log r)}{\log r_2 - \log r_1}$) shows that

$$\frac{\log M(r_2, f) - \log M(r_1, f)}{\log r_2 - \log r_1} \le \frac{\log M(r_2, f) - \log M(r, f)}{\log r_2 - \log r}.$$

On the other hand, by writing Hadamard's result as

$$\begin{aligned} ((\log r_2 - \log r) + (\log r - \log r_1)) \log M(r, f) \le\ & (\log r_2 - \log r) \log M(r_1, f) \\ & + (\log r - \log r_1) \log M(r_2, f), \end{aligned}$$

we obtain
$$\frac{\log M(r,f)-\log M(r_1,f)}{\log r-\log r_1}\le\frac{\log M(r_2,f)-\log M(r_1,f)}{\log r_2-\log r_1}. \tag{1}$$

Now, in a similar way as in exercise 2 in chapter 3, we see that the function $F_1(r)=\frac{\log M(r,f)-\log M(r_1,f)}{\log r-\log r_1}$ is bounded bellow and increasing in some $(r_1,r_1+\delta)$, and the function $F_2(r)=\frac{\log M(r_2,f)-\log M(r,f)}{\log r_2-\log r}$ is bounded above and increasing in some $(r_2-\delta,r_2)$. Therefore the limits

$$\lim_{r\to r_1^+}F_1(r)=\left(\frac{d\log M(r,r)}{d\log r}\right)_+\Bigg|_{r=r_1}\text{ and }\lim_{r\to r_2^-}F_2(r)=\left(\frac{d\log M(r,r)}{d\log r}\right)_-\Bigg|_{r=r_2}$$

both exist. Now we need to show that $\left(\frac{d\log M(r,r)}{d\log r}\right)_-\le\left(\frac{d\log M(r,r)}{d\log r}\right)_+$. But this follows from inequality (1) by letting $r_1\to r^-$ and $r_2\to r^+$, and we are done.