1. Our observations are stated as Theorems **??** and **??** and as an example.

Theorem 0.1. *Consider the complex linear differential equation*

$$f'' + Af = 0. \tag{1}$$

where A is analytic in $D(0,R)$. Let f be non-trivial solution of JM91 in $D(0,R)$. Now, all zeros of f are simple.

Proof. By Theorem 9.2, if A is analytic in $D(0,R)$, then all non-trivial solutions of (**??**) satisfy the pointwise estimate

$$|f(re^{i\theta})| \leq (|f'(0)|R + |f(0)|)\exp\left(\int_0^r |A(te^{i\theta})|(r-t)dt\right), \theta \in [0,2\pi), r \in (0,R). \tag{2}$$

(i) If f has a multiple zero in the origin, the right hand side of (**??**) is identically zero. Now f has to be identically zero, which is a contradiction. Thus if f has a zero at the origin, it must be simple.

(ii) Since $D(0,R)$ is open, we can make the same conclusion in every point of $D(0,R)$ by translation. Namely, let $a \in D(0,R)$ arbitrary and $S = R - |a| > 0$ so that $a \in D(a,S) \subset D(0,R)$. Define $\widetilde{f}, \widetilde{A} : D(0,S) \to \mathbb{C}$, $\widetilde{f} = f(z+a)$, $\widetilde{A} = A(z+a)$. Now, since $\widetilde{A}$ is analytic in $D(0,S)$ and $\widetilde{f}$ is a solution of (**??**) in $D(0,S)$, Theorem 9.2 implies that $\widetilde{f}$ satisfies the pointwise estimate

$$|\widetilde{f}(re^{i\theta})| \leq (|\widetilde{f}'(0)|S + |\widetilde{f}(0)|)\exp\left(\int_0^r |\widetilde{A}(te^{i\theta})|(r-t)dt\right), \theta \in [0,2\pi), r \in (0,S).$$

If f has a multiple zero at $z = a$, $\widetilde{f}$ has a multiple zero at the origin and is identically zero by **??**. Now f is identically zero, which is a contradiction. □

Theorem 0.2. *Consider*

$$f^{(k)} + Af = 0, \tag{3}$$

where A is analytic in $D(0,R)$ and $k \in \mathbb{N}$. Let f be a non-trivial solution of (**??**)*. Now, all zeros of f are atmost of multiplicity $k-1$.*

Proof. Let $a \in D(0,R)$ arbitrary. Now $f(z) = (z-a)^n g(z)$ in $D(0,R)$ for some $n \in \mathbb{N}_0$ and g analytic in $D(0,R)$ such that $g(a) \neq 0$. Let $S = R - |a|$ so that $a \in D(a,S) \subset D(0,R)$. Now g has a power series presentation in the disc $D(a,S)$, that is,

$$g(z) = \sum_{j=0}^{\infty} a_j (z-a)^j,$$

for some $a_j \in \mathbb{C}$, for all $z \in D(a,S)$. Since $g(a) \neq 0$, we have $a_0 \neq 0$. Now

$$f(z) = \sum_{j=0}^{\infty} a_j (z-a)^{n+j}$$

for all $z \in D(a,S)$ and

$$f^{(k)}(z)=\sum_{j=0}^{\infty} b_j(z-a)^{n+j-k},$$

where $b_j=(n+j)(n+j-1)\cdots(n+j-(k-1))a_j$, for all $z\in D(a,S)$. Therefore

$$f^{(k)}(z)=(z-a)^{n-k}h(z),$$

where $h(z)=\sum_{j=0}^{\infty} b_j(z-a)^j$. By (**??**) we have

$$A(z)=-\frac{f^{(k)}(z)}{f(z)}=\frac{1}{(z-a)^k}\frac{h(z)}{g(z)}$$

for all $z\in D(a,S)$. Since A and g are analytic, $h(z)$ has to have a zero atleast of multiplicity k. Therefore, since $a_0\neq 0$ and $b_0=0$, we have $n(n-1)\cdots(n-(k-1))=0$. It follows that either $n=0$ or $n\in\{1,2,\ldots,k-1\}$. In the first case $f(a)\neq 0$. In the second case f has a zero of order $n\leq k-1$ at $z=a$. □

Theorem **??** is a special case of Theorem **??** and can thus be proved by using the power series argument in the proof of Theorem **??**. On the other hand, Theorem **??** can be proved by following the proof of Theorem **??** and using an estimate which is analogous to (**??**), if such an estimate exists.

Let f be as in Theorem **??**. If f has a zero of order k we have in Theorem **??** $S=0$. It follows that f is identically zero.

Example. A non-trivial solution f of (**??**) with an analytic coefficient A can have a zero of multiplicity $k-1$ when (**??**) is considered in a bounded domain D. Let

$$f(z)=z^{2k-1}+az^{k-1}=z^{k-1}(z^k+a),$$

where $a>0$. Now f has a zero of multiplicity $k-1$ at the origin. Moreover,

$$f^{(k)}(z)=\frac{(2k-1)!}{(k-1)!}z^{k-1},$$

so that

$$A(z)=-\frac{f^{(k)}(z)}{f(z)}=-\frac{(2k-1)!}{(k-1)!}\frac{1}{z^k+a}.$$

Taking $a>0$ large enough A is analytic in D.

2. We will first state the results, and then provide the proofs. Bellow we use the notation $\binom{j}{n}=\frac{j!}{n!(j-n)!}$.

Theorem 0.3. *Suppose that f is a solution of $f^{(k)}+A_{k-1}f^{(k-1)}+\ldots+A_1f'+A_0f=0$ in $D(0,R)$, where A_j is analytic in $D(0,R)$ for all j. Then*

$$\left|f(re^{i\theta})\right|\leq S\exp\left(\int_0^r C(te^{i\theta})dt\right),\ \theta\in[0,2\pi),\ r\in(0,R),$$

where

$$S = \sum_{j=0}^{k-1}\left[\sum_{n=0}^{j}\sum_{m=0}^{j-n-1}\binom{j}{n}\frac{\left|(A_j^{(n)}(0)f(0))^{(m)}\right|}{(k-j+n+m)!}R^{k-j+n+m} + \frac{\left|f^{(j)}(0)\right|}{j!}R^j\right]$$

and

$$C(te^{i\theta}) = \sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{j}{n}\left|A_j^{(n)}(te^{i\theta})\right|\frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!}.$$

The term $(A_j^{(n)}(0)f(0))^{(m)}$ above means functions $(A_j^{(n)}f)^{(m)}$ value at the orign. In the nonhomogeneous case we obtain the following result.

Theorem 0.4. *Suppose that f is a solution of $f^{(k)} + A_{k-1}f^{(k-1)} + \ldots + A_1 f' + A_0 f = A_k$ in $D(0,R)$, where A_j is analytic in $D(0,R)$ for all j and $A_k \not\equiv 0$. Then*

$$\left|f(re^{i\theta})\right| \le B(re^{i\theta}) + \int_0^r B(se^{i\theta})C(se^{i\theta})\exp\left(\int_s^r C(te^{i\theta})dt\right)ds,\ \theta\in[0,2\pi),\ r\in(0,R),$$

where

$$\begin{aligned} B(re^{i\theta}) &= \int_0^r \left|A_k(te^{i\theta})\right|\frac{(r-t)^{k-1}}{(k-1)!}dt \\ &+ \sum_{j=0}^{k-1}\left[\sum_{n=0}^{j}\sum_{m=0}^{j-n-1}\binom{j}{n}\frac{\left|(A_j^{(n)}(0)f(0))^{(m)}\right|}{(k-j+n+m)!}R^{k-j+n+m} + \frac{\left|f^{(j)}(0)\right|}{j!}R^j\right] \end{aligned}$$

and

$$C(te^{i\theta}) = \sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{j}{n}\left|A_j^{(n)}(te^{i\theta})\right|\frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!}.$$

In the proofs of these two theorems, we use the following two Lemmas.

Lemma 0.5. *Let f and g be analytic in some domain. Then $gf^{(j)} = \sum_{n=0}^{j}(-1)^n\binom{j}{n}(g^{(n)}f)^{(j-n)}$.*

Proof. The case $j=1$ is a form of Leibniz rule, so suppose that the assertion holds for some $j\in\mathbb{N}$. Then

$$\begin{aligned} gf^{(j+1)} &= (gf^{(j)})' - g'f^{(j)} \\ &= \left(\sum_{n=0}^{j}(-1)^n\binom{j}{n}(g^{(n)}f)^{(j-n)}\right)' - \sum_{n=0}^{j}(-1)^n\binom{j}{n}(g^{(n+1)}f)^{(j-n)} \\ &= (gf)^{(j+1)} + \sum_{n=1}^{j}(-1)^n\left[\binom{j}{n} + \binom{j}{n-1}\right](g^{(n)}f)^{(j+1-n)} + (-1)^{j+1}g^{(j+1)}f. \end{aligned}$$

Since a simple calculation shows that $\binom{j}{n} + \binom{j}{n-1} = \binom{j+1}{n}$, the assertion follows by induction principle. □

Lemma 0.6. *Let $g:(0,R)\to\mathbb{R}_+$ be integrable and $0<t_1<t_2<\ldots<t_n<r<R$. Then*

$$\int_0^r\int_0^{t_n}\cdots\int_0^{t_1}g(t)dtdt_1\cdots dt_n=\int_0^r g(t)\frac{(r-t)^n}{n!}dt.$$

Proof. It is known by Fubini's theorem that the assertion holds for $n=1$, so suppose it holds for some $n\in\mathbb{N}$. Then

$$\begin{aligned}\int_0^r\int_0^{t_{n+1}}\cdots\int_0^{t_1}g(t)dtdt_1\cdots dt_n &= \int_0^r\int_0^{t_{n+1}}g(t)\frac{(t_{n+1}-t)^n}{n!}dtdt_{n+1}\\ &= \int_0^r\int_0^r g(t)\frac{(t_{n+1}-t)^n}{n!}\chi_{\{t\le t_{n+1}\}}(t)dtdt_{n+1}\\ &= \int_0^r g(t)\int_0^r\frac{(t_{n+1}-t)^n}{n!}\chi_{\{t\le t_{n+1}\}}(t_{n+1})dt_{n+1}dt\\ &= \int_0^r g(t)\int_t^r\frac{(t_{n+1}-t)^n}{n!}dt_{n+1}dt\\ &= \int_0^r g(t)\frac{(r-t)^n}{n!}dt,\end{aligned}$$

by Fubini's theorem. The assertion follows by induction principle. □

Now we may prove the theorems above.

Proof of Theorem **??**. By applying the equality

$$f(z)=\int_0^z f'(\xi)d\xi+f(0),\ z\in D(0,R),$$

k times, we obtain

$$f(z)=\int_0^z\int_0^{\xi_1}\cdots\int_0^{\xi_{k-1}}f^{(k)}(\xi_k)d\xi_kd\xi_{k-1}\cdots d\xi_1+\sum_{j=0}^{k-1}\frac{f^{(j)}(0)}{j!}z^j,\ z\in D(0,R).$$

Thus, by using the ODE, we have

$$\begin{aligned}|f(z)| &= \left|\int_0^z\cdots\int_0^{\xi_{k-1}}-\sum_{j=0}^{k-1}A_j(\xi_k)f^{(j)}(\xi_k)d\xi_k\cdots d\xi_1+\sum_{j=0}^{k-1}\frac{f^{(j)}(0)}{j!}z^j\right|\\ &\le \sum_{j=0}^{k-1}\left|\int_0^z\cdots\int_0^{\xi_{k-1}}A_j(\xi_k)f^{(j)}(\xi_k)d\xi_k\cdots d\xi_1\right|+\sum_{j=0}^{k-1}\frac{\left|f^{(j)}(0)\right|}{j!}R^j.\end{aligned}$$

By using Lemma **??**, we may write the integrals as

$$\begin{aligned}
&\int_0^z \cdots \int_0^{\xi_{k-1}} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 \\
&\quad = \int_0^z \cdots \int_0^{\xi_{k-1}} \sum_{n=0}^{j} (-1)^n \binom{j}{n} (A_j^{(n)}(\xi_k) f(\xi_k))^{(j-n)} d\xi_k \cdots d\xi_1 \\
&\quad = \sum_{n=0}^{j} (-1)^n \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \Bigg[A_j^{(n)}(\xi_{k-j+n}) f(\xi_{k-j+n}) \\
&\qquad - \sum_{m=0}^{j-n-1} \frac{(A_j^{(n)}(0) f(0))^{(m)}}{m!} z^m \Bigg] d\xi_{k-j+n} \cdots d\xi_1 \\
&\quad = \sum_{n=0}^{j} (-1)^n \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} A_j^{(n)}(\xi_{k-j+n}) f(\xi_{k-j+n}) d\xi_{k-j+n} \cdots d\xi_1 \\
&\qquad - \sum_{m=0}^{j-n-1} \frac{(A_j^{(n)}(0) f(0))^{(m)}}{(k-j+n+m)!} z^{k-j+n+m},
\end{aligned}$$

so, by denoting

$$S = \sum_{j=0}^{k-1} \left[\sum_{n=0}^{j} \sum_{m=0}^{j-n-1} \binom{j}{n} \frac{\left|(A_j^{(n)}(0) f(0))^{(m)}\right|}{(k-j+n+m)!} R^{k-j+n+m} + \frac{\left|f^{(j)}(0)\right|}{j!} R^j \right],$$

we have

$$|f(z)| \le \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \left|A_j^{(n)}(\xi_{k-j+n})\right| |f(\xi_{k-j+n})| \, |d\xi_{k-j+n}| \cdots |d\xi_1| + S.$$

By setting $z = re^{i\theta}$ and $\xi_j = t_j e^{i\theta}$, Lemma **??** gives

$$\begin{aligned}
&\int_0^z \cdots \int_0^{\xi_{k-j+n-1}} \left|A_j^{(n)}(\xi_{k-j+n})\right| |f(\xi_{k-j+n})| |d\xi_{k-j+n}| \cdots |d\xi_1| \\
&\quad = \int_0^r \cdots \int_0^{t_{k-j+n-1}} \left|A_j^{(n)}(t_{k-j+n} e^{i\theta})\right| \left|f(t_{k-j+n} e^{i\theta})\right| dt_{k-j+n} \cdots dt_1 \\
&\quad = \int_0^r \left|A_j^{(n)}(te^{i\theta})\right| \left|f(te^{i\theta})\right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!} dt,
\end{aligned}$$

so

$$|f(re^{i\theta})| \le \int_0^r \left|f(te^{i\theta})\right| \sum_{j=0}^{k-1} \sum_{n=0}^{j} \binom{j}{n} \left|A_j^{(n)}(te^{i\theta})\right| \frac{(r-t)^{k-j+n-1}}{(k-j+n-1)!} dt + S.$$

The assertion now follows by Gronwall-Bellman inequality. □

Proof of Theorem **??**. Similarly as in the proof of Theorem **??**, we have

$$|f(z)| = \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) - \sum_{j=0}^{k-1} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 + \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j \right|$$
$$\leq \sum_{j=0}^{k-1} \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_j(\xi_k) f^{(j)}(\xi_k) d\xi_k \cdots d\xi_1 \right| + \left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) d\xi_k \cdots d\xi_1 \right|$$
$$+ \sum_{j=0}^{k-1} \frac{\left|f^{(j)}(0)\right|}{j!} R^j.$$

By Lemma **??** we have

$$\left| \int_0^z \cdots \int_0^{\xi_{k-1}} A_k(\xi_k) d\xi_k \cdots d\xi_1 \right| \leq \int_0^r \cdots \int_0^{t_{k-1}} |A_k(t_k e^{i\theta})| dt_k \cdots dt_1$$
$$= \int_0^r |A_k(t_k e^{i\theta})| \frac{(r-t)^{k-1}}{(k-1)!} dt,$$

so, the same calculations that we did in the proof of Theorem **??** now show that

$$\left|f(re^{i\theta})\right| \leq B(re^{i\theta}) + \int_0^r C(te^{i\theta}) \left|f(te^{i\theta})\right| dt.$$

The assertion now follows by exercise 3. □

3. Suppose that

$$u(x) \leq c(x) + \int_a^x u(s)v(s) ds, \quad x \in (a,b), \tag{4}$$

where $u, v, c : (a,b) \to [0,\infty)$ are integrable functions. Then

$$u(x) \leq c(x) + \int_0^x c(s)v(s) \exp\Big(\int_s^x v(r) dr \Big) ds.$$

Proof. Let

$$f(s) = \exp\Big(- \int_a^s v(r) dr \Big) \int_a^s v(r)u(r) dr.$$

Then

$$f'(s) = \Big(u(s) - \int_a^s v(r)u(r) dr \Big) v(s) \exp\Big(- \int_a^s v(r) dr \Big). \tag{5}$$

Hence

$$f(x) \leq \int_a^x c(s)v(s) \exp\Big(- \int_a^s v(r) dr \Big) ds. \tag{6}$$

by (**??**) and (**??**). Now, by (**??**) and (**??**), we obtain

$$\begin{aligned}\int_a^x v(s)u(s)ds &= \exp\Big(\int_a^x v(r)dr\Big)f(x)\\ &\le \int_a^x c(s)v(s)\exp\Big(\int_a^x v(r)dr - \int_a^s v(r)dr\Big)ds\\ &\le \int_a^x c(s)v(s)\exp\Big(\int_s^x v(r)dr\Big)ds.\end{aligned}$$

Thus the assertion follows by the previous inequality and the assumption (**??**).

Suppose that c is non-decreasing. Then the earlier result implies that

$$\begin{aligned}u(x) &\le c(x) + \Big[-c(x)\exp\Big(\int_s^x v(r)dr\Big)\Big]\Big|_{s=a}^{s=t}\\ &= c(x)\exp\Big(\int_a^x v(r)dr\Big).\end{aligned}$$

4. Let $f(z) = f'(z) = f''(z) = e^z$, where $z \in \mathbb{D}$. If $f'' + Af = 0$ and $z = r \in (0,1)$, then

$$e^r \le 2\exp\Big(\int_0^r (r-t)dt\Big) = 2\exp\Big(\frac{r^2}{2}\Big)$$

by the Gronwall-Bellman inequality.

Let $f(z) = \frac{1}{1-z}$, $z \in \mathbb{D}$. Then f satisfies $f'' - \frac{2}{(1-z)^2}f = 0$. Now $f(0) = f'(0) = 1$, and if $\theta = \arg z = 0$, then $|f(re^{i\theta})| = \frac{1}{1-r}$, and the inequality of Theorem 9.2 gets the form

$$\begin{aligned}\frac{1}{1-r} &\le 2\exp\Big(\int_0^r \frac{2(r-t)}{(1-t)^2}dt\Big)\\ &= 2\exp(-2r - 2\log(1-r)) = 2e^{-2r}1(1-r)^2.\end{aligned}$$

Let $f(z) = e^{\frac{1}{1-z}}$, $z \in \mathbb{D}$. Then f satisfies $f'' - \Big(\frac{2}{(1-z)^3} + \frac{1}{(1-z)^4}\Big)f = 0$. Now $f(0) = f'(0) = e$, and if $\theta = 0$, the inequality of Theorem 9.2 holds in the form

$$e^{\frac{1}{1-r}} \le 2e^{-\frac{4}{3}r - \frac{1}{6}}e^{\frac{1}{1-r}}e^{\frac{1}{6}\frac{1}{(1-r)^2}},$$

so

$$1 \le 2e^{-\frac{4}{3}r - \frac{1}{6}}e^{\frac{1}{6}\frac{1}{(1-r)^2}}.$$

In every case above, the right hand side of the inequality grows faster than the left hand side, as $r \to 1^-$. Hence, it looks like the result of the Theorem 9.2 could be improved.