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About my research, long term plans

I am interested to continue my work in all the topics mentioned below. E.g. a discussion with visiting researchers lead to [4]. Many papers similar to [9] can be simplified with methods similar to those in [1]. Studies on Yamanoi's results and 3D graphics have proven to be fruitful.

About my research, current news

I am the second supervisor of Lasse Asikainen's doctoral thesis. Lasse is ready to submit his first manuscript in Spring 2023.

Working with the topics related to Lasse's thesis and Yamanoi's paper are fruitful.

About my publications

In [1], I studied the equation

$$f'' + A_1(z)\exp\left(\frac{a_1}{(1-z)^{p_1}}\right)f' + A_0(z)\exp\left(\frac{a_0}{(1-z)^{p_0}}\right)f = 0, \tag{$*$}$$

where A_j are analytic in $\mathbb{D}\cup\{1\}$, and $p_j \in (0,\infty)$, and $a_j \in \mathbb{C}$. Here $\mathbb{C}$ is the complex plane and $\mathbb{D} = \{z \in \mathbb{C}\,;\, |z| < 1\}$ is the unit disc. It is noteworthy, that behavior of A_j in other parts of $\partial\mathbb{D}$ is not known.

I found cases of suitable constants a_j, and p_j such that all non-trivial solutions of $(*)$ have order of growth

$$\sigma(f) = \lim_{r\to 1^-}\frac{\log\log M(r,f)}{-\log(1-r)} \geq p_0,$$

where $M(r,f) = \max_{|z|=r}|f(z)|$. The main tool was a localization mapping $T : \mathbb{D} \to \mathbb{D}$ such that $T(\mathbb{D})$ is a drop-shaped area touching the point $z = 1$. The motivation of was to show that results in [9] can be proved and improved by using some suitable localization mapping.

In [2], we studied the equation

$$f'' + A(z)f = 0, \tag{1}$$

where A is analytic in $\mathbb{D}$. We found that

$$\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2\log\frac{e}{1-z} < 1 \quad \text{implies} \quad \sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2) < \infty,$$

that is, f belongs to the Bloch space. Similarly,

$$\sup_{z\in\mathbb{D}}|A(z)|(1-|z|^2)^2 < p(p+1) \quad \text{implies} \quad \sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^p < \infty,$$

that is, f belongs to the H_p^∞ space. Both results were shown to be sharp. The results were generalized for the nth order equation. The main method was integration, that is, expressing f in terms of $f^{(n)}$ by using the Fundamental Theorem of calculus. A suitable weight function was introduced and studied.

In [3] we studied the equation (1). We found sufficient conditions for A such that all solutions of (1) belong to certain function space such as H^∞, BMOA or the Bloch space. The main method was to use representation theorems for the function spaces. Also the distribution of zeros of solutions of (1) and a third order equation was studied.

In [4], we studied the equation

$$f'' + A(z)f = 0$$

placing the solutions in the Fock space. The methods were mostly same than in [3].

In [6], we studied locally univalent analytic functions f that satisfy

$$\left|\frac{zf''(z)}{f'(z)}\right|(1-|z|^2) < 1 + C(1-|z|), \quad z \in \mathbb{D},$$

for some $C \in (0, \infty)$. We found that such f is univalent in certain horodiscs. That is, f attains each complex value at most once in such horodiscs. Becker's univalence criterion states that if $(*)$ holds for $C = 0$, then f is univalent in $\mathbb{D}$.

In [5] we studied finite valence criteria for locally univalent functions $f : \mathbb{D} \to \mathbb{D}$. We studied the case where f is analytic in $\mathbb{D}$, and the case where f is harmonic and orientation preserving in $\mathbb{D}$. In the latter case, $f = g + \overline{h}$ for some analytic functions g and h and $|h(z)| < |g(z)|$ for $z \in \mathbb{D}$. The main tool was to use the localization idea from the recent paper [10].

In [7] and [8], I helped with the complex analytic tools. The topic of the papers is photonics.

About my postdoc period 12/2018-7/2019

We studied two papers [11] and [12] by K. Yamanoi. We held a seminar discussing the details of these papers; the materials can be found in [13]. Currently, one PhD student is studying the analogues of the results for the difference operator. I am his second supervisor.

Let me say briefly what are [11] and [12] are about. In [11, Theorem 1.2] where, essentially, the number of poles and a-points of a meromorphic function are bounded by the number of zeros of its k^{th} derivative. This allows Yamanoi to prove the Mues' and Gol'dberg conjectures. The proof is divided into two theorems: we discussed the entire proof of [11, Theorem 1.3] in our seminar; whereas the proof of [11, Theorem 1.4] requires advanced techniques.

To discuss the content of [12], recall that the Second main theorem of Nevanlinna theory reads

$$\sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta} - a_j|} \, d\theta \leq 2T(r, f) - N_1(r, f) + S(r, f).$$

In [12, Theorem 1], Yamanoi lets $k = k(r)$ with suitable growth rate, replaces the sum with a maximum over a_j, and places the maximum inside the integral. Consequently, instead of an inequality, an asymptotic equality is attained. For details, see [12].

About programming 3D visualizations

I gave a talk [15] on my approach to create 3D visualizations by using the JSXGraph JavaScript library. Inspired by this, the developers are implementing the 3D features in the library. For an announcement, see [14].

References

My publications

[1] Huusko, J.-M., *Localisation of Linear Differential Equations in the Unit Disc by a Conformal Map*, Bull. Aust. Math. Soc. 93 (2016), 260–271.

[2] Huusko, J.-M., T. Korhonen, A. Reijonen, *Linear Differential Equations With Solutions in the Growth Space H^∞_ω*, Ann. Acad. Sci. Fenn. Math. 41 (2016), no. 1, 399 – 416.

[3] Gröhn, J., J-M. Huusko, J. Rättyä, *Linear differential equations with slowly growing solutions*, Trans. Amer. Math. Soc. 370 (2018), 7201-7227.

[4] Hu, G., J.-M. Huusko, J. Long, Y. Sun, *Linear differential equations with solutions lying in weighted Fock spaces*, Comp. Var. Ell. Eq., Volume 66, 2021.

[5] Huusko, J.-M. and M. Martin, *Criteria for bounded valence of harmonic mappings*, Comput. Methods Funct. Theory (2017).

[6] Huusko, J.-M., T. Vesikko, *On Becker's univalence criterion*, Journal of Mathematical Analysis and Applications, 458 (1), 781-794.

[7] Pesonen, H., A. Halder, J.-M. Huusko, A.T. Friberg, T. Setälä and J. Turunen, *Spatial coherence effects in second-harmonic generation of scalar light fields*, J. Opt., Volume 23, Number 3.

[8] Pesonen, H.A., J.-M. Huusko, X. Zang, A.T. Friberg, J. Turunen and T. Setälä, *Partial spectral and temporal coherence of plane-wave pulse trains in second-harmonic generation*, J. Opt. (2021)

Other

[9] Hamouda S., *Properties of solutions to linear differential equations with analytic coefficients in the unit disc*, Electron. J. Differential Equations 177 (2012), 1-8.

[10] Becker J. and Ch. Pommerenke, *Locally univalent functions and the Bloch and Dirichlet norm*, Comput. Methods Funct. Theory 16, 43–52 (2016)

[11] Yamanoi, K., *Zeros of higher derivatives of meromorphic functions in the complex plane*, Proc. London Math. Soc. (3) 106 (2013) 703-780.

[12] Yamanoi, K., *On a reversal of the second main theorem for meromorphic functions of finite order*, The proceedings of the 19th ICFIDCAA "Topics in Finite or Infinite Dimensional Complex Analysis", (2013), 75–83.

[13] Notes and exercises of the Yamanoi seminar http://integraali.com/yamanoi/

[14] Release of JSXGraph 1.4.3 https://jsxgraph.org/wp/2022-04-14-release-of-version-1.4.3/

Talks

[15] Huusko, J.-M., *JSXGraph and 3D graphics*, JSXGraph conference, 6.10.2021 (abstract).