Chapter 10.

1. Becauce $(\mathbb{D}, d_h)$ is metric space, every Cauchy sequence is bounded. Since every bounded sequence has converging sub-sequence, Cauchy sequence in $(\mathbb{D}, d_h)$ converges to some point. This point must lie in $\mathbb{D}$, since $\mathbb{T}$ is infinitely far away from each point of $\mathbb{D}$.

With more details: Let $\{z_n\} \subset \mathbb{D}$ be a Cauchy sequence with respect to distance d_h. Then it is bounded, that is, there exists $R \in (0, \infty)$ such that $d_h(0, z_n) \leq R$ for all $n \in \mathbb{N}$. Since $d_h(0, z_n) = \log \frac{1+|z_n|}{1-|z_n|}$, we have $|z_n| \leq \frac{e^R - 1}{e^R + 1} < 1$ for all $n \in \mathbb{N}$. By Bolzano-Weierstrass theorem the bounded sequence $\{z_n\}$ has a converging subsequence, and because $|z_n| \leq K < 1$, the limit point must lie in $\mathbb{D}$.

2.

Lemma 1. *Let $z_1, z_2 \in \triangle_{ph}(a, r)$, where $a \in \mathbb{D}$ and $r \in (0, 1)$. Then*

$$\frac{1}{K} \leq \frac{1 - |z_2|}{1 - |z_1|} \leq K$$

for some constant $K(r) > 0$.

Proof. By the strong form of the triangle inequality,

$$d_{ph}(z_1, z_2) = \frac{d_{ph}(z_1, a) + d_{ph}(z_2, a)}{1 + d_{ph}(z_1, a) d_{ph}(z_2, a)} < \frac{2r}{1 + r^2} := A(r).$$

On the other hand, we can easily prove that

$$1 - d_{ph}(z_1, z_2)^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \overline{z}_1 z_2|^2}, \tag{1}$$

and so

$$\begin{aligned} \frac{1 - |z_2|^2}{1 - |z_1|^2} &= \frac{|1 - \overline{z}_1 z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} \cdot \frac{(1 - |z_2|^2)^2}{|1 - \overline{z}_1 z_2|^2} \\ &< \frac{1}{1 - A^2} \left(\frac{1 - |z_2|^2}{|1 - \overline{z}_1 z_2|} \right)^2. \end{aligned}$$

However, $|1 - \overline{z}_1 z_2| > 1 - |z_2| > (1 - |z_2|^2)/2$, thus

$$\frac{1 - |z_2|}{1 - |z_1|} < 2 \frac{1 - |z_2|^2}{1 - |z_1|^2} < \frac{8}{1 - A^2} := K(r).$$

Since $z_1, z_2 \in \triangle_{ph}(a, r)$ are arbitrary, the assertion follows. $\square$

Theorem 1. *Let $z \in \triangle_{ph}(a, r)$, where $a \in \mathbb{D}$ and $r \in (0, 1)$. Then*

$$\frac{1-|a|}{C} \leq |1-\overline{a}z| \leq C(1-|a|) \tag{2}$$

for some constant $C(r) > 0$.

Proof. It is clear that

$$|1-\overline{a}z| \geq 1-|a||z| \geq 1-|a| \geq \frac{1-|a|}{C}$$

for all $C \geq 1$. So it sufficient that we prove the other side of the inequality (2).

By (1) we obtain

$$\frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2} > 1-r,$$

and so by Lemma 1,

$$\begin{aligned}|1-\overline{a}z|^2 &< \frac{1}{1-r}(1-|z|^2)(1-|a|^2)\\ &< \frac{4}{1-r}(1-|z|)(1-|a|)\\ &< \frac{4K}{1-r}(1-|a|)^2\end{aligned}$$

for some constant $K(r) \geq \frac{1}{4}$. Hence,

$$|1-\overline{a}z| < \sqrt{\frac{4K}{1-r}}(1-|a|) := C(r)(1-|a|)$$

and the assertion follows. $\square$

3. (*Note:* This solution is presented mainly to show what we came up with, we are not sure if it holds or not. The proof covers only the case $p \geq 1$. We didn't manage to proof the case $0 < p < 1$. Also, it is supposed that $\int_0^{R_{z,r}} R_{z,s} \int_0^{2\pi} |f(C_{z,s} + R_{z,s}e^{i\theta})|^p d\theta dR_{z,s} \leq C(r) \int_{\Delta_{ph}(z,r)} |f(w)|^p dA(w)$, which we neither managed to proof, and are not even sure if it holds.)

Let $C_{z,s}$ and $R_{z,s}$ be the Euclidean center and radius of $\Delta_{ph}(z,s)$, $s \in (0,r]$, respectively. Then, by change of variable $\xi = C_{z,s} + R_{z,s}e^{i\theta}$, Cauchy's integral formula gives

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i}\int_{\partial D(C_{z,s},R_{z,s})} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \\ &= \frac{n!}{2\pi} R_{z,s}\int_0^{2\pi} \frac{f(C_{z,s}+R_{z,s}e^{i\theta})}{e^{-i\theta}(C_{z,s}+R_{z,s}e^{i\theta}-z)^{n+1}} d\theta. \end{aligned}$$

Suppose that $p \geq 1$. Then, by Jensen's inequality, we have

$$|f^{(n)}(z)|^p \leq \left(\frac{n!}{2\pi}R_{z,s}\right)^p \int_0^{2\pi} \frac{|f(C_{z,s}+R_{z,s}e^{i\theta})|^p}{|C_{z,s}+R_{z,s}e^{i\theta}-z|^{(n+1)p}} d\theta.$$

Since $C_{z,s}+R_{z,s}e^{i\theta} \in \partial\Delta_{ph}(z,s)$, we have $|C_{z,s}+R_{z,s}e^{i\theta}-z| = s|1-\overline{z}(C_{z,s}+R_{z,s}e^{i\theta})| \geq s(1-|z|)$, and by noticing that $R_{z,s} \leq \frac{2s}{1-s^2}(1-|z|)$, we obtain

$$\begin{aligned} |f^{(n)}(z)|^p &\leq \left(\frac{n!}{2\pi}\right)^p \frac{2^p}{(1-s^2)^p s^{np}} \frac{1}{(1-|z|)^{np}} \int_0^{2\pi} |f(C_{z,s}+R_{z,s}e^{i\theta})|^p d\theta. \\ &\leq \left(\frac{n!}{\pi}\right)^p \frac{1}{(1-r^2)^p s^{np}(1-|z|)^{np}} \int_0^{2\pi} |f(C_{z,s}+R_{z,s}e^{i\theta})|^p d\theta. \end{aligned}$$

Now multiplying by $s^{np}R_{z,s}$ and integrating both sides from 0 to $R_{z,r}$ with respect to $R_{z,s}$ gives

$$\begin{aligned} &|f^{(n)}(z)|^p \int_0^{R_{z,r}} s^{np}R_{z,s}dR_{z,s} \\ &\qquad \leq \left(\frac{n!}{\pi}\right)^p \frac{1}{(1-r^2)^p(1-|z|)^{np}} \int_0^{R_{z,r}} R_{z,s}\int_0^{2\pi} |f(C_{z,s}+R_{z,s}e^{i\theta})|^p d\theta dR_{z,s} \\ &\qquad \leq \left(\frac{n!}{\pi}\right)^p \frac{C(r)}{(1-r^2)^p(1-|z|)^{np}} \int_{\Delta_{ph}(z,r)} |f(w)|^p dA(w). \end{aligned}$$

The integral on the left hand side can be estimated simply with change of variable, $R_{z,s} = \frac{1-|z|^2}{1-s^2|z^2|}s$, $R_{z,s} = \frac{1-|z|^2}{(1-s^2|z|^2)^2}(1+2s^2|z|^2)ds$, as

$$\begin{aligned} \int_0^{R_{z,r}} s^{np}R_{z,s}dR_{z,s} &= \int_0^r s^{np+1}\frac{(1-|z|^2)^2}{(1-s^2|z|^2)^3}(1+2s^2|z|^2)ds \\ &\geq (1-|z|)^2\int_0^r s^{np+1}ds \\ &= (1-|z|)^2\frac{r^{np+2}}{np+2}. \end{aligned}$$

Thus we have

$$|f^{(n)}(z)|^p \leq \left(\frac{n!}{\pi(1-r^2)}\right)^p \frac{np+2}{r^{np+2}} \frac{C(r)}{(1-|z|)^{2+np}} \int_{\Delta_{ph}(z,r)} |f(w)|^p dA(w).$$

Chapter 11.

1. *Solution 1.* The claim is true in the case $\zeta = 1$ and the general case follows by rotation. However, let's be explicite.

Let $\zeta \in \mathbb{T}$ and $k > 0$ be arbitrary. Now $z \in E(k, \zeta)$ if and only if

$$|\zeta - z|^2 \leq k(1 - |z|^2).$$

By writing $z = \zeta w$ we get

$$|\zeta(1-w)|^2 \leq k(1 - |\zeta w|^2)$$

so that

$$|1-w|^2 \leq k(1-|w|^2).$$

Now, since $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + 2\mathrm{Re}(\alpha\overline{\beta})$, for all $\alpha, \beta \in \mathbb{C}$, we get

$$1 - 2\mathrm{Re}(w) + |w|^2 \leq k - k|w|^2.$$

By arranging terms we get

$$-2\mathrm{Re}(w) + (k+1)|w|^2 \leq k-1.$$

By dividing with $k+1$ we get

$$-2\mathrm{Re}\left(\frac{1}{k+1}w\right) + |w|^2 \leq \frac{k-1}{k+1}.$$

By adding $\frac{1}{(k+1)^2}$ on both sides we get

$$\left(\frac{1}{k+1}\right)^2 - 2\mathrm{Re}\left(\frac{1}{k+1}w\right) + |w|^2 \leq \frac{k-1}{k+1} + \frac{1}{(k+1)^2}$$

which gives

$$\left|\frac{1}{k+1} - w\right|^2 \leq \left(\frac{k}{k+1}\right)^2.$$

Recalling that $z = \zeta w$ we get

$$\left| z - \frac{\zeta}{k+1} \right|^2 \leq \left(\frac{k}{k+1} \right)^2.$$

Thus

$$E(k,\zeta) = \overline{D\left(\frac{\zeta}{k+1}, \frac{k}{k+1} \right)}.$$

Moreover, this closed disc is internally tangent to the unit circle $\mathbb{T}$ at ζ.

Solution 2.

Lemma 2. *The Euclidean circle given by the equation $\alpha|z|^2+\beta z+\overline{\beta}\overline{z}+y=0$, where $|\beta|^2 > \alpha y$, has center $-\overline{\beta}/\alpha$ and radius $(\sqrt{|\beta|^2-\alpha y})/|\alpha|$.*

Proof. Set $w = az+b$, so $z = (w-b)/a$. Hence

$$\begin{aligned} \alpha|z|^2 + \beta z + \overline{\beta}\overline{z} + y &= \frac{\alpha}{|a|^2}(w-b)\overline{(w-b)} + \frac{\beta}{a}(w-b) + \overline{\left(\frac{\beta}{a}\right)}\overline{(w-b)} + y \\ &= \frac{\alpha}{|a|^2}\left| w + \frac{\overline{\beta}a}{\alpha} - b \right|^2 + y - \frac{|\beta|^2}{\alpha} = 0, \end{aligned}$$

and so

$$\left| \frac{1}{a}(w-b) + \frac{\overline{\beta}}{\alpha} \right| = \left| z + \frac{\overline{\beta}}{\alpha} \right| = \frac{1}{|\alpha|}\sqrt{|\beta|^2 - \alpha y}.$$

Thus the assertion follows. ☐

Proof of the exercise 1. By choosing $\alpha = k+1$, $\beta = -\overline{\zeta}$ and $y = 1-k$ in Lemma 2, we see that $|z-\zeta|^2 = |z|^2 + 1 - \overline{\zeta}z - \zeta\overline{z} = k(1-|z|^2)$ is the Euclidean disk with center $\zeta/(k+1)$ and radius $k/(k+1)$. On the other hand if $|z| = 1$, then $k(1-|z|^2) = 0 = |\zeta - z|^2$, and so $\zeta = z$. Hence the assertion follows. ☐

2. Suppose that

$$\frac{|\eta - \varphi(z_0)|^2}{1 - |\varphi(z_0)|^2} = d(\zeta)\frac{|\zeta - z_0|^2}{1-|z_0|^2}$$

for some $z_0 \in \mathbb{D}$. Because $d(\zeta) \in (0,\infty)$, we may write the inequality of Julia's lemma as

$$\frac{1}{d(\zeta)}\frac{1-|z|^2}{|\zeta - z|^2} - \frac{1-|\varphi(z)|^2}{|\eta - \varphi(z)|^2} \leq 0, \qquad z \in \mathbb{D}.$$

By noticing that

$$
\begin{aligned}
1-|z|^2 &= \mathrm{Re}(1-|z|^2+i2\mathrm{Im}(\overline{\zeta}z)) = \mathrm{Re}(\zeta\overline{\zeta}-z\overline{z}+\overline{\zeta}z-\zeta\overline{z}) \\
&= \mathrm{Re}((\zeta+z)(\overline{\zeta}-\overline{z})),
\end{aligned}
$$

we see that

$$
\mathrm{Re}\left(\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z}-\frac{\eta+\varphi(z)}{\eta-\varphi(z)}\right) = \frac{1}{d(\zeta)}\frac{1-|z|^2}{|\zeta-z|^2}-\frac{1-|\varphi(z)|^2}{|\eta-\varphi(z)|^2} \leq 0
$$

for all $z \in \mathbb{D}$. Since equality holds at $z_0 \in \mathbb{D}$, the maximum principle for harmonic functions implies that equality holds for all $z \in \mathbb{D}$, and the open mapping theorem then gives

$$
\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z}-\frac{\eta+\varphi(z)}{\eta-\varphi(z)} = ic, \qquad z \in \mathbb{D},
$$

for some constant $c \in \mathbb{R}$. By solving $\varphi(z)$ we get

$$
\begin{aligned}
\varphi(z) &= \eta\left(\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z}-1-ic\right)\Bigg/\left(\frac{1}{d(\zeta)}\frac{\zeta+z}{\zeta-z}+1-ic\right) \\
&= \lambda\frac{z-w}{1-\overline{w}z},
\end{aligned}
$$

where

$$
\lambda = \eta\overline{\zeta}\frac{d(\zeta)+1+icd(\zeta)}{d(\zeta)+1-icd(\zeta)} \quad \textit{and} \quad w = \zeta\frac{d(\zeta)-1+icd(\zeta)}{d(\zeta)+1+icd(\zeta)}.
$$

Since clearly $|\lambda| = 1$ and $|w| < 1$ ($|d(\zeta)-1| < d(\zeta)+1$), we deduce that φ is an automorphism of $\mathbb{D}$.

3. How the set $\Gamma_p(\zeta,\alpha) = \{z \in \mathbb{D} : |z-\zeta|^p < \alpha(1-|z|)\}$, $1 < p, \alpha < \infty$, changes, when p and α change, can be seen in Figure 1.

Now, $\Gamma_p(\zeta,\alpha)$ is an open simply connected subset of $\mathbb{D}$. Here $\overline{\Gamma_p(\zeta,\alpha)} \cap \mathbb{T} = \zeta$. Also $\Gamma_p(\zeta,\alpha)$ is symmetrical with respect to the line $\{\zeta t : t \in \mathbb{R}\}$. Also $\partial\Gamma_p(\zeta,\alpha) \setminus \{\zeta t : t \in \mathbb{R}\}$ consists of two smooth simple curves.

Let $\partial\Gamma_p(\zeta,\alpha) \cap \{\zeta t : t \in \mathbb{R}\} = \{\zeta,\beta\}$. As α increases the 'angle' of $\Gamma_p(\zeta,\alpha)$ at ζ increases and $\partial\Gamma_p(\zeta,\alpha)$ becomes 'smoother' at β. As p increases $\partial\Gamma_p(\zeta,\alpha)$ becomes 'smoother' at ζ.

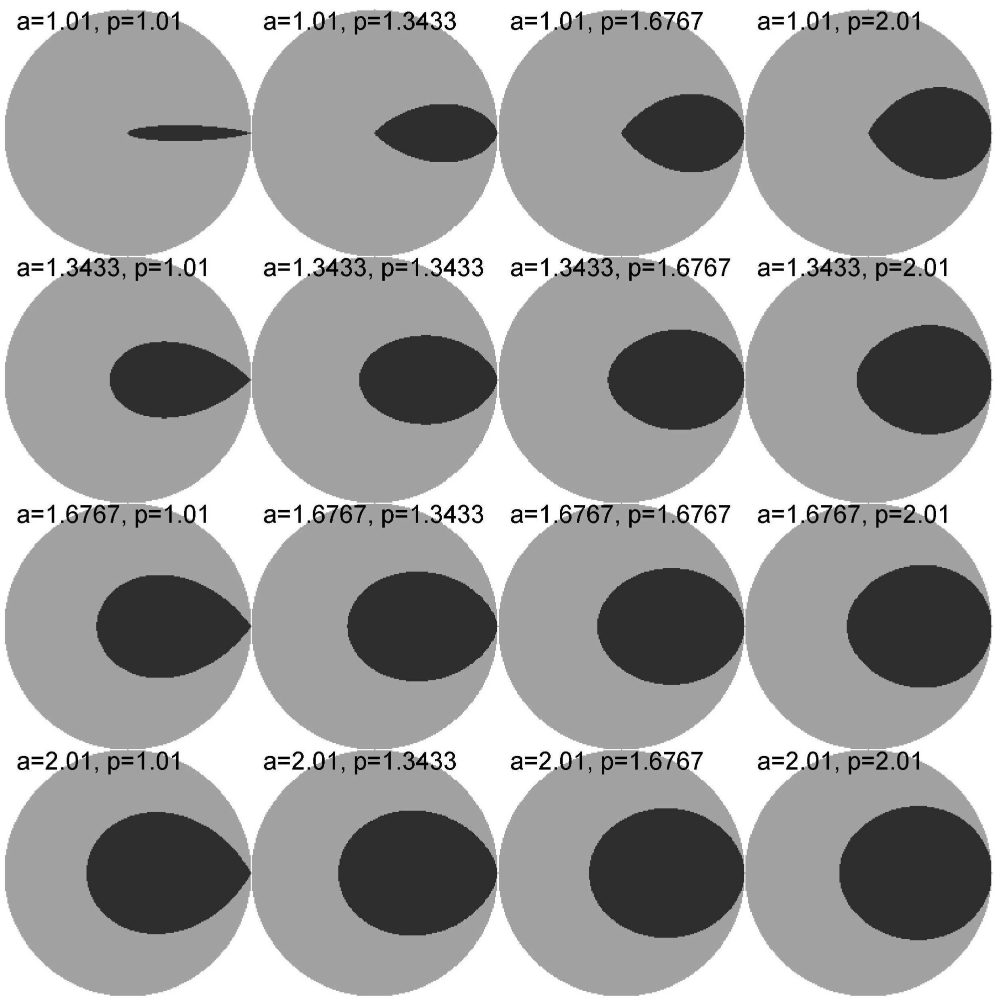

Figure 1: Sets $\Gamma_p(1,\alpha)$ (black) for $\zeta = 1$ and some different α and p in $\mathbb{D}$ (gray discs)

Lemma 3. *The inequality*

$$(x+y)^p \leq 2^{p-1}(x^p+y^p) \tag{3}$$

holds for all $p>1$ and $x,y\geq 0$.

Proof. If $x=0$ or $y=0$, then the statement is trivial, so we can suppose that $0<y\leq x$. Now we can rewrite the inequality (3) to the following form:

$$\left(\frac{x}{y}+1\right)^p \leq 2^{p-1}\left[\left(\frac{x}{y}\right)^p+1\right].$$

Hence, it enough to show that

$$f(t)=2^{p-1}(t^p+1)-(t+1)^p$$

is non-decreasing for all $t\geq 1$.

It is clear that $f(1)=0$ and

$$f'(t)=p((2t)^{p-1}-(t+1)^{p-1})\geq 0$$

for all $t\geq 1$. Thus the assertion follows. □

Proof of the exercise 3. Suppose that $0<\delta<\alpha^{-1}$, $|\lambda|\leq\delta|\zeta-z|^p$ and $z\in\Gamma_p(\zeta,\alpha)$. Then, by Lemma 3 and the triangle inequality, we obtain

$$\begin{aligned}|z+\lambda-\zeta|^p &\leq 2^{p-1}(|z-\zeta|^p+|\lambda|^p)\\ &\leq 2^{p-1}(\alpha(1-|z|)+\delta^p\alpha^p(1-|z|)^p)\\ &\leq 2^{p-1}(1-|z|)(\alpha+\delta^p\alpha^p)\end{aligned}$$

and $1-|z+\lambda|\geq 1-|z|-|\lambda|\geq 1-|z|-\delta\alpha(1-|z|)=(1-|z|)(1-\delta\alpha)$. Hence,

$$\begin{aligned}|z+\lambda-\zeta|^p &\leq 2^{p-1}(1-|z|)(\alpha+\delta^p\alpha^p)\\ &\leq 2^{p-1}\frac{\alpha+\delta^p\alpha^p}{1-\delta\alpha}(1-|z+\lambda|),\end{aligned}$$

and so $z+\lambda\in\Gamma_p(\zeta,\beta)$. □

4. Suppose that $\lim_{n\to\infty} |\arg(1-z_n)| > 0$. Then there exists $\alpha > 1$ such that $z_n \notin \Gamma(1,\alpha)$ for all n sufficiently large. Thus

$$\frac{1-|z_n|}{|1-z_n|} \le \frac{1-|z_n|}{\alpha(1-|z_n|)} = \frac{1}{\alpha}$$

for all n sufficiently large, and hence

$$\lim_{n\to\infty} \frac{1-|z_n|}{|1-z_n|} \le \frac{1}{\alpha} < 1$$

which is a contradiction.

5. Let f be positive ν-integrable function. Then, since $\frac{p+q}{q}, \frac{p+q}{p} > 1$ and $1/\frac{p+q}{q} + 1/\frac{p+q}{p} = 1$, Hölder's inequality gives

$$\begin{aligned} 1 = \int d\nu &= \int \left(\frac{f}{f}\right)^{\frac{pq}{p+q}} d\nu \\ &\le \left(\int \frac{d\nu}{\left(f^{\frac{pq}{p+q}}\right)^{\frac{p+q}{q}}}\right)^{\frac{q}{p+q}} \left(\int \left(f^{\frac{pq}{p+q}}\right)^{\frac{p+q}{p}} d\nu\right)^{\frac{p}{p+q}} \\ &= \left[\left(\int \frac{d\nu}{f^p}\right)^{\frac{1}{p}} \left(\int f^q d\nu\right)^{\frac{1}{q}}\right]^{\frac{pq}{p+q}}. \end{aligned}$$

The assertion follovs by taking the power of $\frac{p+q}{pq}$ on both sides and then dividing by $\left(\int \frac{d\nu}{f^p}\right)^{\frac{1}{p}}$.

Chapter 12.

1. The equality in (12.1) holds at least for all functions $\varphi(z) = \lambda z^2$, where $\lambda \in \mathbb{T}$; $\varphi'(z) = \lambda 2z$,

$$\varphi^*(z) = \lambda 2z \frac{1-|z|^2}{1-|\lambda z^2|^2} = \frac{\lambda 2z}{1+|z|^2},$$

and thus

$$\begin{aligned} d_h(\varphi^*(0), \varphi^*(z)) = d_h(0, \varphi^*(z)) &= \log \frac{1+\left|\frac{\lambda 2z}{1+|z|^2}\right|}{1-\left|\frac{\lambda 2z}{1+|z|^2}\right|} \\ &= \log\left(\frac{1+|z|}{1-|z|}\right)^2 = 2d_h(0,z). \end{aligned}$$

Let $z \in \mathbb{D}$, and suppose that equality in (12.1) holds for function φ. Then

$$\log \frac{1 + d_{ph}(\varphi^*(0), \varphi^*(z))}{1 - d_{ph}(\varphi^*(0), \varphi^*(z))} = 2 \log \frac{1 + |z|}{1 - |z|},$$

and thus

$$(1 - |z|)^2 (1 + d_{ph}(\varphi^*(0), \varphi^*(z))) = (1 + |z|)^2 (1 - d_{ph}(\varphi^*(0), \varphi^*(z))),$$

which is equivalent to

$$d_{ph}(\varphi^*(0), \varphi^*(z))) = \frac{2|z|}{1 + |z|^2} \tag{4}$$

If we suppose that (4) holds, then

$$d_h(\varphi^*(0), \varphi^*(z)) = \log \frac{1 + \frac{2|z|}{1+|z|^2}}{1 - \frac{2|z|}{1+|z|^2}} = 2 d_h(0, z).$$

Hence we see that (4) is necessary and sufficient condition for equality in (12.1) to hold at point z.

2. We didn't succeed in this exercise. It still remained open at 2.8.2013.