1. Let $K > 0$ such that $|g_k(z)| \leq K$ for all $z \in D$ and $k = 1, ..., n$. Since D is simply connected, the lemma of the analytic logarithm shows that there exists an analytic branch of $\log(g_k)$ on D for every $k = 1, ..., n$. Hence $h_k = \exp(\eta \log(g_k))$ is an analytic branch of g_k^η for $\eta > 0$ and $|h_k| = |g_k|^\eta$ on D. Define $F : D \to \mathbb{C}$ by $F(z) = f(z) \prod_{k=1}^n h_k(z) K^{-\eta n}$. Then F is analytic on D and

$$|F(z) = |f(z)| \prod_{k=1}^{n} |g_k(z)|^\eta K^{-\eta n} \leq |f(z)| \ \forall z \in D.$$

But then, by the assumptions a) and b), F satisfies the hypothesis of Theorem 1.6 with $\max\{M, MK^{-\eta}\}$ in the place of M:

$$\limsup_{z \to a} |F(z)| \leq \limsup_{z \to a} |f(z)| \leq M \ a \in A;$$

and

$$\begin{aligned}\limsup_{z \to b} |F(z)| &= \limsup_{z \to b} |f(z)| \prod_{k=1}^{n} |g_k(z)|^\eta K^{-\eta n} \\ &\leq \limsup_{z \to b} |f(z)| \prod_{k=1}^{n} |g_k(z)|^\eta K^{-\eta} \leq MK^{-\eta} \ b \in B_k.\end{aligned}$$

Hence

$$|f(z)| = \frac{|F(z)|}{\prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n}} \leq \frac{\max\{M, MK^{-\eta}\}}{\prod_{k=1}^n |g_k(z)|^\eta K^{-\eta n}} \ \forall z \in D$$

By fixing $z \in D$ arbitrarily and letting $\eta \to 0^+$, we deduce $|f(z)| \leq \forall z \in D$.

2. *Solution 1.* Let $T = \{z : |\arg(z)| < \frac{\pi}{2}\}$ and $g(z) = f(\log(z))$. Then g is analytic in T, $\log(T) = G$ and $\log(\partial T) = \partial G$. Thus

$$\limsup_{z \to \omega \in \partial T} |g(z)| = \limsup_{z \to w \in \partial G} |f(z)| \leq M \ \forall \omega \in \partial T.$$

Also there exists $A > 0$ and $a \in (0, 1)$ such that

$$|g(z)| = |f(\log(z))| < \exp(A \exp[a|Re(\log(z))|]) = \exp A|z|^a < \exp A|z| \ \forall |z| \geq 1.$$

Corollary 8.3 implies $f(z) \leq M \ \forall z \in G$.

Solution 2. The result can also be deduced by using the Phragmen-Lindelöf theorem: Let $b \in (a, 1)$ and $B = \frac{1}{\cos\left(b\frac{\pi}{2}\right)}$, and consider the function $g(z) = \exp\left(-B\left(e^{bz} + e^{-bz}\right)\right)$. Since $\Re\left(e^z + e^{-z}\right) = \left(e^{\Re z} + e^{-\Re z}\right) \cos \Im z$ and $e^x + e^{-x} \geq e^{|x|}$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}|g(z)| &= \exp\left(-B\left(e^{b\Re z} + e^{-b\Re z}\right) \cos(b\Im z)\right) \\ &\leq \exp\left(-Be^{b|\Re z|} \cos\left(b\frac{\pi}{2}\right)\right) \\ &= \exp\left(-\exp(b|\Re z|)\right)\end{aligned}$$

for all $z \in G$. Hence g is bounded in G, and

$$|f(z)||g(z)|^\eta \le \exp\left[A\exp(a|\Re z|) - \eta\exp(b|\Re z|)\right] \to 0,$$

as $z \to \infty$, $z \in G$, for all $\eta > 0$. The assertion follows by Phragmen-Lindelöf theorem.

Let $f(z) = \exp(\exp z)$. Then $|f(z)| = \exp\left(e^{\Re z}\cos\Im z\right) = 1$ for all $z \in \partial G$ and $|f(z)| \le \exp(\exp(\Re z)) \le \exp(\exp(|\Re z|))$ for all $z \in G$, but $\lim_{z\to\infty, z\in\mathbb{R}_+}|f(z)| = \lim_{x\to\infty}\exp(e^x) = \infty$, so the result of the exercise does't hold. Hence the growth condition given is the best possible, and we can not make $a = 1$.

3. Let

$$F(z) = f(z)\frac{1+z}{1-z}.$$

Then $\limsup_{z\to w\in\partial G}|F(z)| \le M$, because $\limsup_{z\to w\in\partial G}|f(z)| \le M$. On the other hand $|f(z)| \le P\exp(|z|^{1-\delta})$ for any $\delta \in (0,1)$. Thus we obtain

$$\begin{aligned}|F(z)| &\le \left|\frac{1+z}{1-z}\right| P\exp(|z|^{1-\delta}) \\ &\le \frac{1+|z|}{|1-|z||}P\exp(|z|^{1-\delta}) \\ &\le 3P\exp(|z|^{1-\delta}),\end{aligned}$$

if $z \in G$ and $|z| > 2$. Hence $|F(z)| \le M$ in G by Corollary 8.2 and the assertion follows.

4. *Solution 1.* Let

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z}, & z \ne 0; \\ f'(0), & z = 0. \end{cases}$$

Moreover, we know that $f(z) \le C|z|^m$, where $C, R \in (0,\infty)$ are constants and $|z| > R$. Hence if $|z|$ is sufficiently large, we obtain the inequality

$$|g(z)| \le A + B|z|^{m-1} < D|z|^{m-1},$$

where $A, B, D \in (0,\infty)$ are constants. Now, by the induction principle, we can easily prove that g is a polynomial with $\deg(g) \le m-1$. Thus f is a polynomial with $\deg(f) \le m$.

Solution 2. Since f is entire, its Maclaurin series $f(z) = \sum_{k=0}^\infty a_k z^k$, $a_k = \frac{f^{(k)}(0)}{k!}$, converges for all $z \in \mathbb{C}$. Now, Cauchy's integral formula gives

$$\begin{aligned}|a_k| &= \left|\frac{1}{2\pi}\int_{\partial D(0,r)}\frac{f(\xi^{k+1})}{\xi}d\xi\right| \le \frac{1}{2\pi}\int_{\partial D(0,r)}\frac{|f(\xi)|}{|\xi|^{k+1}}|d\xi| \\ &\le \frac{1}{2\pi}\int_{\partial D(0,r)}\frac{C|\xi|^m}{|\xi|^{k+1}}|d\xi| = \frac{C}{2\pi}\int_{\partial D(0,r)} r^{m-(k+1)}|d\xi| = Cr^{m-k}\end{aligned}$$

for all $k \in \mathbb{N}$ and $r > R > 0$. Hence, if $k > m$, we have $|a_k| \le \lim_{r\to\infty} Cr^{m-k} = 0$, and thus f is a polynomial of degree at most m.

Solution 3. Since $|f(z)| \leq C|z|^m$ for all $|z| > R$, we have $|f(z)z^{-m}| \leq C$ for all $|z| > R$. By substituting $z = w^{-1}$ we get $|f(\frac{1}{w})w^m| \leq C$ for all $w < \frac{1}{R}$. Hence $f(\frac{1}{w})$ is analytic at $w = 0$ or has a pole of order n, $n \leq m$, at $w = 0$. It follows that f is a polynomial with $\deg(f) \leq m$.

5. Since $f(D(a,r)) \subset D(f(a), R)$, $|f(a+z) - f(a)| \leq R$ for all $z \in D(0,r)$. Consider the function $g : \mathbb{D} \to \mathbb{C}$,

$$g(z) = \frac{f(a+rz) - f(a)}{R}.$$

We see that $g(0) = 0$ and $|g(z)| \leq \frac{R}{R} = 1$ for all $z \in \mathbb{D}$. Thus Schwarz lemma yields $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Hence

$$|f(a+z) - f(a)| \leq \frac{R}{r}|z|$$

for all $z \in D(0,r)$.

To prove Liouville's theorem (every bounded entire function is constant), suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then there exists $R \in (0, \infty)$ such that $f(z) \in D(f(0), R)$ for all $z \in \mathbb{C}$. Hence

$$|f(z) - f(0)| \leq \frac{R}{r}|z|, \;\; z \in D(0,r)$$

for all $r \in (0, \infty)$. By letting $r \to \infty$, we obtain $f(z) = f(0)$ for all $z \in \mathbb{C}$.

6. $\eta_\alpha(\mathbb{D})$ is a "lens" inside $\mathbb{D}$ with its vertices at $\eta_\alpha(1) = 1$ and $\eta_\alpha(-1) = -1$, and with an angle of $\alpha\pi$ at them.

Clearly $\frac{1+z}{1-z}$ is a conformal map of $\mathbb{D}$ onto $D_1 = \{z \in \mathbb{C} : \Re z > 0\}$, z^α is a conformal map of D_1 onto $D_2 = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\frac{\pi}{2}\} \subset D_1$ ($z^\alpha = e^{\alpha \log z}$ has an analytic branch by the lemma of analytic logarithm), and $\frac{z-1}{z+1}$ is a conformal map of D_1 onto $\mathbb{D}$. Thus η_α is a conformal map of $\mathbb{D}$ onto $\eta_\alpha(\mathbb{D}) \subset \mathbb{D}$.

Version of Corollary 8.4. Suppose that $f(z) \to c \in \mathbb{C}$ as $z \to \omega \in \mathbb{T}$, $z \in \mathbb{D}$, along two circular arcs centered at $w \in \mathbb{C}$ and $-w \in \mathbb{C}$ (and intersecting at ω). Let $D \subset \mathbb{D}$ be the domain bounded by these arcs. If f is analytic and bounded in D or $\mathbb{C} \setminus \overline{D}$, then $f(z) \to c$ uniformly as $z \to \omega$ in D or $\mathbb{C} \setminus \overline{D}$ respectively.

Proof. Let $\alpha \in (0,1)$ such that $\alpha\pi$ is the angle at ω formed by the circular arcs bounding D. Then the function $g(z) = \frac{1+\overline{\omega}z}{1-\overline{\omega}z}$ maps D onto the sector $S_+ = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\frac{\pi}{2}\}$ and $\mathbb{C} \setminus \overline{D}$ onto $S_- = \mathbb{C} \setminus \overline{S_+}$. Hence $h = f \circ g$ is bounded and analytic in S_+ or S_- and $h(z) \to c$ as $z \to \infty$ along the rays $\{z \in \mathbb{C} : \arg z = \alpha\frac{\pi}{2}\}$ and $\{z \in \mathbb{C} : \arg z = -\alpha\frac{\pi}{2}\}$. Thus Corollary 8.4 implies $h(z) \to c$ uniformly as $z \to \infty$ in S_+ or S_- respectively, and hence $f(z) \to c$ uniformly as $z \to \infty$ in D or $\mathbb{C} \setminus \overline{D}$.